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Approximation properties of certain linear positive operators of functions of two variables

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ABSTRACT. We introduce certain modified Szasz–Mirakyan operators in polynomial and exponential weighted spaces of functions of two variables and we study the degree of approximation of functions by these operators.
Similar theorems for functions of one variable were given in [4] and [5].

I. Approximation in polynomial weighted spaces

1. Preliminaries

1.1. Let as in [1], for $p \in \mathbb{N}_0 := \{0, 1, 2, ...\}$,

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \quad \text{if} \quad p \geq 1, \quad x \in \mathbb{R}_0 := [0, +\infty). \tag{1}$$

Next, for given $p, q \in \mathbb{N}_0$, we define the weighted function

$$w_{p,q}(x, y) := w_p(x) w_q(y), \quad (x, y) \in \mathbb{R}_0^2 := \mathbb{R}_0 \times \mathbb{R}_0, \tag{2}$$

and the weighted space $C_{p,q}$ of all real-valued functions f continuous on $\mathbb{R}_0^2$ for which $w_{p,q} f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$ and the norm is defined by the formula

$$\|f\|_{p,q} \equiv \|f(\cdot, \cdot)\|_{p,q} := \sup_{(x,y) \in \mathbb{R}_0^2} w_{p,q}(x, y) \, |f(x, y)| \, . \tag{3}$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

$$\omega(f, C_{p,q}; t, s) := \sup_{0 \leq h \leq t, \, 0 \leq \delta \leq s} \|\Delta_{h,\delta} f(\cdot, \cdot)\|_{p,q} \, , \quad t, s \geq 0 \, , \tag{4}$$

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where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ and $(x+h, y+\delta) \in \mathbb{R}_0^2$. Moreover let $C_{p,q}^1$ be the set of all functions $f \in C_{p,q}$ which first partial derivatives belong also to $C_{p,q}$.

From (4) it follows that

$$\lim_{t,s\to 0+} \omega(f, C_{p,q}; t, s) = 0 \tag{5}$$

for every $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$.

1.2. We introduce the following

Definition 1. Let $r, s \in \mathbb{N} := \{1, 2, \cdots\}$ be fixed numbers. For functions $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$, we define operators

$$\begin{aligned} A_{m,n}(f; r, s; x, y) &\equiv A_{m,n}(f; x, y) \\ &:= \frac{1}{g\left((mx+1)^2; r\right) g\left((ny+1)^2; s\right)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(mx+1)^{2j}}{(j+r)!} \frac{(ny+1)^{2k}}{(k+s)!} \\ &\quad \cdot f\left(\frac{j+r}{m(mx+1)}, \frac{k+s}{n(ny+1)}, \right) \end{aligned} \tag{6}$$

for $(x, y) \in \mathbb{R}_0^2$, $m, n \in \mathbb{N}$, where

$$g(t; r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \quad t \in \mathbb{R}_0, \tag{7}$$

i.e.

$$g(0; r) = \frac{1}{r!}, \quad g(t; r) = \frac{1}{t^r} \left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!} \right) \quad \text{if} \quad t > 0.$$

In [4] modified Szasz–Mirakyan operators $A_n(f; r; x)$ were considered, where

$$\begin{aligned} A_n(f; r; x) &:= \frac{1}{g\left((nx+1)^2; r\right)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} \\ &\quad \cdot f\left(\frac{k+r}{n(nx+1)}\right), \quad x \in \mathbb{R}_0, \quad n, r \in \mathbb{N}, \end{aligned} \tag{8}$$

for functions of one variable.

From (6)–(8) we deduce that $A_{m,n}(f; r, s)$ are well defined in every space $C_{p,q}$, $p, q \in \mathbb{N}_0$. Moreover for fixed $r, s \in \mathbb{N}$ we have

$$A_{m,n}(1; r, s; x, y) = 1 \quad \text{for} \quad (x, y) \in \mathbb{R}_0^2, \quad m, n \in \mathbb{N}, \tag{9}$$

and if $f \in C_{p,q}$ and $f(x,y) = f_1(x)f_2(y)$ for all $(x,y) \in \mathbb{R}_0^2$, then

$$A_{m,n}(f;r,s;x,y) = A_m(f_1;r;x)A_n(f_2;s;y) \tag{10}$$

for all $(x,y) \in \mathbb{R}_0^2$ and $m,n \in \mathbb{N}$.

In this paper we shall denote by $M_k(\alpha,\beta)$ suitable positive constants depending only on indicated parameters α, β.

2. Lemmas and theorems

2.1. From (8) and (7) we get for $x \in \mathbb{R}_0$ and $n, r \in \mathbb{N}$

$$A_n(1;r;x) = 1, \tag{11}$$

$$\begin{aligned} A_n(t-x;r;x) &= \frac{1}{n} + \frac{1}{n(nx+1)(r-1)!g((nx+1)^2;r)}, \\ A_n((t-x)^2;r;x) &= \frac{2}{n^2} + \frac{r+(nx+1)^2-2nx(nx+1)}{n^2(nx+1)^2(r-1)!g((nx+1)^2;r)}. \end{aligned} \tag{12}$$

In the paper [4] the following lemma was proved for $A_n(f;r)$ defined by (8).

Lemma 1. *For every fixed $p \in \mathbb{N}_0$ and $r \in \mathbb{N}$ there exist positive constants $M_i \equiv M_i(p,r)$, $i = 1,2$, such that for all $x \in \mathbb{R}_0$, $n \in \mathbb{N}$*

$$w_p(x)\, A_n\left(1/w_p(t);r;x\right) \le M_1, \tag{13}$$

$$w_p(x)\, A_n\left((t-x)^2/w_p(t);r;x\right) \le \frac{M_2}{n^2}. \tag{14}$$

Applying Lemma 1 we shall prove the main lemma on $A_{m,n}$ defined by (6).

Lemma 2. *For fixed $p,q \in \mathbb{N}_0$ and $r,s \in \mathbb{N}$ there exists a positive constant $M_3 \equiv M_3(p,q,r,s)$ such that*

$$\left\|A_{m,n}\left(1/w_{p,q}(t,z);r,s;\cdot,\cdot\right)\right\|_{p,q} \le M_3 \quad \text{for} \quad m,n \in \mathbb{N}. \tag{15}$$

Moreover for every $f \in C_{p,q}$ we have

$$\left\|A_{m,n}\left(f;r,s;\cdot,\cdot\right)\right\|_{p,q} \le M_3 \left\|f\right\|_{p,q} \quad \text{for} \quad m,n \in \mathbb{N}, \quad r,s \in \mathbb{N}. \tag{16}$$

The formulas (6)–(7) and the inequality (16) show that $A_{m,n}$, $m,n \in \mathbb{N}$, defined by (6) are linear positive operators from the space $C_{p,q}$ into $C_{p,q}$.

Proof. The inequality (15) follows immediately from (2), (10) and (13). From (6) and (3) we get for $f \in C_{p,q}$ and $r,s \in \mathbb{N}$

$$\left\|A_{m,n}(f;r,s)\right\|_{p,q} \le \left\|f\right\|_{p,q} \left\|A_{m,n}(1/w_{p,q};r,s)\right\|_{p,q}, \quad m,n \in \mathbb{N},$$

which implies (16) by (15). □

2.2. Now we shall give two theorems on the degree of approximation of functions by $A_{m,n}$ defined by (6).

Theorem 1. *Suppose that* $f \in C^1_{p,q}$ *with fixed* $p, q \in \mathbb{N}_0$. *Then there exists a positive constant* $M_4 = M_4(p,q,r,s)$ *such that for all* $m, n \in \mathbb{N}$ *and* $r, s \in \mathbb{N}$

$$\|A_{m,n}(f;r,s;\cdot,\cdot) - f(\cdot,\cdot)\|_{p,q} \le M_4 \left\{ \frac{1}{m}\|f'_x\|_{p,q} + \frac{1}{n}\|f'_y\|_{p,q} \right\}. \tag{17}$$

Proof. Let $(x,y) \in \mathbb{R}^2_0$ be a fixed point. Then for $f \in C^1_{p,q}$

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)du + \int_y^z f'_v(x,v)dv, \qquad (t,z) \in \mathbb{R}^2_0.$$

Thus by (9)

$$\begin{aligned} &A_{m,n}(f(t,z);r,s;x,y) - f(x,y) \\ &= A_{m,n}\left(\int_x^t f'_u(u,z)du;r,s;x,y\right) + A_{m,n}\left(\int_y^z f'_v(x,v)dv;r,s;x,y\right). \end{aligned} \tag{18}$$

By (1)–(3) we have

$$\begin{aligned} \left|\int_x^t f'_u(u,z)du\right| &\le \|f'_x\|_{p,q}\left|\int_x^t \frac{du}{w_{p,q}(u,z)}\right| \\ &\le \|f'_x\|_{p,q}\left(\frac{1}{w_{p,q}(t,z)} + \frac{1}{w_{p,q}(x,z)}\right)|t-x|, \end{aligned}$$

which by (1), (2), (6) and (8)–(11) implies

$$\begin{aligned} &w_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f'_u(u,z)du;r,s;x,y\right)\right| \\ &\quad\le w_{p,q}(x,y)A_{m,n}\left(\left|\int_x^t f'_u(u,z)du\right|;r,s;x,y\right) \\ &\quad\le \|f'_x\|_{p,q}\, w_{p,q}(x,y)\left\{A_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)};r,s;x,y\right)\right. \\ &\qquad\left. + A_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)};r,s;x,y\right)\right\} \\ &\quad\le \|f'_x\|_{p,q}\, w_q(y)A_n\left(\frac{1}{w_q(z)};s;y\right)\left\{w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};r;x\right)\right. \\ &\qquad\left. + A_m(|t-x|;r;x)\right\}. \end{aligned}$$

Applying the Hölder inequality and (11)–(14), we get

$$A_m\left(|t-x|;r;x\right) \le \left\{A_m((t-x)^2;r;x)A_m(1;r;x)\right\}^{1/2} \le \frac{M_5(p,r)}{m},$$

$$w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};r;x\right) \le \left\{w_p(x)A_m\left(\frac{(t-x)^2}{w_p(t)};r;x\right)\right\}^{1/2} \cdot \left\{w_p(x)A_m\left(\frac{1}{w_p(t)};r;x\right)\right\}^{\frac{1}{2}} \le \frac{M_6(p,r)}{m},$$

for $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$. Consequently,

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f'_u(u,z)du;r,s;x,y\right)\right| \le \frac{M_7(p,q,r,s)}{m}\|f'_x\|_{p,q}, \quad m \in \mathbb{N}.$$

Analogously we obtain

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_y^z f'_v(x,v)dv;r,s;x,y\right)\right| \le \frac{M_8(p,q,r,s)}{n}\|f'_y\|_{p,q}, \quad n \in \mathbb{N}.$$

Combining these, we derive from (18)

$$w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y)-f(x,y)\right| \le M_9\left\{\frac{1}{m}\|f'_x\|_{p,q}+\frac{1}{n}\|f'_y\|_{p,q}\right\},$$

for all $m,n \in \mathbb{N}$, where $M_9 = M_9(p,q,r,s) = const. > 0$. Thus the proof of (17) is completed. □

Theorem 2. *Suppose that $f \in C_{p,q}$, $p,q \in \mathbb{N}_0$. Then there exists a positive constant $M_{10} \equiv M_{10}(p,q,r,s)$ such that*

$$\|A_{m,n}(f;r,s;\cdot,\cdot)-f(\cdot,\cdot)\|_{p,q} \le M_{10}\,\omega\left(f,C_{p,q};\frac{1}{m},\frac{1}{n}\right) \tag{19}$$

for all $m,n \in \mathbb{N}$, $r,s \in \mathbb{N}$.

Proof. We apply the Steklov function $f_{h,\delta}$ for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)dv, \quad (x,y) \in \mathbb{R}_0^2,\ h,\delta > 0. \tag{20}$$

From (20) it follows that

$$f_{h,\delta}(x,y)-f(x,y) = \frac{1}{h\delta}\int_0^h du\int_0^\delta \Delta_{u,v}f(x,y)dv,$$

$$(f_{h,\delta})'_x(x,y) = \frac{1}{h\delta}\int_0^\delta \left(\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y)\right)dv,$$

$$(f_{h,\delta})'_y(x,y) = \frac{1}{h\delta}\int_0^h \left(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y)\right)du.$$

Thus

$$\|f_{h,\delta} - f\|_{p,q} \le \omega\,(f, C_{p,q}; h, \delta)\,, \tag{21}$$

$$\left\|(f_{h,\delta})'_x\right\|_{p,q} \le 2h^{-1}\omega\,(f, C_{p,q}; h, \delta)\,, \tag{22}$$

$$\left\|(f_{h,\delta})'_y\right\|_{p,q} \le 2\delta^{-1}\omega\,(f, C_{p,q}; h, \delta) \tag{23}$$

for all $h, \delta > 0$, which show that $f_{h,\delta} \in C^1_{p,q}$ if $f \in C_{p,q}$ and $h, \delta > 0$.

Now, for $A_{m,n}$ defined by (6), we can write

$$\begin{aligned} &w_{p,q}(x,y)\,|A_{m,n}(f; r, s; x, y) - f(x,y)| \\ &\quad\le w_{p,q}(x,y)\,\{|A_{m,n}\,(f(t,z) - f_{h,\delta}(t,z); r, s; x, y)| \\ &\qquad + |A_{m,n}\,(f_{h,\delta}(t,z); r, s; x, y) - f_{h,\delta}(x,y)| + |f_{h,\delta}(x,y) - f(x,y)|\} \\ &\quad =: T_1 + T_2 + T_3. \end{aligned}$$

By (3), (16) and (21),

$$T_1 \le \|A_{m,n}\,(f - f_{h,\delta}; r, s; \cdot,\cdot)\|_{p,q} \le M_3\,\|f - f_{h,\delta}\|_{p,q} \le M_3\,\omega\,(f, C_{p,q}; h, \delta)\,,$$

$$T_3 \le \omega\,(f, C_{p,q}; h, \delta)\,.$$

Applying Theorem 1 and (22) and (23), we get

$$\begin{aligned} T_2 &\le M_4 \left\{\frac{1}{m}\left\|(f_{h,\delta})'_x\right\|_{p,q} + \frac{1}{n}\left\|(f_{h,\delta})'_y\right\|_{p,q}\right\} \\ &\le 2M_4\omega\,(f, C_{p,q}; h, \delta)\left\{h^{-1}\frac{1}{m} + \delta^{-1}\frac{1}{n}\right\}. \end{aligned}$$

Consequently, there exists $M_{11} \equiv M_{11}(p,q,r,s)$ such that

$$\begin{aligned} &\|A_{m,n}(f; r, s; \cdot,\cdot) - f(\cdot,\cdot)\|_{p,q} \\ &\quad\le M_{11}\omega\,(f, C_{p,q}; h, \delta)\left\{1 + h^{-1}\frac{1}{m} + \delta^{-1}\frac{1}{n}\right\}, \end{aligned} \tag{24}$$

for $m, n \in \mathbb{N}$ and $h, \delta > 0$. Now, for $m, n \in \mathbb{N}$ setting $h = \frac{1}{m}$ and $\delta = \frac{1}{n}$ to (24), we obtain (19). □

From Theorem 2 and the property (5) follows

Corollary 1. *Let $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$. Then for $r, s \in \mathbb{N}$*

$$\lim_{m,n\to\infty} \|A_{m,n}(f; r, s; \cdot,\cdot) - f(\cdot,\cdot)\|_{p,q} = 0. \tag{25}$$

Theorem 2 and Corollary 1 in our paper show that operators $A_{m,n}$, $m, n \in \mathbb{N}$, give better degree of approximation of functions $f \in C_{p,q}$ than the classical Szasz–Mirakyan operator $S_{m,n}$, considered in [3] for continuous and bounded functions.

II. Approximation in exponential weighted spaces

3. Preliminaries

3.1. Let as in [5], for a fixed $p > 0$ and $r \in \mathbb{N}$,

$$v_{2p}(x) := \exp(-2px), \quad x \in \mathbb{R}_0, \tag{26}$$

and let for fixed $p, q > 0$ and $r, s \in \mathbb{N}$

$$v_{2p,2q}(x, y) := v_{2p}(x)v_{2q}(y), \quad (x, y) \in \mathbb{R}_0^2. \tag{27}$$

Denote by $C_{2p,2q}$ the set of all real-valued functions f continuous on $\mathbb{R}_0^2$ for which $v_{2p,2q}f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$ and the norm is defined by

$$\|f\|_{2p,2q} \equiv \|f(\cdot,\cdot)\|_{2p,2q} := \sup_{(x,y)\in\mathbb{R}_0^2} v_{2p,2q}(x, y)\, |f(x, y)|. \tag{28}$$

The modulus of continuity of function $f \in C_{2p,2q}$ is defined as in § 1.1 by formula

$$\omega(f, C_{2p,2q}; t, z) := \sup_{0\le h\le t,\, 0\le\delta\le z} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{2p,2q}, \quad t, z \ge 0,$$

and we have

$$\lim_{t,z\to 0+} \omega(f, C_{2p,2q}; t, z) = 0 \quad \text{for} \quad f \in C_{2p,2q}. \tag{29}$$

Analogously as in § 1.1, for fixed $p, q > 0$, we denote by $C^1_{2p,2q}$ the set of all functions $f \in C_{2p,2q}$ which first partial derivatives belong also to $C_{2p,2q}$.

3.2. Similary as in Section I we introduce

Definition 2. Let $r, s \in \mathbb{N}$ be fixed numbers. For functions $f \in C_{2p,2q}$, $p, q > 0$, we define the operators

$$B_{m,n}(f; p, q, r, s; x, y) \equiv B_{m,n}(f; x, y)$$

$$:= \frac{1}{g\left((mx+1)^2; r\right) g\left((ny+1)^2; s\right)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(mx+1)^{2j}}{(j+r)!}\,\frac{(ny+1)^{2k}}{(k+s)!} \cdot f\left(\frac{j+r}{m(mx+1)+2p}, \frac{k+s}{n(ny+1)+2q}\right), \tag{30}$$

for $(x, y) \in \mathbb{R}_0^2$, $m, n \in \mathbb{N}$.

In [5] were examined the operators

$$B_n(f; x) \equiv B_n(f; q, r; x)$$

$$:= \frac{1}{g\left((nx+1)^2; r\right)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n(nx+1)+2q}\right) \tag{31}$$

for functions f of one variable, belonging to exponential weighted spaces.

In this paper we shall give similar results for operators $B_{m,n}(f)$.

4. Lemmas and theorems

4.1. From (30) and (7) we deduce that $B_{m,n}(f)$ is well defined in every space $C_{2p,2q}$, $p, q > 0$, $r, s \in \mathbb{N}$. In particular

$$B_{m,n}(1; x, y) = 1, \quad (x, y) \in \mathbb{R}_0^2, \quad m, n \in \mathbb{N}, \tag{32}$$

and if $f \in C_{2p,2q}$ and $f(x, y) = f_1(x)f_2(y)$ for all $(x, y) \in \mathbb{R}_0^2$, then

$$B_{m,n}(f; p, q, r, s; x, y) = B_m(f_1; p, r; x)B_n(f_2; q, s; y) \tag{33}$$

for all $(x, y) \in \mathbb{R}_0^2$ and $m, n \in \mathbb{N}$. Moreover, from (31) and (7) we get

$$B_n(1; q, r; x) = 1 \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N}. \tag{34}$$

In the paper [5] were proved the following two lemmas for $B_n(f; q, r; \cdot)$ defined by (31).

Lemma 3. *Let $q > 0$ be a given number and let $r \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have*

$$\begin{aligned} B_n\left(t - x; q, r; x\right) &= \frac{(nx + 1)^2}{n(nx + 1) + 2q} - x \\ &\quad + \frac{1}{(n(nx + 1) + 2q)(r - 1)!g((nx + 1)^2; r)}, \end{aligned}$$

$$\begin{aligned} B_n\left((t - x)^2; q, r; x\right) &= \left(\frac{(nx + 1)^2}{n(nx + 1) + 2q} - x\right)^2 + \left(\frac{nx + 1}{n(nx + 1) + 2q}\right)^2 \\ &\quad + \frac{r + (nx + 1)^2 - 2x(n(nx + 1) + 2q)}{(n(nx + 1) + 2q)^2(r - 1)!g((nx + 1)^2; r)}, \end{aligned}$$

$$\begin{aligned} B_n\left(e^{2qt}; q, r; x\right) &= \frac{g\left((nx + 1)^2 e^{2q/(n(nx+1)+2q)}; r\right)}{g((nx + 1)^2; r)} \\ &\quad \cdot e^{2qr/(n(nx+1)+2q)}, \end{aligned}$$

$$\begin{aligned} B_n\left((t - x)^2 e^{2qt}; q, r; x\right) &= \left[\left(\frac{(nx + 1)^2}{n(nx + 1) + 2q}\, e^{2q/(n(nx+1)+2q)} - x\right)^2\right. \\ &\quad \left. + \left(\frac{nx + 1}{n(nx + 1) + 2q}\right)^2 e^{2q/(n(nx+1)+2q)}\right] B_n\left(e^{2qt}; q, r; x\right) \\ &\quad + \frac{r + (nx + 1)^2 e^{2q/(n(nx+1)+2q)} - 2x(n(nx + 1) + 2q)}{(n(nx + 1) + 2q)^2(r - 1)!g((nx + 1)^2; r)} \\ &\quad \cdot e^{2qr/(n(nx+1)+2q)}. \end{aligned}$$

Lemma 4. *For every fixed $q > 0$ and $r \in \mathbb{N}$ there exist positive constants $M_i \equiv M_i(p,r)$, $i = 12, 13$, such that for all $x \in \mathbb{R}_0$, $n \in \mathbb{N}$,*

$$v_{2q}(x)\, B_n\left(1/v_{2q}(t); q, r; x\right) \le M_{12},$$
$$v_{2q}(x)\, B_n\left(\frac{(t-x)^2}{v_{2q}(t)}; q, r; x\right) \le \frac{M_{13}}{n^2}.$$

Applying (26)–(28) and (32)–(34) and Lemma 4 and arguing as in the proof of Lemma 2, we can prove the basic property of $B_{m,n}(f)$.

Lemma 5. *For fixed $p, q > 0$ and $r, s \in \mathbb{N}$ there exists a positive constant $M_{14} \equiv M_{14}(p,q,r,s)$ such that*

$$\|B_{m,n}\left(1/v_{2p,2q}(t,z); p, q, r, s; \cdot,\cdot\right)\|_{2p,2q} \le M_{14} \quad \text{for} \quad m, n \in \mathbb{N}. \tag{35}$$

Moreover for every $f \in C_{2p,2q}$ we have

$$\|B_{m,n}\left(f; p, q, r, s; \cdot,\cdot\right)\|_{2p,2q} \le M_{14}\, \|f\|_{2p,2q} \quad \text{for} \quad m,n \in \mathbb{N},\ r,s \in \mathbb{N}. \tag{36}$$

The formulas (30) *and* (7) *and the inequality* (36) *show that $B_{m,n}$, $m, n \in \mathbb{N}$, defined by* (30) *are linear positive operators from the space $C_{2p,2q}$ into $C_{2p,2q}$.*

Applying Lemmas 3–5 and (26)–(28) and (32)–(34) and reasoning as in the proof of Theorem 1, we can prove the following

Theorem 3. *Suppose that $f \in C^1_{2p,2q}$ with given $p, q > 0$ and $r, s \in \mathbb{N}$. Then there exists a positive constant $M_{15} = M_{15}(p,q,r,s)$ such that for all $m, n \in \mathbb{N}$,*

$$\|B_{m,n}(f; p, q, r, s; \cdot,\cdot) - f(\cdot,\cdot)\|_{2p,2q} \le M_{15}\left\{\frac{1}{m}\|f'_x\|_{2p,2q} + \frac{1}{n}\|f'_y\|_{2p,2q}\right\}.$$

Theorem 4. *Suppose that $f \in C_{2p,2q}$, $p, q > 0$, $r, s \in \mathbb{N}$. Then there exists a positive constant $M_{16} \equiv M_{16}(p,q,r,s)$ such that*

$$\|B_{m,n}(f; p, q, r, s; \cdot,\cdot) - f(\cdot,\cdot)\|_{2p,2q} \le M_{16}\, \omega\left(f, C_{2p,2q}; \frac{1}{m}, \frac{1}{n}\right), \tag{37}$$

for all $m, n \in \mathbb{N}$.

Proof. Similary as in the proof of Theorem 2 we shall apply the Steklov function $f_{h,\delta}$ for $f \in C_{2p,2q}$, defined by (20). Analogously as in (21)–(23) we get

$$\|f_{h,\delta} - f\|_{2p,2q} \le \omega\left(f, C_{2p,2q}; h, \delta\right), \tag{38}$$
$$\left\|(f_{h,\delta})'_x\right\|_{2p,2q} \le 2h^{-1}\omega\left(f, C_{2p,2q}; h, \delta\right), \tag{39}$$
$$\left\|(f_{h,\delta})'_y\right\|_{2p,2q} \le 2\delta^{-1}\omega\left(f, C_{2p,2q}; h, \delta\right) \tag{40}$$

for all $h, \delta > 0$, which show that $f_{h,\delta} \in C^1_{2p,2q}$ if $f \in C_{2p,2q}$ and $h, \delta > 0$.

Now, for $B_{m,n}$ defined by (30), we can write

$$\begin{aligned} v_{2p,2q}(x,y)\,&|B_{m,n}(f;p,q,r,s;x,y) - f(x,y)| \\ &\le v_{2p,2q}(x,y)\,\{|B_{m,n}\left(f(t,z) - f_{h,\delta}(t,z);p,q,r,s;x,y\right)| \\ &\quad + |B_{m,n}\left(f_{h,\delta}(t,z);p,q,r,s;x,y\right) - f_{h,\delta}(x,y)| \\ &\quad + |f_{h,\delta}(x,y) - f(x,y)|\} =: T_1 + T_2 + T_3. \end{aligned}$$

By (28), (36) and (38),

$$\begin{aligned} T_1 &\le \|B_{m,n}\left(f - f_{h,\delta};p,q,r,s;\cdot,\cdot\right)\|_{2p,2q} \le M_{14}\,\|f - f_{h,\delta}\|_{2p,2q} \\ &\le M_{14}\,\omega\left(f, C_{2p,2q};h,\delta\right), \\ T_3 &\le \omega\left(f, C_{2p,2q};h,\delta\right). \end{aligned}$$

Applying Theorem 3 and (39) and (40), we get

$$\begin{aligned} T_2 &\le M_{15}\left\{\frac{1}{m}\left\|(f_{h,\delta})'_x\right\|_{2p,2q} + \frac{1}{n}\left\|(f_{h,\delta})'_y\right\|_{2p,2q}\right\} \\ &\le 2M_{15}\omega\left(f, C_{2p,2q};h,\delta\right)\left\{h^{-1}\frac{1}{m} + \delta^{-1}\frac{1}{n}\right\}. \end{aligned}$$

From the above we deduce that there exists a positive constant $M_{17} \equiv M_{17}(p,q,r,s)$ such that

$$\begin{aligned} &\|B_{m,n}(f;p,q,r,s;\cdot,\cdot) - f(\cdot,\cdot)\|_{2p,2q} \\ &\qquad \le M_{17}\omega\left(f, C_{2p,2q};h,\delta\right)\left\{1 + h^{-1}\frac{1}{m} + \delta^{-1}\frac{1}{n}\right\}, \end{aligned} \tag{41}$$

for $m, n \in \mathbb{N}$ and $h, \delta > 0$. Now, for $m, n \in \mathbb{N}$ setting $h = \frac{1}{m}$ and $\delta = \frac{1}{n}$ to (41), we obtain (37). □

Theorem 4 and (29) imply

Corollary 2. *Let* $f \in C_{2p,2q}$, $p, q > 0$, $r, s \in \mathbb{N}$. *Then*

$$\lim_{m,n\to\infty} \|B_{m,n}(f;p,q,r,s;\cdot,\cdot) - f(\cdot,\cdot)\|_{p,q} = 0.$$

Theorem 4 and Corollary 2 in our paper show that operators $B_{m,n}$, $m, n \in \mathbb{N}$, give better degree of approximation of functions belonging to exponential weighted spaces than the classical Szasz–Mirakyan operators $S_{m,n}$, examined for continuous and bounded functions in [3].

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