ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 6, 2002

Weakly regular topological algebras and spectral extension property

ARNE KOKK

ABSTRACT. Properties of weakly regular topological algebras are examined. Necessary and sufficient conditions for a commutative semisimple spectral algebra to have the spectral extension property for semisimple algebras are given. Some applications are also considered concerning the automatic continuity, unique uniform norm property and existence of algebra norms on topological algebras with functional spectrum.

1. Introduction

As it is well known in the Banach algebra theory, the algebra $C(X)$ of all continuous functions on a compact Hausdorff space has a unique complete algebra norm in the sense that any complete algebra norm on $C(X)$ induces the same topology that is induced by the sup-norm [16]. Even more, as it was proved in 1949 by I. Kaplansky [18], any algebra norm (not necessarily complete) on $C(X)$ is at least as large as the sup-norm, so that $C(X)$ has the so called spectral extension property, i.e. $r_{C(X)}(f) = r_B(f)$ for every $f \in C(X)$ and Banach algebras B in which $C(X)$ is (not necessarily continuously) embedded (here $r_{C(X)}$ stands for the spectral radius in $C(X)$).

From then on lots of papers have been devoted to the investigation of algebra norms and the spectral extension property of semisimple Banach algebras (as well as of other classes of topological algebras) (see, for example, [8,11–13,26,27]).

In Section 3 of the present paper we examine the spectral extension property for semisimple algebras and prove several necessary and sufficient conditions for a commutative semisimple spectral algebra to have the spectral extension property, as well as the strong spectral extension property, in semisimple extensions (Theorem 1 – Theorem 3). These conditions are given by means of certain subsets of the character space of the algebra under consideration.

Received October 28, 2002.
2000 *Mathematics Subject Classification.* 46H05, 46J05.
Key words and phrases. Topological algebra, weakly regular algebra, spectral extension property.

In conclusion, in Section 4, we apply the results obtained in proving some theorems concerning the automatic continuity, unique uniform norm property and existence of algebra norms on topological algebras with functional spectrum (Theorem 4 – Theorem 6).

Finally, it should be pointed out that our theorems cover many of the related results proved for semisimple Banach algebras in [4,6,7,9,19,25].

2. Preliminaries

Throughout this paper, all algebras are assumed to be associative, unital and over the complex field $\mathbb{C}$.

Let A be an algebra with the identity e_A. A *unital subalgebra* B of A is a subalgebra of A such that $e_A \in B$, $\sigma_A(a)$ is the *spectrum* of an element a in an algebra A and $r_A(a)$ is the *spectral radius* of $a \in A$, i.e. $r_A(a) = \sup\{ |\alpha| : \alpha \in \sigma_A(a)\}$. In case $\sigma_A(a)$ is empty (respectively unbounded) then $r_A(a)$ is defined to be $-\infty$ (respectively $+\infty$); and an algebra A is said to be *spectrally bounded* if $\sigma_A(a)$ is bounded for every $a \in A$ or, equivalently [32], if $\sigma_A(a)$ is compact for every $a \in A$.

In what follows, Rad$A$ will be the Jacobson radical of an algebra A and A is said to be *almost commutative* if $A/\mathrm{Rad}A$ is commutative. Also, Hom A will be the *character space* of A, i.e. the set of all non-zero multiplicative linear functionals on A, endowed with the topology of pointwise convergence. Note that Hom A is compact if A is spectrally bounded. For an algebra A with nonempty character space put $\hat{A} = \{\hat{a} : a \in A\}$, where $\hat{a}$ is the Gelfand transform of $a \in A$, i.e. $\hat{a}(\Lambda) = \Lambda(a)$ for every $\Lambda \in \mathrm{Hom}\, A$. Moreover, if B is any unital subalgebra of A then π_B^A is the restriction map $\pi_B^A : \mathrm{Hom}\, A \to \mathrm{Hom}\, B$.

The union over all $n = 1, 2, \dots$ of the sets A^n of all n-tuples $\mathbf{a} = (a_1, \dots, a_n)$ of elements of A will be denoted by A_∞ and $\hat{\mathbf{a}}(\Lambda) = (\Lambda(a_1), \dots, \Lambda(a_n))$ for every n-tuple $\mathbf{a} = (a_1, \dots, a_n) \in A_\infty$ and $\Lambda \in \mathrm{Hom}A$. The *Harte joint spectrum* $\sigma_A^H(\mathbf{a})$ of an n-tuple $\mathbf{a} = (a_1, \dots, a_n) \in A_\infty$ is defined to be the set of all those $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ for which the n-tuple $(a_1 - \alpha_1 e_A, \dots, a_n - \alpha_n e_A)$ generates a proper left or right ideal in A; and the Harte joint spectrum σ_A^H is said to have the *projection property* if

(I) $\pi(\sigma_A^H(\mathbf{a})) = \sigma_A^H(\pi(\mathbf{a}))$ for each $\mathbf{a} \in A_\infty$,

where $\pi(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_m})$ for $1 \le i_1 \le \dots \le i_m \le n$.

If A is an algebra such that $\sigma_A(a) = \hat{a}(\mathrm{Hom}A)$ for every $a \in A$ then it is said that A is an algebra with *functional spectrum* [2]. An algebra with functional spectrum is always almost commutative. In addition, a spectrally bounded algebra A is an algebra with functional spectrum if and only if the Harte joint spectrum σ_A^H of A admits the projection property[1] [21].

If X is a topological space then $C(X)$ is the algebra of all continuous complex-valued functions on X equipped with the compact-open topology

[1]Note that this may not be true for algebras having elements with unbounded spectra. Namely, there exist commutative algebras with empty character space but with Harte joint spectrum possessing the projection property [22].

and if A is an algebra with nonempty character space, we shall let G_A denote the *Gelfand map* of A, i.e. $G_A : A \to \hat{A} \subset C(\mathrm{Hom}A)$, $G_A(a) = \hat{a}$ $(a \in A)$.

Recall that a *topological algebra* is an algebra which is also a topological vector space in such a way that the ring multiplication is separately continuous; and a *locally convex algebra* is a topological algebra topology of which is defined by a family $\{p_i : i \in I\}$ of seminorms. In particular, when every seminorm p_i $(i \in I)$ is submultiplicative (i.e. $p_i(ab) \leq p_i(a)p_i(b)$ for every $a, b \in A$ and $i \in I$) then a locally convex algebra A is called a *locally m-convex algebra.* Note that every locally m-convex algebra is a topological algebra with jointly continuous ring multiplication. Moreover, a *uniform topological algebra* is a locally m-convex algebra topology of which is defined by a family of uniform seminorms (i.e. seminorms p satisfying $p(a^2) = p(a)^2$ for all $a \in A$).

A topological algebra A is called a Q-*algebra* if the set Inv$A$ of its invertible elements is open in the topology of A. Also, A is said to be *functionally continuous* if hom$A$ = Hom$A$, where hom$A$ stands for the space of continuous characters of A. Every Q-algebra is functionally continuous and spectrally bounded (see, for instance, [23, p. 60])[2]. In addition, $\{a \in A : r_A(a) < \alpha\}$ $(\alpha > 0)$ is a neighbourhood of zero in every Q-algebra A [15,34].

An algebra seminorm (i.e. submultiplicative seminorm) q on an algebra A is called *spectral* if $r_A(a) \leq q(a)$ for all $a \in A$; and if A can be equipped with a spectral seminorm then A is called a *spectral algebra* [30]. So, if A is a spectral algebra and q is a spectral seminorm on A then (A, q) is a locally m-convex Q-algebra. Moreover, an algebra norm $\|\ \|$ on A is said to be a Q-*norm* if $(A, \|\ \|)$ is a Q-algebra [27,33]. An easy calculation shows that an algebra norm on a commutative algebra is a Q-norm if and only if it is spectral[3].

Now, a Gelfand-Mazur algebra is a topological algebra A such that for every proper closed two-sided maximal modular ideal M of A the quotient algebra A/M is topologically isomorphic to $\mathbb{C}$; and if τ is a topology on A such that (A, τ) is a Gelfand-Mazur Q-algebra then we will say that τ is a Gelfand-Mazur Q-topology on A. For different classes of Gelfand-Mazur algebras see, for example, [1,23]. In particular, every commutative locally m-convex algebra is a Gelfand-Mazur algebra. Furthermore, one can easily check that an almost commutative Gelfand-Mazur Q-algebra is always spectral. In fact, it is well known that the following takes place (see [5,21]).

Proposition 1. *Let A be a spectrally bounded algebra. The following statements are equivalent:*

(a) *A is almost commutative and spectral,*

(b) *A is almost commutative and there is a uniform* Q-*norm on* $A/\mathrm{Rad}A$,

(c) *A is an algebra with functional spectrum,*

(d) $\sigma_A^H(\mathbf{a}) = \hat{\mathbf{a}}(\mathrm{Hom}A)$ *for any* $\mathbf{a} \in A_\infty$,

(e) *r_A is submultiplicative on A.*

---

[2]There exist spectrally bounded topological algebras, as well as functionally continuous topological algebras, which are not Q-algebras [10,14,29].

[3]Concerning this and several other equivalent conditions see, for instance, [3,15,19,24].

Let A be an algebra. A compact set $K \subset \mathrm{Hom}A$ is called a *set of uniqueness* for A if for any non-zero a in A there exists $\Lambda \in K$ such that $\Lambda(a) \neq 0$. Clearly, if there is a set of uniqueness for A then A is commutative and semisimple. A topological algebra A is said to be *weakly regular* [4] if given a closed subset $F \subset \mathrm{hom}A$, $F \neq \mathrm{hom}A$, there exists an element $a \neq 0$ in A such that $\hat{a}(\Lambda) = 0$ for every $\Lambda \in F$. Weak regularity arises naturally in the study of uniqueness of the uniform norm in commutative Banach algebras [4,7] and, in general, is weaker than regularity[4] [25].

A closed set $F \subset \mathrm{hom}A$ is called a *boundary set* (or else a *maximizing set*) for a topological algebra A if for every $a \in A$, there exists an element $\Lambda_0 \in F$ such that $|\hat{a}(\Lambda_0)| = \sup\{|\hat{a}(\Lambda)| : \Lambda \in \mathrm{hom}A\}$. The minimal (with respect to inclusion) boundary set for A is called the *Shilov boundary* of A. It is well known that every commutative Banach algebra A has a unique Shilov boundary $\Gamma(A)$. In fact, this is true for all commutative Gelfand-Mazur Q-algebras (see [23, pp. 189–193]). If A is a semisimple commutative Gelfand-Mazur Q-algebra then every compact $K \subset \mathrm{Hom}A$ containing the Shilov boundary $\Gamma(A)$ is a set of uniqueness for A, but there may be sets of uniqueness for A that do not contain the Shilov boundary [25].

Following [36], we call a compact set $K \subset \mathrm{Hom}A$ a *spectral set* for A if $\sigma_A(a) = \hat{a}(K)$ for each $a \in A$. A standard calculation shows that every commutative Gelfand-Mazur Q-algebra possesses a minimal spectral set. However, in general such a set is not determined in a unique way [36]. In addition, any spectral set K for a commutative Gelfand-Mazur Q-algebra A contains its Shilov boundary $\Gamma(A)$, so that $\Gamma(A)$ is contained in any minimal spectral set for A.

Let now A be an algebra with functional spectrum. An *extension* of A is an algebra B with functional spectrum such that there is a one-to one algebra homomorphism T of A into B with $T(e_A) = e_B$. If B is an extension of A then we shall view A as a unital subalgebra of B. Finally, analogously to [25,7], we say that A has the *spectral extension property for semisimple algebras* (respectively *strong spectral extension property for semisimple algebras*) if for any $a \in A$ and spectrally bounded semisimple extension B of A one has $r_A(a) = r_B(a)$ (respectively $\sigma_A(a) = \sigma_B(a)$).

In the sequel we shall need the following easy lemmas.

Lemma 1. *Let A be an algebra with functional spectrum and let B be a spectrally bounded extension of A such that $A \cap \mathrm{Rad}B = \{0\}$. Then $\pi_A^B(\mathrm{Hom}B)$ is a set of uniqueness for A.*

Proof. Note that, by definition, B is an algebra with functional spectrum. In addition, $\mathrm{Hom}B$ is compact, so that $K = \pi_A^B(\mathrm{Hom}B)$ is compact as well. Now $\hat{a}(K) = 0$ implies $a \in A \cap \mathrm{Rad}B = \{0\}$, so that K is a set of uniqueness for A. □

---

[4]A commutative topological algebra A is said to be *regular* if for any closed set $F \subset \mathrm{hom}A$ and continuous character $\Lambda_0 \notin F$ there exists an element $a \in A$ such that $\hat{a}(\Lambda_0) = 1$ and $\hat{a}(\Lambda) = 0$ for every $\Lambda \in F$.

Lemma 2. *Let A be an algebra with functional spectrum and suppose that K is a set of uniqueness for A. Then there exists a commutative semisimple Gelfand-Mazur* Q*-algebra B which is an extension of A such that a character $\lambda \in \mathrm{Hom}A$ has a multiplicative linear extension to B if and only if $\lambda \in K$.*

Proof. Let K be a set of uniqueness for A and put $B = C(K)$. Then B is a semisimple Banach algebra, hence a commutative semisimple Gelfand-Mazur Q-algebra, and because K is a set of uniqueness for A, the map $a \to \hat{a}_{|K}$ $(a \in A)$ is a one-to-one homomorphism of A into B. So, B is a semisimple spectrally bounded extension of A. Now, $\lambda \in \mathrm{Hom}A$ has a multiplicative linear extension to B if and only if there is $\Lambda \in K$ such that $\Lambda(a) = \hat{a}(\lambda) = \lambda(a)$ for all $a \in A$. Hence $\lambda \in \mathrm{Hom}A$ has a multiplicative linear extension to B if and only if $\lambda \in K$. □

Lemma 3. *Let A be an algebra with functional spectrum and let $\lambda \in \mathrm{Hom}A$. The following statements are equivalent:*

(a) *λ is contained in any set of uniqueness for A,*

(b) *λ has a multiplicative linear extension to any spectrally bounded extension B of A such that $A \cap \mathrm{Rad}B = \{0\}$.*

Proof. (a) $\Rightarrow$ (b). Let B be any spectrally bounded extension of A such that $A \cap \mathrm{Rad}B = \{0\}$. Then, by Lemma 1, $\pi_A^B(\mathrm{Hom}B)$ is a set of uniqueness for A, so that $\lambda \in \pi_A^B(\mathrm{Hom}B)$ or, equivalently, λ has a multiplicative linear extension to B.

(b) $\Rightarrow$ (a). Clear by Lemma 2. □

3. Weak regularity and spectral extension property for semisimple algebras

In this section we characterize weakly regular algebras and give, among others, several necessary and sufficient conditions for a commutative semisimple spectral algebra to have the spectral extension property for semisimple algebras.

Theorem 1. *Let A be a commutative unital semisimple spectral algebra. The following are equivalent:*

(a) *A has the spectral extension property for semisimple algebras,*

(b) *every set of uniqueness for A is a boundary set for A,*

(c) $\Gamma(A) \subset \cap\{K \subset \mathrm{Hom}A : K\, is\, a\, set\, of\, uniqueness\, for\, A\}$,

(d) *every character $\lambda \in \Gamma(A)$ has a multiplicative linear extension to any semisimple spectrally bounded extension B of A,*

(e) *if B is any commutative semisimple Gelfand-Mazur* Q*-algebra and $T : A \to B$ is one-to-one homomorphism then $G_A \circ T^{-1}$ is continuous on $T(A)$.*

Proof. (a) $\Rightarrow$ (b). Suppose on the contrary that there exists a set of uniqueness K for A that is not a boundary set for A. Then there is an element $a_0 \in A$ such that $|\hat{a_0}(\lambda)| < r_A(a_0)$ for every $\lambda \in K$. By Lemma 2, there is now an extension B of A such that

$$r_A(a_0) = r_B(a_0) = \sup\{|\hat{a_0}(\Lambda)| : \Lambda \in K\} < r_A(a_0).$$

This contradiction proves that K is a boundary set for A.

(b) $\Rightarrow$ (c). By Proposition 1, A has a unique Shilov boundary. The rest is clear.

(c) $\Rightarrow$ (d). Use Lemma 3.

(d) $\Rightarrow$ (e). Let B be a commutative semisimple Gelfand-Mazur Q-algebra and let $T : A \to B$ be a one-to-one homomorphism. If U is any neighbourhood of zero in $C(\mathrm{Hom}A)$ then there is $\epsilon > 0$ such that $U_\epsilon = \{\hat{a} \in \hat{A} : r_A(a) < \epsilon\} \subset U$. Now, as it was mentioned above, $V_\epsilon = \{b \in T(A) : r_B(b) < \epsilon\}$ is a neighbourhood of zero in $T(A)$, and an easy calculation shows that $(G_A \circ T^{-1})(V_\epsilon) \subset U_\epsilon$. We conclude that $G_A \circ T^{-1}$ is continuous on $T(A)$.

(e) $\Rightarrow$ (a). Let B be any semisimple spectrally bounded extension of A. Since B is a semisimple spectrally bounded algebra with functional spectrum, r_B is a Q-norm on B (see Proposition 1). Consider now A as a unital subalgebra of a semisimple Gelfand-Mazur Q-algebra (B, r_B). Then, by condition (e), the Gelfand map G_A is continuous, so that $r_A(a) \leq r_B(a)$ for any $a \in A$. □

By means of Theorem 1, we prove now the following characterization of commutative semisimple weakly regular Gelfand-Mazur Q-algebras.

Theorem 2 (cf. [25, Theorem 1]). *Let A be a commutative unital semisimple Gelfand-Mazur* Q*-algebra. The following are equivalent:*

(a) *A is weakly regular,*

(b) $\Gamma(A) = \mathrm{hom}A$ *and A has the spectral extension property for semisimple algebras,*

(c) $\Gamma(A) = \mathrm{hom}A$ *is a minimal set of uniqueness for A,*

(d) *every character $\lambda \in \mathrm{hom}A$ has a multiplicative linear extension to any spectrally bounded extension B of A satisfying $A \cap \mathrm{Rad}B = \{0\}$,*

(e) $\sigma_A^H(\mathbf{a}) = \sigma_B^H(\mathbf{a})$ *for any semisimple spectrally bounded extension B of A and $\mathbf{a} \in A_\infty$.*

Proof. (a) $\Rightarrow$ (b). If A is semisimple then $\Gamma(A)$ is the set of uniqueness for A. So, $\Gamma(A) = \mathrm{hom}A$ and, by Theorem 1, A has the spectral extension property for semisimple algebras.

(b) $\Rightarrow$ (c). Clear by Theorem 1 because $\Gamma(A) = \mathrm{hom}A$ is a set of uniqueness for A.

(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (e) are clear by Lemma 3 and Proposition 1.

(e) $\Rightarrow$ (a). If B is any semisimple spectrally bounded extension of A then σ_B^H possesses the projection property and, since $\sigma_A^H(\mathbf{a}) = \hat{\mathbf{a}}(\mathrm{Hom}A)$, every character in HomA has a multiplicative linear extension to B [20]. So, by Lemma 2, Hom$A$ is the only set of uniqueness for A, i.e. A is weakly regular. □

So, the weak regularity is equivalent to the "permanence" of the Harte joint spectrum in semisimple spectrally bounded extensions. This clearly implies the strong spectral extension property for A.

Theorem 3 (cf. [25, Theorem 2]). *Let A be a commutative semisimple Gelfand-Mazur* Q*-algebra. The following are equivalent:*

(a) *A has the strong spectral extension property for semisimple algebras,*

(b) *every set of uniqueness for A is a spectral set for A,*

(c) $\Gamma(A) = \bigcap\{K \subset \mathrm{Hom}A : K$ *is a set of uniqueness for* $A\}$ *is the unique minimal spectral set for A,*

(d) *every character $\lambda \in \mathrm{Hom}A$ belonging to the minimal spectral set for A has a multiplicative linear extension to any semisimple spectrally bounded extension B of A.*

Proof. (a) $\Rightarrow$ (b). Suppose on the contrary that there exists a set of uniqueness K for A that is not a spectral set for A. Then there is an element $a_0 \in A$ such that $\hat{a_0}(K) \neq \sigma_A(a_0)$. By Lemma 2, there is now a semisimple Gelfand-Mazur Q-algebra extension B of A such that $\sigma_A(a_0) = \sigma_B(a_0) = \hat{a_0}(K)$. This contradiction proves that K is a spectral set for A.

(b) $\Rightarrow$ (c). Since A is semisimple, every boundary set for A is a set of uniqueness for A. Thus $\Gamma(A)$ is a spectral set for A and, moreover, $\Gamma(A) \subset \bigcap\{K \subset \mathrm{Hom}A : K \text{ is a set of uniqueness for } A\} \subset \Gamma(A)$. On the other hand, since every spectral set for A is a boundary set for A, $\Gamma(A)$ is contained in any spectral set for A, so that $\Gamma(A)$ is the unique minimal spectral set for A.

(c) $\Rightarrow$ (d). Clear by Lemma 3.

(d) $\Rightarrow$ (a). Since A is a commutative Gelfand-Mazur Q-algebra, there is a minimal spectral set for A. The rest is clear. □

4. Applications

In conclusion, we would like to apply the theorems proved in the previous section to obtain some results concerning the automatic continuity, unique uniform norm property and existence of algebra norms on topological algebras with functional spectrum.

We begin with the following automatic continuity theorem.

Theorem 4 (cf. [4, Theorem 1], [19, Theorem 8]). *Let A be a uniform weakly regular* Q*-algebra, let B be a commutative semisimple Gelfand-Mazur* Q*-algebra and suppose that $T : A \to B$ is a continuous one-to-one homomorphism. Then T is a topological isomorphism of A into B.*

Proof. Note that a uniform commutative weakly regular Q-algebra A is a commutative semisimple topological algebra with continuous G_A^{-1} [23] and then apply Theorem 1 and Theorem 2. □

Let now A be a spectrally bounded semisimple algebra with functional spectrum. Then A is said to have the *unique uniform norm property* if r_A is the only uniform norm on A [7].

From Theorem 1 we easily have the following characterization of the unique uniform norm property.

Theorem 5 (cf. [7, Theorem 2.3]). *Let A be a commutative semisimple Gelfand-Mazur* Q*-algebra. The following are equivalent:*

(a) *A has the unique uniform norm property,*

(b) *A has the spectral extension property for semisimple algebras,*

(c) *every set of uniqueness for A is a boundary set for A,*

(d) $\Gamma(A) \subset \bigcap\{K \subset \mathrm{Hom}A : K\, is\, a\, set\, of\, uniqueness\, for\, A\}$,

(e) *every character $\lambda \in \Gamma(A)$ has a multiplicative linear extension to any semisimple spectrally bounded extension B of A.*

Finally, to consider the existence of algebra norms on semisimple algebras with functional spectrum, we need the following definition and lemma.

A topological algebra A is said to be *normal* if for any disjoint closed subsets S and T in hom$A$ there exists $a \in A$, such that $\hat{a}(\Lambda) = 0$ for all $\Lambda \in T$ and $\hat{a}(\Lambda) = 1$ for all $\Lambda \in S$.

Lemma 4. *Let A be a normal and semisimple Gelfand-Mazur* Q*-algebra and let B be a spectrally bounded extension of A. Then $A \cap \mathrm{Rad}B = \{0\}$.*

Proof. If $K = \pi_A^B(\mathrm{Hom}B) \neq \mathrm{Hom}A$ then, by [19, Lemma 7], there are nonzero elements $a, b \in A$ such that $ab = 0$ and $\hat{a}(\Lambda) = 1$ for all $\Lambda \in K$. But this is impossible, so that $\pi_A^B(\mathrm{Hom}B) = \mathrm{Hom}A$ which gives us the desired equality $A \cap \mathrm{Rad}B = \{0\}$. □

Theorem 6. *Let A be a semisimple algebra with functional spectrum and let $C \subset A$ be a unital semisimple subalgebra of A which can be equipped with a topology τ so that (C, τ) is a normal Gelfand-Mazur* Q*-algebra. If π_C^A is one-to-one then the following are equivalent:*

(a) *A is spectrally bounded,*

(b) *A has an algebra norm.*

Proof. (a) $\Rightarrow$ (b). Use Proposition 1.

(b) $\Rightarrow$ (a). Let $\|\ \|$ be an algebra norm on A and denote by B the completion of $(A, \|\ \|)$. If $\Lambda \in \mathrm{Hom}A$ then $\Lambda_{|_C} \in \mathrm{Hom}C$ so that, by Theorem 2 and Lemma 4, $\Lambda_{|_C}$ has a multiplicative linear extension $\Phi \in \mathrm{Hom}B$. Now $\Phi_{|_C} = \Lambda_{|_C}$ and, by assertions, $\Lambda = \Phi_{|_A}$. We conclude that $\sigma_A(a) = \hat{a}(\mathrm{Hom}B)$ is compact for every $a \in A$. □

Corollary ([17,28,31,35]). *Let X be a completely regular topological space. Then there exists an algebra norm on $C(X)$ if and only if every continuous function on X is bounded.*

Proof. Put $C = C_b(X)$, where $C_b(X)$ is the subalgebra of $C(X)$ consisting of bounded functions on X. Now apply Theorem 6. □

Acknowledgement. The author is grateful to the referee for useful remarks.

References

[1] M. Abel, *Gelfand-Mazur algebras*; In: Topological vector spaces, algebras and related areas. Pitman Research Notes in Math. Series **316**. Longman Group Ltd. Harlow, 1994, pp. 116–129.

[2] M. Abel, *Advertive topological algebras*; In: General Topological Algebras. Proc. Intern. Workshop, Tartu, Oct. 4–7, 1999. Mathematics Studies **1** (2001), pp. 14–24.

[3] H. Arizmendi and V. Valov, *Some characterizations of Q-algebras*, Comment. Math. Prace Mat. **39** (1999), 11–21.

[4] A. Beddaa, S. J. Bhatt and M. Oudadess, *On automatic continuity of homomorphisms*, Proc. Amer. Math. Soc. **128** (2000), 1039–1045.

[5] V. A. Belfi and R. S. Doran, *Norm and spectral characterizations in Banach algebras*, Enseign. Math. **26** (1980), 103–130.

[6] S. J. Bhatt, *Automatic continuity of homomorphisms in topological algebras*, Proc. Amer. Math. Soc. **119** (1993), 135–139.

[7] S. J. Bhatt and H. V. Dedania, *Banach algebras with unique uniform norm*, Proc. Amer. Math. Soc. **124** (1996), 579–584.

[8] S. J. Bhatt and H. V. Dedania, *Banach algebras with unique uniform norm. II.*, Studia Math. **147** (2001), 211–235.

[9] S. J. Bhatt and D. J. Karia, *Uniqueness of the uniform norm with an application to topological algebras*, Proc. Amer. Math. Soc. **116** (1992), 499–503.

[10] R. Choukri, *Sur certaines questions concernant les Q-algèbres*, Extracta Math. **16** (2001), 79–82.

[11] H. G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. **10** (1978), 129–183.

[12] H. G. Dales and J. Esterle, *Discontinuous homomorphisms from C(X)*, Bull. Amer. Math. Soc. **83** (1977), 257–259.

[13] M. Fragoulopoulou, *Automatic continuity of *-morphisms between non-normed topological *-algebras*, Pacific J. Math. **147** (1991), 57–70.

[14] M. Fragoulopoulou, *Tensor products of enveloping locally C^*-algebras*, Schriftenreihe Math. Inst. Univ. Münster, 3. Ser. **21** (1997), 1–81.

[15] M. Fragoulopoulou, *Q-algebras: how close are they to Banach algebras? A survey*; In: General Topological Algebras. Proc. Intern. Workshop, Tartu, Oct. 4–7, 1999. Mathematics Studies **1** (2001), pp. 125–134.

[16] I. Gelfand, D. Raikov and G. Shilov, *Commutative Normed Rings*, Chelsea, New York, 1964.

[17] K. Jarosz, *Norms on C(X)*, Preprint, 1996.

[18] I. Kaplansky, *Normed algebras*, Duke Math. J. **16** (1949), 399–418.

[19] E. Kissin and V. S. Shulman, *Dense Q-subalgebras of Banach and C^*-algebras and unbounded derivations of Banach and C^*-algebras*, Proc. Edinburgh Math. Soc. **36** (1993), 261–276.

[20] A. Kokk, *Joint spectrum and extension of homomorphisms*, Tartu Ül. Toimetised **878** (1990), 67–82 (Russian).

[21] A. Kokk, *Almost commutativity of spectrally bounded algebras*, Tartu Ül. Toimetised **960** (1993), 29–40.

[22] A. Kokk, *Spectral systems on countably generated algebras*; In: General Topological Algebras. Proc. Intern. Workshop, Tartu, Oct. 4–7, 1999. Mathematics Studies **1** (2001), pp. 161–164.

[23] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland Publ. Co., Amsterdam, 1986.

[24] V. Mascioni, *Some characterizations of complex normed Q-algebras*, Elem. Math. **42** (1987), 10–14.

[25] M. J. Meyer, *The spectral extension property and extension of multiplicative linear functionals*, Proc. Amer. Math. Soc. **112** (1991), 855–861.

[26] M. J. Meyer, *Continuous dense embeddings of strong Moore algebras*, Proc. Amer. Math. Soc. **116** (1992), 727–735.

[27] M. J. Meyer, *Minimal incomplete norms on Banach algebras*, Studia Math. **102** (1992), 77–85.

[28] M. J. Meyer, *Some algebras without submultiplicative norms or positive functionals*, Studia Math. **116** (1995), 299–302.

[29] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. **11** (1952).

[30] T. W. Palmer, *Spectral algebras*, Rocky Mountain J. Math. **22** (1992), 293–328.

[31] A. R. Pruss, *A remark on non-existence of an algebra norm for the algebra of continuous functions on a topological space admitting an unbounded continuous function*, Studia Math. **116** (1995), 295–297.

[32] M. Roitman and Y. Sternfeld, *When is a linear functional multiplicative?* Trans. Amer. Math. Soc. **267** (1981), 111–124.

[33] B. J. Tomiuk and B. Yood, *Incomplete normed algebra norms on Banach algebras*, Studia Math. **95** (1989), 119–132.

[34] Y. Tsertos, *Representations and extensions of positive functionals on *-algebras*, Boll. Un. Mat. Ital. B(7) **8** (1994), 541–555.

[35] B. Yood, *On the non-existence of norms for some algebras of functions*, Studia Math. **111** (1994), 97–101.

[36] W. Żelazko, *An axiomatic approach to joint spectra I*, Studia Math. **64** (1979), 249–261.

INSTITUTE OF PURE MATHEMATICS, UNIVERSITY OF TARTU, LIIVI 2–616, 50409 TARTU, ESTONIA

E-mail address: arne@math.ut.ee