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Stability of the spline collocation method for Volterra integro-differential equations

PEETER OJA AND MARE TARANG

ABSTRACT. Numerical stability of the spline collocation method for Volterra integro-differential equation is investigated and the connection between this theory and corresponding theory for Volterra integral equation is explored. A series of numerical tests is presented.

1. Introduction

One of the most natural methods for solving Volterra integral and integro-differential equations is the standard step-by-step collocation method with polynomial splines. An important property of this method is the numerical stability which means the boundedness of approximate solutions in uniform norm when the number of knots increases. From general considerations it turns out that it is relevant to find out stability conditions for certain test equations. While for Volterra integral equations such conditions are established in [6], similar results for Volterra integro-differential equations were missing. We show the connection between stability conditions for integral and integro-differential equations when the splines to be used are at least continuous. In some cases we get explicit formulae showing the dependence of the stability on collocation parameters. A series of numerical tests is given to support the theoretical results.

2. The spline collocation method

Consider the Volterra integro-differential equation (VIDE)

$$y'(t) = f(t, y(t)) + \int_0^t \mathcal{K}(t, s, y(s))ds, \quad t \in [0, T], \tag{2.1}$$

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with the initial condition $y(0) = y_0$. Here functions $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{K}\colon S \times \mathbb{R} \to \mathbb{R}$ (where $S = \{(t,s)\colon 0 \le s \le t \le T\}$) with number y_0 are supposed to be given. In order to describe this method, let $0 = t_0 < t_1 < \ldots < t_N = T$ (with t_n depending on N) be a mesh on the interval $[0,T]$.

Denote $h_n = t_n - t_{n-1}$, $\sigma_n = (t_{n-1}, t_n]$, $n = 1,\ldots,N$, and $\Delta_N = \{t_1,\ldots,t_{N-1}\}$. Let $\mathcal{P}_k$ denote the space of polynomials of degree not exceeding k. Then, for given integers $m \ge 1$ and $d \ge -1$, we define

$$S^d_{m+d}(\Delta_N) = \{u\colon\ u|_{\sigma_n} \in \mathcal{P}_{m+d},\ n = 1,\ldots,N,\ u^{(j)}_{n-1}(t_n) = u^{(j)}_n(t_n),\\ t_n \in \Delta_N,\ j = 0,1,\ldots,d\}$$

to be the space of polynomial splines of degree $m + d$ which are d-times continuously differentiable on $[0,T]$.

An element $u \in S^d_{m+d}(\Delta_N)$ as a polynomial spline of degree not greater than $m + d$ for all $t \in \sigma_n$, $n = 1,\ldots,N$, can be represented in the form

$$u_n(t) = \sum_{k=0}^{m+d} b_{nk}(t - t_{n-1})^k. \tag{2.2}$$

The smoothness conditions impose certain linear restrictions on the coefficients b_{nk} and we will present them in the next section.

Suppose that there is a given fixed selection of collocation parameters $0 < c_1 < \ldots < c_m \le 1$. Then we define collocation points $t_{nj} = t_{n-1} + c_j h_n$, $j = 1,\ldots,m$, $n = 1,\ldots,N$, forming a set $X(N)$. In order to determine the approximate solution $u \in S^d_{m+d}(\Delta_N)$ of the equation (2.1) we impose the following collocation conditions

$$u'(t) = f(t,u(t)) + \int_0^t \mathcal{K}(t,s,u(s))ds, \quad t \in X(N). \tag{2.3}$$

Starting the calculations by this method we assume also that we can use the initial values $u^{(j)}_1(0) = y^{(j)}(0)$, $j = 0,\ldots,d$, which is justified by the requirement $u \in C^d[0,T]$. Another possible approach is to use only $u_1(0) = y(0)$ and more collocation points (if $d \ge 1$) to determine u_1. Thus, on every interval σ_n we have $d+1$ conditions of smoothness and m collocation conditions to determine $m + d + 1$ parameters b_{nk}. This allows us to implement the method step-by-step going from an interval σ_n to the next one.

In this paper we will analyze the stability of the collocation method where the splines are at least continuous. Thus, we suppose in the sequel that $d \ge 0$.

3. The method in the case of a test equation

Consider the well-known test equation

$$y'(t) = \alpha y(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0,T], \tag{3.1}$$

where, in general, λ and α may be any complex numbers. Assume that the mesh sequence $\{\Delta_N\}$ is uniform, i.e., $h_n = h = T/N$ for all n. Representing $t \in \sigma_n$ as $t = t_{n-1} + \tau h$, $\tau \in (0,1]$, we have on σ_n

$$u_n(t_{n-1} + \tau h) = \sum_{k=0}^{m+d} a_{nk}\tau^k, \qquad \tau \in (0,1], \tag{3.2}$$

where we passed to the parameters $a_{nk} = b_{nk}h^k$.

The smoothness conditions (for any $u \in S^d_{m+d}(\Delta_N)$)

$$u_n^{(j)}(t_n - 0) = u_{n+1}^{(j)}(t_n + 0), \quad j = 0, \dots, d, \ n = 1, \dots, N-1,$$

can be expressed in the form

$$a_{n+1,j} = \sum_{k=j}^{m+d} \frac{k!}{(k-j)!j!} a_{nk}, \quad j = 0, \dots, d, \ n = 1, \dots, N-1. \tag{3.3}$$

The collocation conditions (2.3), applied to the test equation (3.1), give

$$\begin{aligned} u'(t_{nj}) &= f(t_{nj}) + \alpha u(t_{nj}) \\ &\quad + \lambda \int_0^{t_{nj}} u(s)ds, \quad j = 1, \dots, m, \ n = 1, \dots, N. \end{aligned} \tag{3.4}$$

From (3.2) we get

$$u_n(t_{nj}) = \sum_{k=0}^{m+d} a_{nk}c_j^k$$

and

$$u_n'(t_{nj}) = \frac{1}{h}\sum_{k=1}^{m+d} a_{nk}kc_j^{k-1}.$$

Now the equation (3.4) becomes

$$\begin{aligned}
&\frac{1}{h}\sum_{k=0}^{m+d} a_{nk}kc_j^{k-1}\\
&=\alpha\sum_{k=0}^{m+d} a_{nk}c_j^k+\sum_{r=1}^{n-1}\lambda\int_{t_{r-1}}^{t_r} u_r(s)ds+\lambda\int_{t_{n-1}}^{t_{nj}} u_n(s)ds+f(t_{nj})\\
&=\alpha\sum_{k=0}^{m+d} a_{nk}c_j^k+\sum_{r=1}^{n-1}\lambda h\int_0^1\Big(\sum_{k=0}^{m+d} a_{rk}\tau^k\Big)d\tau\\
&\quad+\lambda h\int_0^{c_j}\Big(\sum_{k=0}^{m+d} a_{nk}\tau^k\Big)d\tau+f(t_{nj})\\
&=\alpha\sum_{k=0}^{m+d} a_{nk}c_j^k+\sum_{r=1}^{n-1}\lambda h\Big(\sum_{k=0}^{m+d}\frac{1}{k+1}a_{rk}\Big)\\
&\quad+\lambda h\sum_{k=0}^{m+d} a_{nk}\frac{c_j^{k+1}}{k+1}+f(t_{nj}).
\end{aligned}\tag{3.5}$$

Using the notation $\alpha_n=(a_{nk})_{k=0}^{m+d}$, we write (3.5) as follows:

$$\begin{aligned}
&\sum_{k=0}^{m+d} a_{nk}kc_j^{k-1}-\alpha h\sum_{k=0}^{m+d} a_{nk}c_j^k-\lambda h^2\sum_{k=0}^{m+d} a_{nk}\frac{c_j^{k+1}}{k+1}\\
&=\lambda h^2\langle q,\sum_{r=1}^{n-1}\alpha_r\rangle+hf(t_{nj}),
\end{aligned}\tag{3.6}$$

where $q=(1,1/2,\dots,1/(m+d+1))$ and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^{m+d+1}$. The difference of the equations (3.6) with n and $n+1$ yields

$$\begin{aligned}
&\sum_{k=0}^{m+d} a_{n+1,k}kc_j^{k-1}-\alpha h\sum_{k=0}^{m+d} a_{n+1,k}c_j^k-\lambda h^2\sum_{k=0}^{m+d} a_{n+1,k}\frac{c_j^{k+1}}{k+1}\\
&=\sum_{k=0}^{m+d} a_{nk}kc_j^{k-1}-\alpha h\sum_{k=0}^{m+d} a_{nk}c_j^k-\lambda h^2\sum_{k=0}^{m+d} a_{nk}\frac{c_j^{k+1}}{k+1}+\lambda h^2\langle q,\alpha_n\rangle\\
&\quad+hf(t_{n+1,j})-hf(t_{nj}),\quad j=1,\dots,m,\ n=1,\dots,N-1.
\end{aligned}\tag{3.7}$$

Now we may write together the equations (3.3) and (3.7) in matrix form

$$
\begin{aligned}
&(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_{n+1} \\
&= (V_0 - \alpha h V_1 - \lambda h^2 (V_2 - V_3))\alpha_n + h g_n, \quad n = 1, \dots, N-1, \qquad (3.8)
\end{aligned}
$$

with $(m+d+1) \times (m+d+1)$ matrices V, V_0, V_1, V_2, V_3 as follows:

$$
V = \begin{pmatrix} I \mid 0 \\ \hline C \end{pmatrix}, \qquad V_0 = \begin{pmatrix} A \\ \hline C \end{pmatrix},
$$

I being the $(d+1) \times (d+1)$ identity matrix,

$$
C = \begin{pmatrix} 0 & 1 & 2c_1 & \dots & (m+d)c_1^{m+d-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2c_m & \dots & (m+d)c_m^{m+d-1} \end{pmatrix},
$$

A being a $(d+1) \times (m+d+1)$ matrix

$$
A = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 0 & 1 & 2 & \dots & \dots & m+d \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & \dots & \binom{m+d}{d} \end{pmatrix},
$$

$$
V_1 = \begin{pmatrix} & & 0 & & \\ 1 & c_1 & c_1^2 & \dots & c_1^{m+d} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & c_m & c_m^2 & \dots & c_m^{m+d} \end{pmatrix},
$$

$$
V_2 = \begin{pmatrix} & & 0 & \\ c_1 & c_1^2/2 & \dots & c_1^{m+d+1}/(m+d+1) \\ \dots & \dots & \dots & \dots \\ c_m & c_m^2/2 & \dots & c_m^{m+d+1}/(m+d+1) \end{pmatrix},
$$

V_3 having first $d+1$ rows 0 and last m rows the vector q, and, finally, the $m{+}d{+}1$ dimensional vector $g_n = (0, \dots, 0, f(t_{n+1,1}) - f(t_{n1}), \dots, f(t_{n+1,m}) - f(t_{nm}))$. Thus $g_n = O(h)$ for $f \in C^1$.

Proposition 1. *The matrix $V - \alpha h V_1 - \lambda h^2 V_2$ is invertible for sufficiently small h.*

Proof. Since

$$\det V = \det \begin{pmatrix} (d+1)c_1^d & \dots & (m+d)c_1^{m+d-1} \\ \dots & \dots & \dots \\ (d+1)c_m^d & \dots & (m+d)c_m^{m+d-1} \end{pmatrix}$$

$$= (d+1)c_1^d \cdot \ldots \cdot (m+d)c_m^d \cdot \det \begin{pmatrix} 1 & c_1 & \dots & c_1^{m-1} \\ \dots & \dots & \dots & \dots \\ 1 & c_m^d & \dots & c_m^{m-1} \end{pmatrix} \neq 0,$$

and $d \geq 0$, the matrix V is invertible. Such is also $V - \alpha h V_1 - \lambda h^2 V_2$ for small h, which completes the proof. □

Therefore, the equation (3.8) can be written in the form

$$\alpha_{n+1} = (V^{-1}V_0 + W)\alpha_n + r_n,$$

where $W = O(h)$ and $r_n = O(h^2)$ for $f \in C^1$. Note that $W = 0$ if $\alpha = 0$ and $\lambda = 0$. Set $M = V^{-1}V_0$.

4. Stability of the method

We have seen that the spline collocation method (2.3) for the test equation (3.1) leads to the iteration process

$$\alpha_{n+1} = (V^{-1}V_0 + W)\alpha_n + r_n, \quad n = 1, \dots, N-1, \tag{4.1}$$

with $W = O(h)$ and $r_n = O(h^2)$.

We distinguish the method with initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \dots, d$, and another method which uses only $u_1(0) = y(0)$ and additional collocation points $t_{0j} = t_0 + c_{0j}h$, $j = 1, \dots, d$, with fixed $c_{0j} \in (0,1] \setminus \{c_1, \dots, c_m\}$ on the first interval σ_1. Denote, in addition, $d_0 = \max\{d-1, 0\}$ for the method with initial values and $d_0 = 0$ for the method with additional initial collocation.

Definition 2. We say that the spline collocation method is *stable* if for any $\alpha, \lambda \in \mathbb{C}$ and any $f \in C^{d_0}[0,T]$ the approximate solution u remains uniformly bounded in $C[0,T]$ as $h \to 0$.

Let us notice that the boundedness of $||u||_{C[0,T]}$ is equivalent to the boundedness of $||\alpha_n||$ in n and h in any fixed norm of $\mathbb{R}^{m+d+1}$.

The principle of uniform boundedness allows to establish

Proposition 3. *The spline collocation method is stable if and only if*

$$||u||_{C[0,T]} \leq \text{const}||f||_{C^{d_0}[0,T]} \quad \forall f \in C^{d_0}[0,T], \tag{4.2}$$

where the constant may depend only on T, α, λ and on the parameters c_j and c_{0j}.

In order to formulate and prove the results concerning the numerical stability properties of the polynomial spline collocation method, we need the following results for Volterra integral equations (VIE) (see [6]). The step-by-step collocation method for VIE is supposed to determine the approximate solution in $S^d_{m+d}(\Delta_N)$ by the collocation conditions similarly to (2.3) at the points t_{nj}.

(1) The stability for VIE depends on the matrix $\overline{M} = U_0^{-1}U$, where U_0 and U are $(m+d+1)\times(m+d+1)$ matrices as follows:

$$U = \left(\frac{I \mid 0}{G}\right), \qquad U_0 = \left(\frac{A}{G}\right),$$

I and A being defined as in V and V_0,

$$G = \begin{pmatrix} 1 & c_1 & \dots & c_1^{m+d} \\ \dots & \dots & \dots & \dots \\ 1 & c_m & \dots & c_m^{m+d} \end{pmatrix}.$$

(2) If all eigenvalues of $\overline{M}$ are in the closed unit disk and if those which lie on the unit circle have equal algebraic and geometric multiplicities, then the spline collocation method is stable.
(3) If $\overline{M}$ has an eigenvalue outside of the closed unit disk, then the method is unstable (u has exponential growth: $\| u \|_\infty \geq \text{const}\, e^{KN}$, $K > 0$).
(4) If all eigenvalues of $\overline{M}$ are in the closed unit disk and there is an eigenvalue on the unit circle with different algebraic and geometric multiplicities, then the method is weakly unstable (u may have polynomial growth: $\| u \|_\infty \sim \text{const}\, N^k, k \in N$).

Theorem 4. *For fixed c_j the eigenvalues of M for VIDE in the case m and $d+1$ and eigenvalues of $\overline{M}$ for VIE in the case m and d coincide and have the same algebraic and geometric multiplicities, except $\mu = 1$ whose algebraic multiplicity for VIDE is greater by one than for VIE.*

Proof. The eigenvalue problem for M is equivalent to the generalized eigenvalue problem for V_0 and V, i.e., $(M - \mu I)v = 0$ for $v \neq 0$ if and only if $(V_0 - \mu V)v = 0$, and $(M - \mu I)w = v$ takes place if and only if $(V_0 - \mu V)w = Vv$. Denote $\nu = 1 - \mu$. Then for VIDE with the parameters

m and $d+1$ we have

$$V_0 - \mu V = \begin{pmatrix} \nu & 1 & 1 & 1 & \dots & \dots & 1 \\ 0 & \nu & 2 & 3 & \dots & \dots & m+d+1 \\ 0 & 0 & \nu & \binom{3}{2} & \dots & \dots & \binom{m+d+1}{2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \nu & \dots & \binom{m+d+1}{d+1} \\ 0 & \nu & \nu \cdot 2c_1 & \dots & \dots & \dots & \nu(m+d+1)c_1^{m+d} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \nu & \nu \cdot 2c_m & \dots & \dots & \dots & \nu(m+d+1)c_m^{m+d} \end{pmatrix}. \quad (4.3)$$

Let $I_{i,p}$ be the diagonal matrix obtained from an identity matrix, replacing the i-th diagonal element by the number p. Thus, the products $I_{i,p}A$ and $AI_{i,p}$ mean the multiplication of i-th row and i-th column of A, respectively, by p. Consider also the matrices U_0 and U arising in the study of VIE with the parameters m and d. A direct calculation and the observation that $\binom{p}{q}\frac{q}{p} = \binom{p-1}{q-1}$, allows us to get from (4.3)

$$I_{d+2,d+1} \dots I_{3,2}(V_0 - \mu V)I_{3,1/2} \dots I_{d+m+2,1/(m+d+1)}$$
$$= \begin{pmatrix} \nu & 1 & 1/2 & \dots & 1/(m+d+1) \\ 0 & & & U_0 - \mu U & \end{pmatrix}$$

or

$$S(V_0 - \mu V)S^{-1} = R \begin{pmatrix} \nu & 1 & 1/2 & \dots & 1/(m+d+1) \\ 0 & & & U_0 - \mu U & \end{pmatrix}, \quad (4.4)$$

where $S = I_{d+2,d+1} \dots I_{3,2}$ and $R = I_{d+m+2,d+m+1} \dots I_{d+3,d+2}$. Now (4.4) gives

$$\det(V_0 - \mu V) = (d+2) \dots (d+m+1)\nu \det(U_0 - \mu U)$$

which permits to get the assertion about algebraic multiplicities of eigenvalues of M and $\overline{M}$. The eigenvalue $\mu = 1$ of M and $\overline{M}$ has geometric multiplicity m (this is proved for $\overline{M}$ in [6], but the proof for M is identical).

It remains to consider the geometric multiplicity of eigenvalues $\mu \neq 1$. Thus, suppose $\nu \neq 0$. Using (4.4), the equation $(V_0 - \mu V)v = 0$ can be written as

$$\begin{pmatrix} \nu & 1 & \dots \\ 0 & U_0 - \mu U & \end{pmatrix} Sv = 0$$

or, denoting $w = Sv$, equivalently

$$\nu w_1 + w_2 + \ldots + w_{m+d+2}/(m+d+1) = 0, \tag{4.5}$$

$$(U_0 - \mu U)\overline{w} = 0 \tag{4.6}$$

with $\overline{w} = (w_2, \ldots, w_{m+d+2})$.

Let $\overline{w}^1, \ldots, \overline{w}^k$ be linearly independent solutions of (4.6). Extending these vectors with the first components defined by (4.5), we get vectors $w^1, \ldots, w^k$ and then $S^{-1}w^1, \ldots, S^{-1}w^k$ as linearly independent solutions of $(V_0 - \mu V)v = 0$.

Conversely, consider $v^1, \ldots, v^k$ as linearly independent solutions of $(V_0 - \mu V)v = 0$. Dropping the first components of the vectors $w^i = Sv^i$ we get the solutions $\overline{w}^1, \ldots, \overline{w}^k$ of (4.6). Suppose $\gamma_1\overline{w}^1 + \ldots + \gamma_k\overline{w}^k = 0$ with at least one $\gamma_i \neq 0$. Then, (4.5) allows to get $\gamma_1 w^1 + \ldots + \gamma_k w^k = 0$ or $\gamma_1 v^1 + \ldots + \gamma_k v^k = 0$. This contradiction shows that the geometric multiplicities of $\mu \neq 1$ as an eigenvalue of M and $\overline{M}$ coincide. The proof is complete. □

Proposition 5. *If M has an eigenvalue outside of the closed unit disk, then the spline collocation method is not stable with possible exponential growth of the approximate solution.*

Proof. The structure of the proof is similar to that of Proposition 5 in [6] and we will deal only with the main ingredients. Consider an eigenvalue μ of $M + W$ such that $|\mu| \geq 1 + \delta$ with some fixed $\delta > 0$ for any sufficiently small h. For $\alpha_1 \neq 0$, being an eigenvector of $M + W$, we have here

$$(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_1 = h g_0, \tag{4.7}$$

where $g_0 = (a_{10}, \ldots, a_{1d}, f(t_{11}), \ldots, f(t_{1m}))$ and $a_{1j} = h^j y^{(j)}(0)/j!$, $j = 0, \ldots, d$. Because of

$$y'(0) = \alpha y(0) + f(0),$$

$$y^{(j)}(0) = \alpha y^{(j-1)}(0) + \lambda y^{(j-2)}(0) + f^{(j-1)}(0), j = 2, \ldots, d, \tag{4.8}$$

the vector α_1 determines via (4.7) and (4.8) the values $f^{(j)}(0)$, $j = 0, \ldots, d-1$, $f(t_{11}), \ldots, f(t_{1m})$. We take f on $[0, h]$ as the polynomial interpolating the values $f^{(j)}(0)$, $j = 0, \ldots, d-1$, $f(t_{1j})$, $j = 1, \ldots, m$, and $f^{(j)}(h) = 0$, $j = 0, \ldots, d_0$ (if $c_m = 1$, then $f^{(j)}(h) = 0$, $j = 1, \ldots, d_0$). In the case of the method of additional knots let f be on $[0, h]$ the interpolating polynomial for the data $f(0)$, $f(t_{0j})$, $j = 0, \ldots, d$, $f(t_{1j})$, $j = 1, \ldots, m$, and $f^{(j)}(h) = 0$, (here $d_0 = 0$ and if $c_m = 1$, then $f(t_{1m}) = f(h)$ is already given and we drop the requirement $f(h) = 0$). In both cases we ask f to be on $[nh, (n+1)h]$, $n \geq 1$, the interpolating polynomial for the values $f^{(j)}(nh) = 0$ and $f^{(j)}((n+1)h) = 0$, $j = 0, \ldots, d_0$ (if $c_m = 1$, then for $j = 1, \ldots, d_0$), and also $f(t_{n+1,j}) = f(t_{1j})$, $j = 1, \ldots, m$.

This guarantees that $f \in C^{d_0}[0,T]$ and $r_n = 0$, $n \geq 1$. The interpolant f can be represented on $[t_n, t_{n+1}]$ by the formula

$$f(t) = f(t_n + \tau h) = \sum_{i=0}^{\kappa} \left(\sum_{l=0}^{k_i} h^{s_l} p_{il} f^{(s_l)}(\xi_l) \right) \prod_{r=0}^{i-1} (\tau - b_r) \qquad (4.9)$$

with b_r being c_j or c_{0j}, ξ_l being t_{nj} or t_j, $0 \leq s_l \leq d_0$, $k_i \leq i$, constants p_{il} depending on c_j and c_{0j}. In the case of initial conditions $\kappa = m + d + d_0$ ($\kappa = m+d+d_0-1$ if $c_m = 1$) and in the case of additional knots $\kappa = m+d+1$ ($\kappa = m+d$, if $c_m = 1$) on the interval $[0,h]$ and $\kappa = m+2d_0+1$ ($\kappa = m+2d_0$ if $c_m = 1$) on the interval $[nh, (n+1)h]$, $n \geq 1$.

Replacing h by h/k, $k = 1, 2, \ldots$, and keeping $||\alpha_1|| = h/k$, we have $||g_0||_\infty$ bounded which means that $f(t_{1j})$, $j = 1, \ldots, m$, and $h^j y^{(j)}(0)/k^j$, $j = 0, \ldots, d$, or $h^j f^{(j)}(0)/k^j$, $j = 0, \ldots, d_0$, are bounded, too, in the process $k \to \infty$. Thus, (4.9) gives

$$||f||_{C^{d_0}[0,T]} \leq \text{const } k^{d_0}. \qquad (4.10)$$

On the other hand, $||\alpha_{n+1}|| \geq (1+\delta)^n ||\alpha_1||$ yields

$$||\alpha_{kN}|| \geq \frac{h}{k}(1+\delta)^{kN-1} \qquad (4.11)$$

and (4.2) cannot be satisfied. The inequalities (4.10) and (4.11) mean also the exponential growth of the approximate solution if we keep the norm of f bounded in C^{d_0}. The proof is complete. □

The case where all eigenvalues of M are in the closed unit disk and there is at least one of them on the unit circle having different algebraic and geometric multiplicities can be treated as for VIE. In fact, for VIDE the eigenvalue $\mu = 1$ has always different algebraic and geometric multiplicities. Thus, the collocation method is always at least weakly unstable. This weak instability, however, cannot be observed for low order splines which we confirm with numerical examples. In practice, the method is stable if and only if all eigenvalues of M are in the closed unit disk which we keep in mind describing the examples.

5. Examples

Let us consider some special cases of d and m.

Case $d = 0$, $m \geq 1$ being arbitrary. We have

$$V = \left(\frac{1\ 0\ \dots\ 0}{C} \right), \qquad V_0 = \left(\frac{1\ 1\ \dots\ 1}{C} \right)$$

and $\det(V_0 - \mu V) = (1-\mu)^{m+1} \det C_0$ where C_0 is obtained from C omitting the first column. This means that the method is always stable.

Case $d = 1$, $m = 1$ *(quadratic splines).* The equation $\det(V_0 - \mu V) = 0$ besides $\mu = 1$ has the solution $\mu = 1 - 1/c_1$. The method is stable if and only if $1/2 \leq c_1 \leq 1$.

Case $d = 1$, $m = 2$ *(Hermite cubic splines).* Now the equation $\det(V_0 - \mu V) = 0$ has a root $\mu = 1$ with geometric multiplicity 2 and algebraic multiplicity at least 3. The solution $\mu(c_1, c_2) = 1 - (c_1 + c_2 - 1)/c_1 c_2$ yields (see [6]) that the method is stable if and only if $c_1 + c_2 \geq 1$.

Case $d = 2$, $m = 1$ *(cubic splines).* Here the geometric multiplicity of $\mu = 1$ as solution of $\det(V_0 - \mu V) = 0$ is 1 and its algebraic multiplicity is 2. Two other solutions $\mu = 1 - (1 + 2c_1 \pm (1 + 4c_1(1-c_1))^{1/2})/2c_1^2$ show (see [6]) that the method is stable if and only if $c_1 = 1$.

6. Numerical tests

We chose the initial function $f(t) = (\cos t - 3 \sin t - e^t)/2$ and $\alpha = 1$, $\lambda = 1$ in the equation (3.1) on the interval $[0, 1]$. This equation has the exact solution $y(t) = (\sin t + \cos t + e^t)/2$. As an approximate value of $||u||_\infty$ we actually calculated $\max_{1 \leq n \leq N} \max_{0 \leq k \leq 10} \left| u_n(t_{n-1} + kh/10) \right|$. The results are presented in following tables.

Case $d = 0$, $m = 1$ *(linear splines).*

N	4	16	64	256	4096
$c_1 = 1.0$	2.105018	2.059782	2.052299	2.050586	2.050062
$c_1 = 0.5$	2.049933	2.050022	2.050027	2.050028	2.050028

Case $d = 0$, $m = 2$

N	4	16	64	256	4096
$c_1 = 0.7$, $c_2 = 1.0$	2.042611	2.049641	2.050004	2.050026	2.050028
$c_1 = 0.4$, $c_2 = 0.6$	2.047681	2.049882	2.050018	2.050027	2.050028

Case $d = 1$, $m = 1$ (quadratic splines)

N	4	16	64	256	4096
$c_1 = 1.0$	2.055503	2.050359	2.050048	2.050029	2.050028
$c_1 = 0.5$	2.047524	2.049863	2.050017	2.050027	2.050028
$c_1 = 0.4$	2.047418	2.049880	8.962233	$2.69 \cdot 10^{32}$	$1.83 \cdot 10^{165}$

Case $d = 1$, $m = 2$ (Hermite cubic splines)

N	4	16	64	256
$c_1 = 0.5$, $c_2 = 1.0$	2.050006	2.050027	2.050028	2.050028
$c_1 = 0.3$, $c_2 = 0.7$	2.049615	2.050001	2.050026	2.050027
$c_1 = 0.2$, $c_2 = 0.5$	2.043332	$3.21 \cdot 10^{2}$	$9.21 \cdot 10^{28}$	$1.39 \cdot 10^{142}$

Case $d = 2$, $m = 1$ (cubic splines)

N	4	16	64	256
$c_1 = 1.0$	2.050148	2.050028	2.050028	2.050028
$c_1 = 0.9$	2.049806	2.049999	5.773942	$1.60 \cdot 10^{29}$
$c_1 = 0.5$	2.054945	$3.30 \cdot 10^{4}$	$7.30 \cdot 10^{38}$	$2.77 \cdot 10^{183}$

7. Notes

The numerical solution of VIEs and VIDEs by collocation methods in certain piecewise polynomial spaces is discussed in detail in [1]. The methods which use polynomial splines as approximate solutions are considered in [4]. The most systematic attempt to study the numerical stability for Volterra integro-differential equations seems to be [3]. It should be remarked that the proof of the main result of [3] (Theorem 2.3) is not correct. In [3] this Theorem 2.3 is also applied to the particular cases and there are obtained stability conditions which are disproved by our results.

The test equation (3.1) in the case $f(t) = 0$ can be found in [2], see also [1].

Our technique was applied to VIE of the second kind in [6] in general setting. Additional information about the behaviour of the method with smooth polynomial splines is presented in [7]. We studied the numerical stability of the collocation method for VIDE in the case $d = -1$, $m \geq 1$ in [8]. It turns out that the stability depends essentially on the collocation parameters c_j and the numbers α, λ in connection with m.

The collocation with multiple collocation nodes (if $m \geq 2$) coinciding with spline knots for the Cauchy problem $y' = f(x, y)$, $y(0) = y_0$, is studied in [5]. In particular, it is proved that such a method is divergent for $d \geq m + 2$ and convergent for $d \leq m + 1$.

The case $m = 1$ in [5] coincides with our setting for $c_1 = 1$. Taking into account Theorem 4 of the current paper and the results of [7] for smooth splines ($m = 1$, $c_1 = 1$) we can find the complete consistence of the results, i.e., the method for VIDE with $m = 1$ and $c_1 = 1$ is stable if and only if $d \leq 2$ (until cubic splines).

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Institute of Applied Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia

E-mail address: `peeter_o@ut.ee; mare@ut.ee`