On transitivity of $M(r, s)$-inequalities and geometry of higher duals of Banach spaces

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ABSTRACT. In this note we study transitivity of $M(r, s)$-inequalities and geometry of higher duals of Banach spaces. Our main theorem shows that if $X$ and $Y$ are closed subspaces of a Banach space $Z$ such that $X$ is an ideal satisfying the $M(r, s)$-inequality in $Y$ and $Y$ is an ideal satisfying the $M(r, s)$-inequality in $Z$, then $X$ is an ideal satisfying the $M(r, s)$-inequality in $Z$. This extends the corresponding result for $M$-ideals. As an application we show that if a Banach space $X$ is an ideal satisfying the $M(r, s)$-inequality in its bidual, then $X$ is an ideal satisfying the $M(r, s)$-inequality in its dual space of order $2n$ for every $n \in \mathbb{N}$. It follows that if $X$ is an ideal satisfying the $M(1, s)$-inequality in its bidual, then $X$ has the strong uniqueness property $SU$ in all its duals of even order. These results generalize the corresponding results for $M$-ideals established by Rao [R].

1. Introduction

Let $Y$ be a (real or complex) Banach space, and let $X$ be a closed subspace of $Y$. The subspace $X$ is called an ideal (cf. [GKS]) if there exists a norm one projection $P$ (an ideal projection) on the dual space $Y^*$ with $\ker P = X^\perp = \{ y^* \in Y^* : y^*(x) = 0 \ \forall x \in X \}$ (the annihilator of $X$ in $Y^*$). Given $r, s \in (0, 1]$, the subspace $X$ is said to be an ideal satisfying the $M(r, s)$-inequality in $Y$ (cf. [CNO]) if $X$ is an ideal in $Y$ and a corresponding ideal projection $P$ satisfies

$$\|y^*\| \geq r \|Py^*\| + s \|y^* - Py^*\| \ \forall y^* \in Y^*.$$ 

If

$$\|y^*\| = \|Py^*\| + \|y^* - Py^*\| \ \forall y^* \in Y^*,$$

then $X$ is said to be an $M$-ideal in $Y$ (cf. [AE] or e.g. [HWW]).

Clearly, the class of ideals satisfying the $M(r, s)$-inequality is more general than the class of $M$-ideals. In fact, $X$ is an ideal satisfying the $M(1, 1)$-inequality in $Y$ if and only if $X$ is an $M$-ideal in $Y$. In this case the projection $P$ is unique, and in the case $Y = X^{**}$, it is the canonical projection $\pi_X =$
where $j_X : X \to X^{**}$ denotes the canonical embedding) from $X^{***}$ onto $X^*$. According to the terminology in [CN1] (cf. also [CNO] and [HO]), we say that $X$ satisfies the $M(r,s)$-inequality if

$$
\|x^{***}\| \geq r \|\pi_X x^{***}\| + s \|x^{***} - \pi_X x^{***}\| \quad \forall x^{***} \in X^{***}.
$$

In [CN1] and [CNO] examples of spaces satisfying $M(r,s)$-inequality for some $r,s \in (0,1]$ that are not $M$-ideals are given, among them there are also spaces satisfying $M(1,s)$- or $M(r,1)$-inequalities.

The ideal projections are closely related to the Hahn-Banach extension operators. Recall that a Hahn-Banach extension operator is a linear operator $\varphi : X^* \to Y^*$ such that $\varphi x^*$ is a norm-preserving extension of $x^*$ for all $x^* \in X^*$. In this case note that $P = \varphi i_Y^* X^*$, where $i_Y^* X^*$ is denoting the inclusion map of $X$ into $Y$, is an ideal projection on $Y^*$ with $\ker P = X^1$. Also note that $i_Y^* X^*$ denotes the identity map on $X^*$. On the other hand, if $P$ is an ideal projection on $Y$ with $\ker P = X^1$, then $\varphi : X^* \to Y^*$ defined by $\varphi x^* = Py^*$, where $y^* \in Y^*$ is a norm-preserving extension of $x^* \in X^*$, is a Hahn-Banach extension operator.

In Section 2, we study transitivity of $M(r,s)$-inequalities. It is well known (see e.g. [HWW, Proposition 1.17 (b)]) that if $X$ and $Y$ are closed subspaces of a Banach space $Z$ such that $X$ is an $M$-ideal in $Y$ and $Y$ is an $M$-ideal in $Z$, then $X$ is an $M$-ideal in $Z$. Our main theorem (see Theorem 1) generalizes this result to $M(r,s)$-inequalities, thereby the connection between ideal projections and Hahn-Banach extension operators is used.

If $X$ is an $M$-ideal in $Y$, then the projection $I_Y - P$ has norm one, where $P$ is a corresponding ideal projection on $Y^*$. In general, this is not true for ideals satisfying the $M(r,s)$-inequality (see [CN1, Example 4.3] and [JW]). The above-mentioned property of $M$-ideals is used in the recent paper by Rao [R], where it is proved (see [R, Theorem 1]) that any Banach space which is an $M$-ideal in its bidual is an $M$-ideal in its duals of even order. In Section 3, as an application of our main theorem, a corresponding extended result for $M(r,s)$-inequalities is established (see Theorem 6). It contains the Rao's result as a particular case.

Let us fix some more notation. In a Banach space $X$, we denote the closed unit ball by $B_X$. For $n \in \mathbb{N}$, the dual space of $X$ of order $n$ is denoted by $X^{(n)}$. As usual, we identify Banach space $X$ and its canonical image $j_X : (n-1) \to j_X(X)$ in $X^{(2n)}$.

2. Transitivity of $M(r,s)$-inequalities

The following is our main result.

**Theorem 1.** Let $X$ and $Y$ be closed subspaces of a Banach space $Z$. Let $X$ be an ideal satisfying the $M(r,s)$-inequality in $Y$ and let $Y$ be an ideal satisfying the $M(u,v)$-inequality in $Z$ for some $r,s,u,v \in (0,1]$. If $su \leq v$, then $X$ is an ideal satisfying the $M(\frac{ru}{1-u} v, \frac{su}{1-u} v)$-inequality in $Z$. If $su \geq v$, then $X$ is an ideal satisfying the $M(\frac{ru}{s}, v)$-inequality in $Z$. 

Proof. Let $P$ and $Q$ be corresponding ideal projections on $Y^*$ and $Z^*$, respectively. Let $P = \varphi i_{XY}^*$ and $Q = \psi i_{YZ}^*$, where $\varphi: X^* \to Y^*$ and $\psi: Y^* \to Z^*$ are the Hahn-Banach extension operators (cf. the Introduction). Then $\psi\varphi: X^* \to Z^*$ is also a Hahn-Banach extension operator and hence $R = \psi \varphi i_{XZ}^*$ is an ideal projection on $Z^*$ with $\ker R = X^\perp$. We also note that $R = \psi P i_{YZ}^*$. For every $z^* \in Z^*$ we have

$$rv\|Rz^*\| + su\|z^* - Rz^*\|$$

$$= rv\|\psi P i_{YZ}^* z^*\| + su\|z^* - \psi P i_{YZ}^* z^*\| + \psi i_{YZ}^* z^* - \psi i_{YZ}^* z^*\|$$

$$\leq rv\|\psi P i_{YZ}^* z^*\| + su\|\psi (i_{YZ}^* z^* - P i_{YZ}^* z^*)\| + su\|z^* - \psi i_{YZ}^* z^*\|$$

$$= v(r\|P i_{YZ}^* z^*\| + s\|i_{YZ}^* z^* - P i_{YZ}^* z^*\|) + su\|z^* - Q z^*\|$$

$$\leq (v\|s\| i_{YZ}^* z^*\| + s\|z^* - Q z^*\|)$$

$$= (v - su\|i_{YZ}^* z^*\| + s\|z^*\|)$$

If $su \leq v$, then

$$rv\|Rz^*\| + su\|z^* - Rz^*\|$$

$$\leq (v - su\|i_{YZ}^* z^*\| + s\|z^*\|)$$

$$\leq (s(1 - u) + v\|z^*\|.$$

Hence $X$ is an ideal satisfying the $M(\frac{rv}{s(1-u)+v}, \frac{su}{s(1-u)+v})$-inequality in $Z$.

If $su \geq v$, then

$$rv\|Rz^*\| + su\|z^* - Rz^*\|$$

$$\leq (v - su\|i_{YZ}^* z^*\| + s\|z^*\|)$$

$$\leq s\|z^*\|.$$

Hence $X$ is an ideal satisfying the $M(\frac{rv}{s}, v)$-inequality in $Z$. 

One immediately obtains the following two corollaries.

**Corollary 2.** If $X$ is an ideal satisfying the $M(r, s)$-inequality in $Y$ and $Y$ is an ideal satisfying the $M(r, s)$-inequality in $Z$, then $X$ is an ideal satisfying the $M(\frac{rv}{s}, \frac{rv}{s})$-inequality in $Z$.

**Corollary 3** (see e.g. [HWW, Proposition 1.17 (b)]). If $X$ is an $M$-ideal in $Y$ and if $Y$ is an $M$-ideal in $Z$, then $X$ is an $M$-ideal in $Z$.

The next lemma leads us to a more general version of our main theorem.

**Lemma 4.** Let $X$ be an ideal satisfying the $M(r, s)$-inequality in a Banach space $Y$. Let $T$ be a linear isometry from $Y$ onto a Banach space $W$. Then $T(X)$ is an ideal satisfying the $M(r, s)$-inequality in $W$.

Proof. Let $P$ be a corresponding ideal projection on $Y^*$ with ker $P = X^\perp$. Let $P = \varphi i_{XY}^*$, where $\varphi: X^* \to Y^*$ is the Hahn-Banach extension operator. Take $R = (T^{-1})^* \varphi S^* i_{T(X)^*}^*$, where $S: X \to T(X)$ is defined by $Sx = Tx$ for every $x \in X$. Note that $(T^{-1})^* \varphi S^*: T(X)^* \to W^*$ is a
Hahn-Banach extension operator and $i^*_{T(X)W} = (S^{-1})^*i^*_{XY}T^*$. It follows that $R = (T^{-1})^*PT^*$ and for every $w^* \in W^*$ we have

$$r\|R w^*\| + s\|w^* - Rw^*\| \\
\quad = r\|(T^{-1})^*\phi S^*i^*_{S_{T(X)W} w^*} + s\|w^* - (T^{-1})^*\phi S^*i^*_{S_{T(X)W} w^*}\|
\quad \leq r\|(T^{-1})^*\| \|\phi S^*i^*_{S_{T(X)W} w^*}\| + s\|(T^{-1})^*\| \|T w^* - \phi S^*i^*_{S_{T(X)W} w^*}\|
\quad = r\|PT^*w^*\| + s\|T^*w^* - PT^*w^*\|
\quad \leq \|T^*w^*\|
\quad \leq \|T\| \|w^*\|
\quad = \|w^*\|.$$ 

Hence, $T(X)$ is an ideal satisfying the $M(r, s)$-inequality in $W$. \qed

Remark. If $r = s = 1$ in Lemma 4 and we consider all the pertinent spaces as subspaces of a certain Banach space, then we get exactly Lemma 2.1 in [R] without the assumption $Tx = x$, for all $x \in X$. Thus this assumption is superfluous.

Combining Theorem 1 and Lemma 4 we immediately get the following result.

**Theorem 5.** Let $X$ be an ideal satisfying the $M(r, s)$-inequality in a Banach space $Y$ and let $W$ be an ideal satisfying the $M(u, v)$-inequality in a Banach space $Z$ for some $r, s, u, v \in (0, 1]$. Assume further that there exists a linear isometry $T$ from $Y$ onto $W$. If $su \leq v$, then $T(X)$ is an ideal satisfying the $M\left(\frac{ru}{s(1-u)+v}, \frac{sv}{s(1-u)+v}\right)$-inequality in $Z$. If $su \geq v$, then $T(X)$ is an ideal satisfying the $M\left(\frac{ru}{s}, v\right)$-inequality in $Z$.

### 3. Applications

Our main result in this section is the following.

**Theorem 6.** Let $X$ be a Banach space and let $r, s \in (0, 1]$. If $X$ is an ideal satisfying the $M(r, s)$-inequality in its bidual, then $X$ is an ideal satisfying the $M\left(\frac{r}{r+n(1-r)}, \frac{s}{r+n(1-r)}\right)$-inequality in its dual space $X^{(2n)}$ for every $n \in \mathbb{N}$.

To prove this theorem we need the following lemmas. The first one gives us a sufficient condition for a subspace to be an ideal satisfying the $M(r, s)$-inequality.

**Lemma 7.** Let $Y$ be a Banach space and let $X$ be a closed subspace of $Y$. Let $r, s \in (0, 1]$. If there exists a norm one projection $Q$ on $Y$ with $\text{ran} Q = X$ and

$$\|rQy + s(I-Q)z\| \leq \max\{\|y\|, \|z\|\} \quad \forall y, z \in Y,$$

then $X$ is an ideal satisfying the $M(r, s)$-inequality in $Y$. 
It follows that

\[ \| \phi \| \leq \| \psi \| \]

and

\[ \| \psi \| \geq \| \phi \| \frac{r}{s} \]

since \( \phi \) and \( \psi \) are defined on the same subspace of \( X \) with \( \| \phi \| \leq \| \psi \| \).
Therefore the astriction $T$ of $A^{**}$ is a linear isometry from $X^{**}$ onto $(\text{ran } A)^{\perp}$ satisfying

$$T(j_X(X)) = (j_{X^{(2n)}} A)(X) = (j_{X^{(2n)}} \ldots j_X)(X)$$

as desired.

By Theorem 5, $T(j_X(X)) = (j_X^{(2n)} \ldots j_X)(X)$ is an ideal satisfying the $M(u, v)$-inequality in $X^{(2n+2)}$, where

$$u = \frac{r_{r+n(1-r)}^s}{s(1 - \frac{s}{r+n(1-r)}) + \frac{s}{r+n(1-r)}} = \frac{r}{r + n(1-r) - r + 1}$$

and, similarly,

$$v = \frac{s}{r + (n+1)(1-r)}.$$

The next results are immediate from Theorem 6.

**Corollary 9.** If $X$ satisfies the $M(1, s)$-inequality for some $s \in (0, 1]$, then $X$ is an ideal satisfying the $M(1, s)$-inequality in all its duals of even order.

**Corollary 10** (see [R, Theorem 2.2]). If $X$ is an $M$-ideal in its bidual, then $X$ is an $M$-ideal in all its duals of even order.

A subspace $X$ of a Banach space $Y$ is said to have property $SU$ (the strong uniqueness property) in $Y$ (cf. [O]) if there exists a linear projection $P$ on the dual space $Y^*$ with $\ker P = X^{\perp}$ such that for each $y^* \in Y^*$ with $y^* \neq Py^*$, one has $\|Py^*\| < \|y^*\|$. Let us remark (cf. [O]) that $X$ has property $SU$ in $Y$ if and only if $X$ is an ideal in $Y$ and $X$ has Phelps’ property $U$ (the uniqueness property) in $Y$ (cf. [Ph]) meaning that every functional $x^* \in X^*$ has a unique norm-preserving extension $y^* \in Y^*$. Subspaces having property $U$ or $SU$ have been studied e.g. in [O], [OP$_1$], [OP$_2$].

It is clear that if a subspace $X$ is an ideal satisfying the $M(1, s)$-inequality in a Banach space $Y$ for some $s \in (0, 1]$, then $X$ has property $SU$ in $Y$. Therefore the next result is immediate from Corollary 9.

**Corollary 11.** If $X$ satisfies the $M(1, s)$-inequality for some $s \in (0, 1]$, then $X$ has property $SU$ in all its duals of even order.

**Remark.** In [R] it is shown (see [R, Remark 2.3]) that if $X$ is an $M$-ideal in its bidual then all unit vectors of $X^*$ are points of weak*-weak continuity for the identity map on all the unit balls of duals of odd order, whereas only the fact that the unit vectors of $X^*$ have unique norm-preserving extensions to $X^{(4)}$ is used. Hence, the same conclusion holds if $X$ is an ideal satisfying the $M(1, s)$-inequality in its bidual.
Acknowledgement

This article is a part of my Ph.D. thesis, written under the direction of Eve Oja at Tartu University. I am very grateful to her valuable help.

References


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