ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 6, 2002

# On transitivity of $M(r,s)$-inequalities and geometry of higher duals of Banach spaces

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ABSTRACT. In this note we study transitivity of $M(r,s)$-inequalities and geometry of higher duals of Banach spaces. Our main theorem shows that if $X$ and $Y$ are closed subspaces of a Banach space $Z$ such that $X$ is an ideal satisfying the $M(r,s)$-inequality in $Y$ and $Y$ is an ideal satisfying the $M(r,s)$-inequality in $Z$, then $X$ is an ideal satisfying the $M(\frac{r}{2-r}, \frac{s}{2-r})$-inequality in $Z$. This extends the corresponding result for $M$-ideals. As an application we show that if a Banach space $X$ is an ideal satisfying the $M(r,s)$-inequality in its bidual, then $X$ is an ideal satisfying the $M(\frac{r}{r+n(1-r)}, \frac{s}{r+n(1-r)})$-inequality in its dual space of order $2n$ for every $n \in \mathbb{N}$. It follows that if $X$ is an ideal satisfying the $M(1,s)$-inequality in its bidual, then $X$ has the strong uniqueness property $SU$ in all its duals of even order. These results generalize the corresponding results for $M$-ideals established by Rao [R].

## 1. Introduction

Let $Y$ be a (real or complex) Banach space, and let $X$ be a closed subspace of $Y$. The subspace $X$ is called an *ideal* (cf. [GKS]) if there exists a norm one projection $P$ (an *ideal projection*) on the dual space $Y^*$ with $\ker P = X^\perp = \{y^* \in Y^* \colon y^*(x) = 0 \ \forall x \in X\}$ (the annihilator of $X$ in $Y^*$). Given $r, s \in (0,1]$, the subspace $X$ is said to be an *ideal satisfying the $M(r,s)$-inequality* in $Y$ (cf. [CNO]) if $X$ is an ideal in $Y$ and a corresponding ideal projection $P$ satisfies

$$\|y^*\| \geq r\,\|Py^*\| + s\,\|y^* - Py^*\| \quad \forall y^* \in Y^*.$$

If

$$\|y^*\| = \|Py^*\| + \|y^* - Py^*\| \quad \forall y^* \in Y^*,$$

then $X$ is said to be an *$M$-ideal* in $Y$ (cf. [AE] or e.g. [HWW]).

Clearly, the class of ideals satisfying the $M(r,s)$-inequality is more general than the class of $M$-ideals. In fact, $X$ is an ideal satisfying the $M(1,1)$-inequality in $Y$ if and only if $X$ is an $M$-ideal in $Y$. In this case the projection $P$ is unique, and in the case $Y = X^{**}$, it is the canonical projection $\pi_X =$

Received February 8, 2002.

2000 *Mathematics Subject Classification.* 46B20.

*Key words and phrases.* Ideals, $M(r,s)$-inequalities, higher duals, uniqueness properties.

Supported in part by Estonian Science Foundation Grant 4400.

$j_{X^*}(j_X)^*$ (where $j_X : X \to X^{**}$ denotes the canonical embedding) from $X^{***}$ onto $X^*$. According to the terminology in [CN$_1$] (cf. also [CNO] and [HO]), we say that $X$ *satisfies the $M(r,s)$-inequality* if

$$\|x^{***}\| \geq r\,\|\pi_X x^{***}\| + s\,\|x^{***} - \pi_X x^{***}\| \quad \forall x^{***} \in X^{***}.$$

In [CN$_1$] and [CNO] examples of spaces satisfying $M(r,s)$-inequality for some $r, s \in (0,1]$ that are not $M$-ideals are given, among them there are also spaces satisfying $M(1,s)$- or $M(r,1)$-inequalities.

The ideal projections are closely related to the Hahn-Banach extension operators. Recall that a *Hahn-Banach extension operator* is a linear operator $\varphi\colon X^* \to Y^*$ such that $\varphi x^*$ is a norm-preserving extension of $x^*$ for all $x^* \in X^*$. In this case note that $P = \varphi i^*_{XY}$, where $i_{XY}$ is denoting the inclusion map of $X$ into $Y$, is an ideal projection on $Y^*$ with $\ker P = X^\perp$. Also note that $i^*_{XY}\varphi = I_{X^*}$, where $I_{X^*}$ denotes the identity map on $X^*$. On the other hand, if $P$ is an ideal projection on $Y$ with $\ker P = X^\perp$, then $\varphi\colon X^* \to Y^*$ defined by $\varphi x^* = Py^*$, where $y^* \in Y^*$ is a norm-preserving extension of $x^* \in X^*$, is a Hahn-Banach extension operator.

In Section 2, we study transitivity of $M(r,s)$-inequalities. It is well known (see e.g. [HWW, Proposition 1.17 (b)] that if $X$ and $Y$ are closed subspaces of a Banach space $Z$ such that $X$ is an $M$-ideal in $Y$ and $Y$ is an $M$-ideal in $Z$, then $X$ is an $M$-ideal in $Z$. Our main theorem (see Theorem 1) generalizes this result to $M(r,s)$-inequalities, hereby the connection between ideal projections and Hahn-Banach extension operators is used.

If $X$ is an $M$-ideal in $Y$, then the projection $I_{Y^*} - P$ has norm one, where $P$ is a corresponding ideal projection on $Y^*$. In general, this is not true for ideals satisfying the $M(r,s)$-inequality (see [CN$_1$, Example 4.3] and [JW]). The above-mentioned property of $M$-ideals is used in the recent paper by Rao [R], where it is proved (see [R, Theorem 1]) that any Banach space which is an $M$-ideal in its bidual is an $M$-ideal in its duals of even order. In Section 3, as an application of our main theorem, a corresponding extended result for $M(r,s)$-inequalities is established (see Theorem 6). It contains the Rao's result as a particular case.

Let us fix some more notation. In a Banach space $X$, we denote the closed unit ball by $B_X$. For $n \in \mathbb{N}$, the dual space of $X$ of order $n$ is denoted by $X^{(n)}$. As usual, we identify Banach space $X$ and its canonical image $j_{X^{(2n-2)}} \dots j_X(X)$ in $X^{(2n)}$.

## 2. Transitivity of $M(r,s)$-inequalities

The following is our main result.

**Theorem 1.** *Let $X$ and $Y$ be closed subspaces of a Banach space $Z$. Let $X$ be an ideal satisfying the $M(r,s)$-inequality in $Y$ and let $Y$ be an ideal satisfying the $M(u,v)$-inequality in $Z$ for some $r,s,u,v \in (0,1]$. If $su \leq v$, then $X$ is an ideal satisfying the $M(\frac{rv}{s(1-u)+v}, \frac{sv}{s(1-u)+v})$-inequality in $Z$. If $su \geq v$, then $X$ is an ideal satisfying the $M(\frac{rv}{s}, v)$-inequality in $Z$.*

*Proof.* Let $P$ and $Q$ be corresponding ideal projections on $Y^*$ and $Z^*$, respectively. Let $P = \varphi i^*_{XY}$ and $Q = \psi i^*_{YZ}$, where $\varphi\colon X^* \to Y^*$ and $\psi\colon Y^* \to Z^*$ are the Hahn-Banach extension operators (cf. the Introduction). Then $\psi\varphi\colon X^* \to Z^*$ is also a Hahn-Banach extension operator and hence $R = \psi\varphi i^*_{XZ}$ is an ideal projection on $Z^*$ with $\ker R = X^\perp$. We also note that $R = \psi P i^*_{YZ}$. For every $z^* \in Z^*$ we have

$$\begin{aligned}
&rv\|Rz^*\| + sv\|z^* - Rz^*\| \\
&\quad = rv\|\psi P i^*_{YZ} z^*\| + sv\|z^* - \psi P i^*_{YZ} z^* + \psi i^*_{YZ} z^* - \psi i^*_{YZ} z^*\| \\
&\quad \le rv\|\psi P i^*_{YZ} z^*\| + sv\|\psi(i^*_{YZ} z^* - P i^*_{YZ} z^*)\| + sv\|z^* - \psi i^*_{YZ} z^*\| \\
&\quad = v(r\|P i^*_{YZ} z^*\| + s\|i^*_{YZ} z^* - P i^*_{YZ} z^*\|) + sv\|z^* - Qz^*\| \\
&\quad \le v\|i^*_{YZ} z^*\| + sv\|z^* - Qz^*\| \\
&\quad \le v\|i^*_{YZ} z^*\| + s(\|z^*\| - u\|Qz^*\|) \\
&\quad = v\|i^*_{YZ} z^*\| + s(\|z^*\| - u\|i^*_{YZ} z^*\|) \\
&\quad = (v - su)\|i^*_{YZ} z^*\| + s\|z^*\|.
\end{aligned}$$

If $su \le v$, then

$$\begin{aligned}
&rv\|Rz^*\| + sv\|z^* - Rz^*\| \\
&\quad \le (v - su)\|i^*_{YZ} z^*\| + s\|z^*\| \\
&\quad \le (s(1-u) + v)\|z^*\|.
\end{aligned}$$

Hence $X$ is an ideal satisfying the $M(\frac{rv}{s(1-u)+v}, \frac{sv}{s(1-u)+v})$-inequality in $Z$.

If $su \ge v$, then

$$\begin{aligned}
&rv\|Rz^*\| + sv\|z^* - Rz^*\| \\
&\quad \le (v - su)\|i^*_{YZ} z^*\| + s\|z^*\| \\
&\quad \le s\|z^*\|.
\end{aligned}$$

Hence $X$ is an ideal satisfying the $M(\frac{rv}{s}, v)$-inequality in $Z$. □

One immediately obtains the following two corollaries.

**Corollary 2.** *If $X$ is an ideal satisfying the $M(r,s)$-inequality in $Y$ and $Y$ is an ideal satisfying the $M(r,s)$-inequality in $Z$, then $X$ is an ideal satisfying the $M(\frac{r}{2-r}, \frac{s}{2-r})$-inequality in $Z$.*

**Corollary 3** (see e.g. [HWW, Proposition 1.17 (b)]). *If $X$ is an $M$-ideal in $Y$ and if $Y$ is an $M$-ideal in $Z$, then $X$ is an $M$-ideal in $Z$.*

The next lemma leads us to a more general version of our main theorem.

**Lemma 4.** *Let $X$ be an ideal satisfying the $M(r,s)$-inequality in a Banach space $Y$. Let $T$ be a linear isometry from $Y$ onto a Banach space $W$. Then $T(X)$ is an ideal satisfying the $M(r,s)$-inequality in $W$.*

*Proof.* Let $P$ be a corresponding ideal projection on $Y^*$ with $\ker P = X^\perp$. Let $P = \varphi i^*_{XY}$, where $\varphi\colon X^* \to Y^*$ is the Hahn-Banach extension operator. Take $R = (T^{-1})^*\varphi S^* i^*_{T(X)W}$, where $S\colon X \to T(X)$ is defined by $Sx = Tx$ for every $x \in X$. Note that $(T^{-1})^*\varphi S^*\colon T(X)^* \to W^*$ is a

Hahn-Banach extension operator and $i^*_{T(X)W} = (S^{-1})^* i^*_{XY} T^*$. It follows that $R = (T^{-1})^* P T^*$ and for every $w^* \in W^*$ we have

$$\begin{aligned}
r\|Rw^*\| &+ s\|w^* - Rw^*\| \\
&= r\|(T^{-1})^* \varphi S^* i^*_{T(X)W} w^*\| + s\|w^* - (T^{-1})^* \varphi S^* i^*_{T(X)W} w^*\| \\
&\le r\|(T^{-1})^*\| \, \|\varphi S^* i^*_{T(X)W} w^*\| + s\|(T^{-1})^*\| \, \|Tw^* - \varphi S^* i^*_{T(X)W} w^*\| \\
&= r\|PT^*w^*\| + s\|T^*w^* - PT^*w^*\| \\
&\le \|T^*w^*\| \\
&\le \|T\| \, \|w^*\| \\
&= \|w^*\|.
\end{aligned}$$

Hence, $T(X)$ is an ideal satisfying the $M(r,s)$-inequality in $W$. □

*Remark.* If $r = s = 1$ in Lemma 4 and we consider all the pertinent spaces as subspaces of a certain Banach space, then we get exactly Lemma 2.1 in [R] without the assumption $Tx = x$, for all $x \in X$. Thus this assumption is superfluous.

Combining Theorem 1 and Lemma 4 we immediately get the following result.

**Theorem 5.** *Let $X$ be an ideal satisfying the $M(r,s)$-inequality in a Banach space $Y$ and let $W$ be an ideal satisfying the $M(u,v)$-inequality in a Banach space $Z$ for some $r,s,u,v \in (0,1]$. Assume further that there exists a linear isometry $T$ from $Y$ onto $W$. If $su \le v$, then $T(X)$ is an ideal satisfying the $M(\frac{rv}{s(1-u)+v}, \frac{sv}{s(1-u)+v})$-inequality in $Z$. If $su \ge v$, then $T(X)$ is an ideal satisfying the $M(\frac{rv}{s}, v)$-inequality in $Z$.*

## 3. Applications

Our main result in this section is the following.

**Theorem 6.** *Let $X$ be a Banach space and let $r,s \in (0,1]$. If $X$ is an ideal satisfying the $M(r,s)$-inequality in its bidual, then $X$ is an ideal satisfying the $M(\frac{r}{r+n(1-r)}, \frac{s}{r+n(1-r)})$-inequality in its dual space $X^{(2n)}$ for every $n \in \mathbb{N}$.*

To prove this theorem we need the following lemmas. The first one gives us a sufficient condition for a subspace to be an ideal satisfying the $M(r,s)$-inequality.

**Lemma 7.** *Let $Y$ be a Banach space and let $X$ be a closed subspace of $Y$. Let $r,s \in (0,1]$. If there exists a norm one projection $Q$ on $Y$ with $\operatorname{ran} Q = X$ and*

$$\|rQy + s(I-Q)z\| \le \max\{\|y\|, \|z\|\} \quad \forall y, z \in Y,$$

*then $X$ is an ideal satisfying the $M(r,s)$-inequality in $Y$.*

*Proof.* Take $P = Q^*$. Clearly, $X$ is an ideal in $Y$ with respect to the projection $P$. Let $y^* \in Y^*$ and $\epsilon > 0$. Choose $y, z \in B_Y$ so that

$$\operatorname{Re}(Py^*)(y) \geq \|Py^*\| - \epsilon$$

and

$$\operatorname{Re}((I_{Y^*} - P)y^*)(z) \geq \|(I_{Y^*} - P)y^*\| - \epsilon.$$

Then

$$\begin{aligned} \|y^*\| &\geq |y^*(rQx + s(I_Y - Q)z)| \\ &\geq \operatorname{Re} r(Py^*)(y) + \operatorname{Re} s((I_{Y^*} - P)y^*)(z) \\ &\geq r\|Py^*\| + s\|(I_{Y^*} - P)y^*\| - (r+s)\epsilon. \end{aligned}$$

Since $\epsilon$ was arbitrary, we have

$$\|y^*\| \geq r\|Py^*\| + s\|(I_{Y^*} - P)x^*\| \quad \forall y^* \in Y^*.$$

Hence, $X$ is an ideal satisfying the $M(r,s)$-inequality in $Y$. □

**Lemma 8.** *If $X$ is an ideal satisfying the $M(r,s)$-inequality in $Y$, then $X^{\perp\perp}$ is an ideal satisfying the $M(r,s)$-ineqality in $Y^{**}$.*

*Proof.* Let $X$ be an ideal satisfying the $M(r,s)$-inequality in $Y$, and let $P$ be a corresponding ideal projection on $Y^*$. Let $y^{**}, z^{**} \in Y^{**}$. Then for every $y^* \in Y^*$, we have

$$\begin{aligned} &|(rP^*y^{**} + s(I_{Y^{**}} - P^*)z^{**})y^*| \\ &\quad\leq |y^{**}(rPy^*)| + |z^{**}(s(I_{Y^*} - P)y^*)| \\ &\quad\leq \max\{\|y^{**}\|, \|z^{**}\|\}\,(r\|Py^*\| + s\|(I_{Y^*} - P)y^*\|) \\ &\quad\leq \max\{\|y^{**}\|, \|z^{**}\|\}\,\|y^*\|. \end{aligned}$$

Hence,

$$\|rP^*y^{**} + s(I_{Y^{**}} - P^*)z^{**}\| \leq \max\{\|y^{**}\|, \|z^{**}\|\}.$$

Clearly, $\operatorname{ran} P^* = (\ker P)^\perp = (X^\perp)^\perp = X^{\perp\perp}$. By Lemma 7, $X^{\perp\perp}$ is an ideal satisfying the $M(r,s)$-inequality in $Y^{**}$. □

*Proof of Theorem* 6. Let $X$ be an ideal satisfying the $M(r,s)$-inequality in its bidual. The proof goes by induction. Assume that $X$ is an ideal satisfying the $M(\frac{r}{r+n(1-r)}, \frac{s}{r+n(1-r)})$-inequality in the dual space $X^{(2n)}$ for some $n \in \mathbb{N}$, or more precisely, that $(j_{X^{(2n-2)}} \ldots j_X)(X)$ is an ideal satisfying the $M(\frac{r}{r+n(1-r)}, \frac{s}{r+n(1-r)})$-inequality in $X^{(2n)}$. Note that this is true for $n = 1$. By Lemma 8, $((j_{X^{(2n-2)}} \ldots j_X)(X))^{\perp\perp}$ is an ideal satisfying the $M(\frac{r}{r+n(1-r)}, \frac{s}{r+n(1-r)})$-inequality in $X^{(2n+2)}$.

Using an idea of Rao (cf. [R, proof of Theorem 2.2]) it can be shown that there exists a linear isometry $T$ from $X^{**}$ onto $((j_{X^{(2n-2)}} \ldots j_X)(X))^{\perp\perp}$ such that $T(j_X(X)) = (j_{X^{(2n)}} j_{X^{(2n-2)}} \ldots j_X)(X)$. We, however, prefer the following simpler argument. Consider the linear into isometry

$$A := j_{X^{(2n-2)}} \ldots j_X \colon X \to X^{(2n)}.$$

Then $A^{**}\colon X^{**} \to X^{(2n+2)}$ is also a linear into isometry. Since $\operatorname{ran} A^* = (\ker A)^\perp$ (because $\operatorname{ran} A$ is closed) is closed, we have $\operatorname{ran} A^{**} = (\ker A^*)^\perp =$

$(\operatorname{ran} A)^{\perp\perp}$. Therefore the astriction $T$ of $A^{**}$ is a linear isometry from $X^{**}$ onto $(\operatorname{ran} A)^{\perp\perp}$ satisfying

$$T(j_X(X)) = (j_{X^{(2n)}}A)(X) = (j_{X^{(2n)}} \dots j_X)(X)$$

as desired.

By Theorem 5, $T(j_X(X)) = (j_{X^{(2n)}} \dots j_X)(X)$ is an ideal satisfying the $M(u,v)$-inequality in $X^{(2n+2)}$, where

$$u = \frac{r\frac{s}{r+n(1-r)}}{s(1-\frac{r}{r+n(1-r)}) + \frac{s}{r+n(1-r}} = \frac{r}{r+n(1-r)-r+1}$$
$$= \frac{r}{r+(n+1)(1-r)}$$

and, similarly,

$$v = \frac{s}{r+(n+1)(1-r)}.$$

□

The next results are immediate from Theorem 6.

**Corollary 9.** *If $X$ satisfies the $M(1,s)$-inequality for some $s \in (0,1]$, then $X$ is an ideal satisfying the $M(1,s)$-inequality in all its duals of even order.*

**Corollary 10** (see [R, Theorem 2.2]). *If $X$ is an $M$-ideal in its bidual, then $X$ is an $M$-ideal in all its duals of even order.*

A subspace $X$ of a Banach space $Y$ is said to have *property SU* (the strong uniqueness property) in $Y$ (cf. [O]) if there exists a linear projection $P$ on the dual space $Y^*$ with $\ker P = X^{\perp}$ such that for each $y^* \in Y^*$ with $y^* \neq Py^*$, one has $\|Py^*\| < \|y^*\|$. Let us remark (cf. [O]) that $X$ has property $SU$ in $Y$ if and only if $X$ is an ideal in $Y$ and $X$ has Phelps' *property U* (the uniqueness property) in $Y$ (cf. [Ph]) meaning that every functional $x^* \in X^*$ has a unique norm-preserving extension $y^* \in Y^*$. Subspaces having property $U$ or $SU$ have been studied e.g. in [O], [OP$_1$], [OP$_2$].

It is clear that if a subspace $X$ is an ideal satisfying the $M(1,s)$-inequality in a Banach space $Y$ for some $s \in (0,1]$, then $X$ has property $SU$ in $Y$. Therefore the next result is immediate from Corollary 9.

**Corollary 11.** *If $X$ satisfies the $M(1,s)$-inequality for some $s \in (0,1]$, then $X$ has property SU in all its duals of even order.*

*Remark.* In [R] it is shown (see [R, Remark 2.3]) that if $X$ is an $M$-ideal in its bidual then all unit vectors of $X^*$ are points of weak*-weak continuity for the identity map on all the unit balls of duals of odd order, whereas only the fact that the unit vectors of $X^*$ have unique norm-preserving extensions to $X^{(4)}$ is used. Hence, the same conclusion holds if $X$ is an ideal satisfying the $M(1,s)$-inequality in its bidual.

## Acknowledgement

This article is a part of my Ph.D. thesis, written under the direction of Eve Oja at Tartu University. I am very grateful to her valuable help.

## References

[AE] E. M. Alfsen, E. G. Effros, *Structure in real Banach spaces*, Ann. Math. **68** (1972), 98–173.

[CN$_1$] J. C. Cabello, E. Nieto, *On properties of $M$-ideals*, Rocky Mountain J. Math. **28** (1998), 61–93.

[CN$_2$] J. C. Cabello, E. Nieto, *An ideal characterization of when a subspace of certain Banach spaces has the metric compact approximation property*, Studia Math. **129** (1998), 185–196.

[CNO] J. C. Cabello, E. Nieto, and E. Oja, *On ideals of compact operators satisfying the $M(r,s)$-inequality*, J. Math. Anal. Appl. **220** (1998), 334–348.

[GKS] G. Godefroy, N. J. Kalton, and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59.

[HO] R. Haller, E. Oja, *Geometric characterizations of positions of Banach spaces in their biduals*, Arch. Math. **69** (1997), 227–233.

[HWW] P. Harmand, D. Werner, and W. Werner, *$M$-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math., vol. 1547, Springer, Berlin, 1993.

[JW] J. Johnson and J. Wolfe, *On the norm of the canonical projection of $E^{***}$ onto $E^{\perp}$*, Proc. Amer. Math. Soc. **75** (1979), 50–52.

[O] E. Oja, *Strong uniqueness of the extension of linear continuous functionals according to the Hahn-Banach theorem*, Mat. Zametki **43** (1988), 237–246 (Russian); English transl. in Math. Notes **43** (1988), 134–139.

[OP$_1$] E. Oja, M. Põldvere, *On subspaces of Banach spaces where every functional has a unique norm-preserving extension*, Studia Math. **117** (1996), 289–306.

[OP$_2$] E. Oja, M. Põldvere, *Intersection properties of ball sequences and uniqueness of Hahn-Banach extensions*, Proc. Royal Soc. Edinburgh **129A** (1999), 1–12.

[Ph] R. R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. **95** (1960), 238–255.

[R] T. S. S. R. K. Rao, *On the geometry of higher duals of a Banach space*, Illinois J. Math. **45** (2001), 1389–1392.

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