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Metric locally constant functions

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Abstract. Given an ultrametric space E, a function $f : E \to [0, \infty]$ is said to be metric locally constant (m.l.c.) provided that for any $x \in E$ and any y in the open ball $B(x, f(x))$ one has $f(x) = f(y)$. Given two ultrametric spaces E and F, we investigate the maps $\varphi : E \to F$, for which $f \circ \varphi$ is m.l.c. for any m.l.c. function $f : F \to [0, \infty]$.

1. Introduction

Let E be an ultrametric space, that is a metric space on which the distance d_E satisfies the triangle inequality in the strong form

$$d_E(x, z) \leq \max\{d_E(x, y),\, d_E(y, z)\} \tag{1}$$

for any x, y, $z \in E$. We denote by $\mathcal{F}_E$ the set of maps $f : E \to [0, \infty]$. A function $f \in \mathcal{F}_E$ is said to be metric locally constant (m.l.c. for short) provided that for any $x \in E$ and for any y in the open ball $B(x, f(x))$ one has $f(x) = f(y)$. We denote by $\widetilde{\mathcal{F}}_E$ the set of m.l.c. functions. Metric locally constant functions often appear in practice. For instance, it is easy to see that for any ultrametric space E and any isometry $\psi : E \to E$, the distance function between an element and its image, that is the function $f : E \to [0, \infty)$ given by $f(x) = d_E(x, \psi(x))$, is an m.l.c. function. The notion of metric locally constant function has been introduced in [14] in order to study certain groups of isometries on a given ultrametric space. In particular, various Galois groups over local fields can be described in this way. Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value $|\cdot|$. If $f \in \mathcal{F}_{\mathbb{C}_p}$, then

$$\mathrm{Gal}(f) := \{\sigma \in \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p) : |\sigma(x) - x| \leq f(x),\ x \in \mathbb{C}_p\}$$

is a subgroup of the Galois group $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p) \simeq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Some important groups are given by such metric constraints. For example, if L is

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a finite Galois extension of $\mathbb{Q}_p$ and π_L is a uniformising element of L, then the ramification groups (see [12])

$$G_n := \{\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p) : \sigma(\pi_L) \equiv \pi_L (\mathrm{mod}\ \pi_L^n)\}$$

are of this type. To be precise, if $f_n \in \mathcal{F}_{\mathbb{C}_p}$ is given by

$$f_n(x) = \begin{cases} |\pi_L|^n, & \text{if } x = \pi_L, \\ \infty, & \text{if } x \neq \pi_L, \end{cases}$$

then the restrictions to L of the automorphisms $\sigma \in \mathrm{Gal}(f_n)$ produce the above group G_n. For another example, recall that by Galois theory in $\mathbb{C}_p$ (see [3], [11], [13]), each closed subgroup $\mathcal{H}$ of $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$ corresponds to a closed subfield F of $\mathbb{C}_p$. We may write $\mathcal{H}$ in the form $\mathcal{H} = \mathrm{Gal}(f)$, with

$$f(x) = \begin{cases} 0, & \text{if } x \in F, \\ \infty, & \text{if } x \in \mathbb{C}_p \setminus F. \end{cases}$$

If we now choose a generating element T of F (see [1], [2], [7]), that is, an element $T \in F$ for which $\mathbb{Q}_p[T]$ is dense in F, then one also has $\mathcal{H} = \mathrm{Gal}(g)$, where

$$g(x) = \begin{cases} 0, & \text{if } x = T, \\ \infty, & \text{if } x \neq T. \end{cases}$$

The nature of the above equality $\mathrm{Gal}(f) = \mathrm{Gal}(g)$ has both an algebraic aspect and a metric one. The m.l.c. functions on $\mathbb{C}_p$ play an important role in the metric aspect of the theory. In particular, as shown in [14] and [15], there is a canonical way to associate to any $f \in \mathcal{F}_{\mathbb{C}_p}$ an m.l.c. function $\tilde{f} \in \widetilde{\mathcal{F}}_{\mathbb{C}_p}$, and this process leaves the Galois group unchanged, i.e., one has $\mathrm{Gal}(f) = \mathrm{Gal}(\tilde{f})$.

There are various natural maps between local fields, such as the trace or the norm, and it would be interesting to study the effect of these maps on m.l.c. functions.

In the present paper we take a general point of view. Given an ultrametric space E, together with the associated set of functions $\mathcal{F}_E$ and the canonical map $f \mapsto \tilde{f}$ from $\mathcal{F}_E$ to $\widetilde{\mathcal{F}}_E$, we look for natural ways to transport this metric structure from E to another ultrametric space F. The main questions and results are presented in Section 2 below.

2. Statement of results

Let E and F be two ultrametric spaces. Any map $\varphi : E \to F$ gives rise to a map $\varphi^* : \mathcal{F}_F \to \mathcal{F}_E$ given by $\varphi^*(f) = f \circ \varphi$. We are interested to find circumstances under which φ^* sends m.l.c. functions to m.l.c. functions, and moreover to describe the image of $\widetilde{\mathcal{F}}_F$ through φ^*. A partial answer to the problem is provided by the following result. First recall that given a positive real number λ and two metric spaces E and F, a map $\varphi : E \to F$ is said to be

λ-Lipschitzian provided that $d_F(\varphi(x), \varphi(y)) \leq \lambda d_E(x, y)$ for any $x, y \in E$. A map $\varphi : E \to E$ is called a contraction if $d_E(\varphi(x), \varphi(y)) \leq d_E(x, y)$.

Proposition 1. *Let E and F be two ultrametric spaces, and let $\varphi : E \to F$. If φ is 1-Lipschitzian then $\varphi^*(\widetilde{\mathcal{F}}_F) \subseteq \widetilde{\mathcal{F}}_E$.*

Corollary 1. *Let E be an ultrametric space, and let $\varphi : E \to E$. If φ is a contraction then $\varphi^*(\widetilde{\mathcal{F}}_E) \subseteq \widetilde{\mathcal{F}}_E$.*

We say that a map $\varphi : E \to F$ is a quasi-isometry provided that for any $x, y \in E$ one has

$$d_F(\varphi(x), \varphi(y)) = d_E(x, \varphi^{-1}(\varphi(y))). \tag{2}$$

Here the right hand side of (2) is defined as usual as a distance between a point and a set:

$$d_E(x, \varphi^{-1}(\varphi(y))) = \inf_{z \in \varphi^{-1}(\varphi(y))} d_E(x, z) = \inf_{\varphi(z) = \varphi(y)} d_E(x, z).$$

Note that if φ is injective then $\varphi^{-1}(\varphi(y))$ consists of y alone, thus an injective quasi-isometry is an isometry. Note also that any quasi-isometry is 1-Lipschitzian. A suggestive example is the following. Let $\varphi : \mathbb{Q}_p \to \mathbb{Q}_p$ be given by $\varphi(x) = x^{p-1}$, where $\mathbb{Q}_p$ is the field of p-adic numbers. Observe that $\varphi(\mathbb{Z}_p) \subseteq \mathbb{Z}_p$, $\varphi(U) \subseteq U$, $\varphi(U_1) \subseteq U_1$ where $\mathbb{Z}_p$, U and U_1 denote respectively the ring of p-adic integers, the group of units in $\mathbb{Z}_p$ and the group of principal units (i.e. those which are $\equiv 1 \pmod p$). Now, on $\mathbb{Q}_p$ the map φ is continuous but it is not 1-Lipschitzian. On $\mathbb{Z}_p$, φ is 1-Lipschitzian but it is not a quasi-isometry. On U, φ is a quasi-isometry but it is not an isometry. On U_1, φ is an isometry. A less trivial example of a quasi-isometry is the following. Let $\mathbb{C}_p$ denote the topological closure of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let B_{dR}^+ be Fontaine's ring (see [4], [5]) and let $B_n = B_{dR}^+/I^n$, where I denotes the maximal ideal of B_{dR}^+. As was shown in [6], the canonical projections $B_n \to \mathbb{C}_p$ are quasi-isometries.

The notion of quasi-isometry is useful in our present investigation. It is proved in [14] that for any ultrametric space E and any function $f \in \mathcal{F}_E$ there is a largest m.l.c. function g satisfying $g \leq f$. Moreover, this function g is denoted by $\tilde{f}$ and it is given by

$$\tilde{f}(x) = \inf_{y \in E} \max\{d_E(x, y), f(y)\}$$

for any $x \in E$. In the terminology from [15], the map $(\sim) : \mathcal{F}_E \to \mathcal{F}_E$, $f \mapsto \tilde{f}$ is an inferior regularization on the set $\mathcal{F}_E$. The set of regular elements, i.e., elements f for which $\tilde{f} = f$, coincides with the set $\widetilde{\mathcal{F}}_E$ of m.l.c. functions. Now let E, F be two ultrametric spaces and $\varphi : E \to F$. If we take a function $f \in \mathcal{F}_F$, send it via φ^* to $\varphi^*(f)$ and take the regularization $\widetilde{\varphi^*(f)}$, this element of $\mathcal{F}_E$ is an m.l.c. function. If we first take the regularization of f in $\mathcal{F}_F$ and then send it via φ^* we obtain the element $\varphi^*(\tilde{f})$ of $\mathcal{F}_E$. Under the assumption that φ is 1-Lipschitzian we know from Proposition

1 that $\varphi^*(\tilde{f})$ is m.l.c., but it might not coincide with $\widetilde{\varphi^*(f)}$. We do have $\widetilde{\varphi^*(f)} = \varphi^*(\tilde{f})$ under the stronger assumption that φ is a surjective quasi-isometry.

Proposition 2. *Let E, F be ultrametric spaces and let $\varphi : E \to F$ be a surjective quasi-isometry. Then the diagram*

$$\begin{array}{ccc} \mathcal{F}_F & \xrightarrow{\varphi^*} & \mathcal{F}_E \\ {\scriptstyle (\sim)}\downarrow & & \downarrow{\scriptstyle (\sim)} \\ \mathcal{F}_F & \xrightarrow[\varphi^*]{} & \mathcal{F}_E \end{array}$$

is commutative.

We want to describe the image of $\widetilde{\mathcal{F}}_F$ through φ^*. In order to do this, we introduce the following subset $\check{\mathcal{F}}_E$ of $\mathcal{F}_E$,

$$\check{\mathcal{F}}_E = \{f \in \mathcal{F}_E \colon f(x) = f(y) \text{ for any } x,\, y \in E \text{ with } \varphi(x) = \varphi(y)\}.$$

Proposition 3. *Let E, F be ultrametric spaces and let $\varphi : E \to F$ be a surjective quasi-isometry. Then $\varphi^*(\widetilde{\mathcal{F}}_F) = \check{\mathcal{F}}_E \cap \widetilde{\mathcal{F}}_E$.*

3. Proof of the results

We start with the proof of Proposition 1. Let E, F be ultrametric spaces and let $\varphi : E \to F$ be 1-Lipschitzian. Choose an m.l.c. function $f \in \mathcal{F}_F$. We need to show that $\varphi^*(f)$ is m.l.c.. For, let $x, y \in E$ such that $d_E(x,y) \leq \varphi^*(f)(x)$. Note that $\varphi^*(f)(x) = f(\varphi(x))$ and $d_F(\varphi(x), \varphi(y)) \leq d_E(x,y) \leq \varphi^*(f)(x) = f(\varphi(x))$. Since f is m.l.c. it follows that $f(\varphi(x)) = f(\varphi(y))$, i.e., $\varphi^*(f)(x) = \varphi^*(f)(y)$. This shows that $\varphi^*(f)$ is m.l.c. and Proposition 1 is proved.

Next we turn to the proof of Proposition 2. Because φ is a surjective quasi-isometry we have for any $x \in E$ that

$$\begin{aligned} \widetilde{\varphi^*(f)}(x) &= \inf_{y\in E} \max\{d_E(x,y), f(\varphi(y))\} = \inf_{\substack{z\in F \\ y\in\varphi^{-1}(z)}} \max\{d_E(x,y), f(z)\} \\ &= \inf_{z\in F} \max\{\inf_{y\in\varphi^{-1}(z)} d_E(x,y), f(z)\} = \inf_{z\in F} \max\{d_E(x,\varphi^{-1}(z)), f(z)\} \\ &= \inf_{z\in F} \max\{d_F(\varphi(x),z), f(z)\} = \tilde{f}(\varphi(x)) = \varphi^*(\tilde{f})(x). \end{aligned}$$

It follows that the diagram from the statement of Proposition 2 is commutative, and this completes the proof of Proposition 2.

We now turn to the proof of Proposition 3. Before we start the proof, let us recall the notion of regularization from [15].

Let $(\mathcal{M}, \leq)$ be a partially ordered set. A map $\alpha : \mathcal{M} \to \mathcal{M}$ is called an *inferior regularization* on $\mathcal{M}$ if

$$\alpha(x) \leq \alpha(y) \qquad \text{for any } x, y \in \mathcal{M} \text{ with } x \leq y, \tag{3}$$

$$\alpha(\alpha(x)) = \alpha(x) \qquad \text{for any } x \in \mathcal{M}, \tag{4}$$

$$\alpha(x) \leq x \qquad \text{for any } x \in \mathcal{M}. \tag{5}$$

If instead of (5) above the map α satisfies

$$\alpha(x) \geq x \qquad \text{for any } x \in \mathcal{M} \tag{6}$$

then we call α a *superior regularization* on $\mathcal{M}$.

Theorem 2.4 of [14] shows that for any ultrametric space E, the map $f \mapsto \tilde{f}$ is an inferior regularization on $\mathcal{F}_E$. We need the following result.

Lemma 1. *Let A, B be two partially ordered sets, let $f : A \to B$ and $g : B \to A$ be increasing (i.e., $f(x) \leq f(y)$ for any $x, y \in A$ with $x \leq y$, and similarly for g), and let α_A and α_B be inferior regularizations on A, respectively B, such that:* (i) $f \circ g = Id_B$, (ii) $g \circ \alpha_B = \alpha_A \circ g$ *and* (iii) $h(x) \geq x$ *for any $x \in A$, where $h = g \circ f$. Then:*

(a) *h is a superior regularization on A.*

(b) Im g = *Im* h.

(c) *If $x \in A$ is h-regular (i.e. $h(x) = x$) then $\alpha_A(x)$ is also h-regular.*

(d) Im $(g \circ \alpha_B) = \text{Im } \alpha_A \cap \text{Im } h$.

Proof. (a) From (iii) and the fact that f and g are increasing it follows that h satisfies properties (3) and (6) of a superior regularization on A. The equality (i) gives $(g \circ f \circ g \circ f)(x) = (g \circ f)(x)$, i.e., $h(h(x)) = h(x)$ for any $x \in A$ and this implies that h is a superior regularization on A.

(b) Let b be an arbitrary element of B. Since $f \circ g = Id_B$ it follows that f is surjective and hence there exists $a \in A$ such that $f(a) = b$. We have $h(a) = g(f(a)) = g(b)$ and this implies Im $g \subseteq$ Im h. The reverse inclusion is an immediate consequence of the definition of h.

(c) Using (ii) in the equality $f \circ g \circ \alpha_B = \alpha_B$ it follows that $f \circ \alpha_A \circ g = \alpha_B$. Since x is h-regular we have $(f \circ \alpha_A)(x) = (f \circ \alpha_A)(h(x)) = (f \circ \alpha_A \circ g \circ f)(x) = (\alpha_B \circ f)(x)$. Now it is easy to see that $h(\alpha_A(x)) = (g \circ f \circ \alpha_A)(x) = (g \circ \alpha_B \circ f)(x) = (\alpha_A \circ g \circ f)(x) = (\alpha_A \circ h)(x) = \alpha_A(x)$ so $\alpha_A(x)$ is h-regular.

(d) Using again (ii) and (b), it is enough to prove that Im $(\alpha_A \circ h) =$ Im $\alpha_A \cap$ Im h. Clearly Im $(\alpha_A \circ h) \subseteq$ Im α_A. Also, we observe that Im $(\alpha_A \circ h) =$ Im $(\alpha_A \circ g \circ f) =$ Im $(g \circ \alpha_B \circ f) \subseteq$ Im $g =$ Im h. Therefore Im $(\alpha_A \circ h) \subseteq$ Im $\alpha_A \cap$ Im h.

For the reverse inclusion let $\alpha_A(u) = h(v) \in$ Im $\alpha_A \cap$Im h, where $u, v \in A$. Because α_A is an inferior regularization we have $\alpha_A(u) = \alpha_A(\alpha_A(u)) = \alpha_A(h(v)) \in$ Im $(\alpha_A \circ h)$ and the proof is complete. □

We now prove Proposition 3 in the following more precise form. Let E, F be two ultrametric spaces and $\varphi : E \to F$ be a surjective quasi-isometry. Consider the map $\varphi_* : \mathcal{F}_E \to \mathcal{F}_F$ defined by

$$\varphi_*(f)(x) = \sup_{y \in \varphi^{-1}(x)} f(y).$$

For any $f \in \mathcal{F}_E$, denote $\check{f} = \varphi^*(\varphi_*(f))$ and note that $\check{f} \geq f$. Note also that the set $\check{\mathcal{F}}_E$ defined in the statement of Proposition 3 can be described as

$$\check{\mathcal{F}}_E = \{f \in \mathcal{F}_E \colon f = \check{f}\}.$$

Recall that the maps $(\sim) : \mathcal{F}_E \to \mathcal{F}_E$ and $(\sim) : \mathcal{F}_F \to \mathcal{F}_F$ are inferior regularizations, and moreover, $(\sim)\circ\varphi^* = \varphi^*\circ(\sim)$ by Proposition 2. Observe also that φ^* and φ_* are increasing and moreover $\varphi_*\circ\varphi^* = Id_{\mathcal{F}_F}$. Then by Lemma 1 we obtain the following result, which proves in particular Proposition 3.

Theorem 1. *Let E, F be ultrametric spaces and let $\varphi : E \to F$ be a surjective quasi-isometry. Then*

1) *The map $(\vee)\colon \mathcal{F}_E \to \mathcal{F}_E$, $f \mapsto \check{f}$ is a superior regularization on $\mathcal{F}_E$.*
2) *$Im\ \varphi^* = \check{\mathcal{F}}_E$.*
3) *If $f \in \check{\mathcal{F}}_E$ then $\tilde{f} \in \check{\mathcal{F}}_E$.*
4) *$\varphi^*(\tilde{\mathcal{F}}_F) = \tilde{\mathcal{F}}_E \cap \check{\mathcal{F}}_E$.*

4. An example

We end the paper with an example.

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, K a finite extension of $\mathbb{Q}_p$, $\bar{K}$ a fixed algebraic closure of K and $\mathbb{C}_p$ the completion of $\bar{K}$ with respect to the p-adic absolute value. For any $a \in \bar{K}$ we denote by $\deg_K a$ the degree of a over K. A pair (a, b) of elements from $\bar{K}$ is called a *distinguished pair* if the following conditions hold:

$$\deg_K a > \deg_K b, \tag{7}$$

$$\text{if } c \in \bar{K} \text{ and } \deg_K c < \deg_K a \text{ then } |a - c| \geq |a - b|, \tag{8}$$

and

$$\text{if } c \in \bar{K} \text{ and } \deg_K c < \deg_K b \text{ then } |a - c| > |a - b|. \tag{9}$$

Let now (a, b) be a distinguished pair and consider the K-linear map $\varphi : K(a) \to K(b)$ which sends a^n to b^n for any $n \in \{0, 1, \ldots, (\deg_K a) - 1\}$. Thus $\varphi(P(a)) = P(b)$ for any polynomial $P(X) \in K[X]$ of degree $\deg P(X) < \deg_K a$. Note that the map φ is surjective, since $\deg_K a > \deg_K b$. Moreover, for any element $z \in K(a)$ we have

$$|\varphi(z)| \leq |z|. \tag{10}$$

Indeed, choose $P(X) \in K[X]$ with $\deg P(X) < \deg_K a$ such that $P(a) = z$. Then $\varphi(z) = P(b)$. Next, decompose the polynomial $P(X)$ over $\bar{K}$, say

$$P(X) = c(X - \theta_1) \cdots (X - \theta_r), \tag{11}$$

with $c \in K$ and $\theta_1, \ldots, \theta_r \in \bar{K}$. Here $\deg_K \theta_j < \deg_K a$ for any $j \in \{1, \ldots, r\}$, hence $|a - \theta_j| \geq |a - b|$ by (8), and since we are in an ultrametric space it follows that $|a - \theta_j| \geq |b - \theta_j|$. Therefore

$$|z| = |c| \prod_{j=1}^{r} |a - \theta_j| \geq |c| \prod_{j=1}^{r} |b - \theta_j| = |\varphi(z)|, \tag{12}$$

which proves (10). Using (10) and the fact that the map φ is linear, it follows immediately that φ is 1-Lipschitzian. Next, take any two elements $x, y \in K(a)$. There exists a polynomial $G(X) \in K[X]$ with $\deg G(X) < \deg_K b$ such that $G(b) = \varphi(x) - \varphi(y)$. Denote $y_1 = x - G(a)$. Let us note that $\varphi(y_1) = \varphi(x) - \varphi(G(a)) = \varphi(x) - G(b) = \varphi(y)$, thus $y_1 \in \varphi^{-1}(\varphi(y))$. Now decompose $G(X)$ over $\bar{K}$, say

$$G(X) = c'(X - \eta_1) \cdots (X - \eta_l), \tag{13}$$

with $c' \in K$ and $\eta_1, \ldots, \eta_l \in \bar{K}$. For any $j \in \{1, \ldots, l\}$ one has $\deg_K \eta_j < \deg_K b$, and from (9) we find that $|a - \eta_j| > |a - b|$, which further implies $|a - \eta_j| = |b - \eta_j|$. We deduce that

$$\begin{aligned} |\varphi(x) - \varphi(y)| = |G(b)| = |c'| \prod_{j=1}^{l} |b - \eta_j| = |c'| \prod_{j=1}^{l} |a - \eta_j| \\ = |G(a)| = |x - y_1|. \end{aligned} \tag{14}$$

Combining (14) with the fact that $y_1 \in \varphi^{-1}(\varphi(y))$, and recalling that φ is 1-Lipschitzian, we conclude that φ satisfies the equality (2), so φ is a quasi-isometry.

If we start with a given element $a \in \bar{K}$, we may construct chains of elements from $\bar{K}$ such that any two consecutive elements in the chain form a distinguished pair. An $(s+1)$-tuple $(a_0, \ldots, a_s)$ of elements from $\bar{K}$ such that $a_0 = a$, (a_{j-1}, a_j) is a distinguished pair for any $j \in \{1, \ldots, s\}$ and $a_s \in K$, is said to be a *saturated distinguished chain for a over K*. Saturated distinguished chains have been introduced in [10] in order to investigate the structure of irreducible polynomials in one variable over a local field, and were also studied in [1], [8] and [9]. From the above discussion we know that for any $a \in \bar{K}$, any saturated distinguished chain $(a_0, \ldots, a_s)$ for a over K and any $j \in \{1, \ldots, s\}$, the linear map $\varphi_j : K(a_{j-1}) \to K(a_j)$ given by $\varphi_j(P(a_{j-1})) = P(a_j)$ for any polynomial in one variable over K of degree $\deg P < \deg_K a_{j-1}$ is a surjective quasi-isometry. The chain of maps $\varphi_1, \ldots, \varphi_s$ produces a chain $\varphi_1^*, \ldots, \varphi_s^*$, with $\varphi_j^* : \mathcal{F}_{K(a_j)} \to \mathcal{F}_{K(a_{j-1})}$. In particular, at each step $j \in \{1, \ldots, s\}$, the metric locally constant functions defined on the field $K(a_j)$ are sent via φ_j^* to metric locally constant functions defined on $K(a_{j-1})$, and the image of $\widetilde{\mathcal{F}}_{K(a_j)}$ in $\widetilde{\mathcal{F}}_{K(a_{j-1})}$ is described by Theorem 1.

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