ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 7, 2003

Statistically pre-Cauchy sequences and bounded moduli

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ABSTRACT. Let $x = (x_k)$ be a sequence and let f be a bounded modulus. We prove that x is statistically pre-Cauchy if and only if

$$\lim_n \frac{1}{n^2} \sum_{j,k \leq n} f\left(|x_k - x_j|\right) = 0.$$

This implies a theorem due to Connor, Fridy and Kline [4].

Introduction

The concept of statistical convergence was first defined by Steinhaus [11] at a conference held at Wroclaw University, Poland, in 1949 and also independently by Fast [5], Buck [1] and Schoenberg [10] for real and complex sequences. Šalát [9] used the idea of bounded statistical convergence to construct the sequence space which is a nowhere dense subset of the linear normed space l_∞ of all bounded sequences of real numbers. Maddox [7] gave some basic properties of statistical convergence of a sequence $x = (x_k)$ in a locally convex Hausdorff topological linear space. Fridy [6] obtained the statistical analogue of Cauchy criterion of convergence for real sequences.

Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is called statistically convergent to L if

$$\lim_n \frac{1}{n} \left|\{k : |x_k - L| \geq \varepsilon, \quad k \leq n\}\right| = 0,$$

and statistically pre-Cauchy if

$$\lim_n \frac{1}{n^2} \left|\{(j,k) : |x_k - x_j| \geq \varepsilon, \quad j,k \leq n\}\right| = 0$$

Received October 29, 2002.

2000 *Mathematics Subject Classification.* 40A05.

Key words and phrases. Statistically convergent sequence, statistically pre-Cauchy sequence, modulus function.

for every $\varepsilon > 0$. Connor, Fridy and Kline [4] proved that statistically convergent sequences are statistically pre-Cauchy and any bounded statistically pre-Cauchy sequence with a nowhere dense set of limit points is statistically convergent. They also gave an example showing statistically pre-Cauchy sequences are not necessarily statistically convergent.

The notion of a modulus function was introduced by Nakano [8]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that (i) $f(x)=0$ if and only if $x=0$, (ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$, (iii) f is increasing, (iv) f is continuous from the right at 0. Because of (ii), $|f(x)-f(y)|\leq f(x-y)$ so that in view of (iv), f is continuous on $[0,\infty)$. A modulus may be bounded or unbounded. For example, $f(x)=x^p$ $(0<p\leq 1)$ is unbounded and $f(x)=\frac{x}{1+x}$ is bounded.

Results

In [4] Connor, Fridy and Kline proved that a bounded sequence $x=(x_k)$ is statistically pre-Cauchy if and only if

$$\lim_n \frac{1}{n^2}\sum_{j,k\leq n}|x_k-x_j|=0.$$

We establish the following criterion for arbitrary sequences to be statistically pre-Cauchy.

Theorem 1. *Let $x=(x_k)$ be a sequence and let f be a bounded modulus. Then x is statistically pre-Cauchy if and only if*

$$\lim_n \frac{1}{n^2}\sum_{j,k\leq n} f(|x_k-x_j|)=0.$$

Proof. First suppose that

$$\lim_n \frac{1}{n^2}\sum_{j,k\leq n} f(|x_k-x_j|)=0.$$

Observe that for each $\varepsilon>0$ and $n\in\mathbb{N}$ we have that

$$\begin{aligned}\frac{1}{n^2}\sum_{j,k\leq n} f(|x_k-x_j|) &= \frac{1}{n^2}\sum_{\substack{j,k\leq n\\ |x_k-x_j|<\varepsilon}} f(|x_k-x_j|)+\frac{1}{n^2}\sum_{\substack{j,k\leq n\\ |x_k-x_j|\geq\varepsilon}} f(|x_k-x_j|)\\ &\geq \frac{1}{n^2}\sum_{\substack{j,k\leq n\\ |x_k-x_j|\geq\varepsilon}} f(|x_k-x_j|)\\ &\geq f(\varepsilon)\left(\frac{1}{n^2}\left|\{(j,k):|x_k-x_j|\geq\varepsilon,\quad j,k\leq n\}\right|\right)\geq 0\end{aligned}$$

and thus x is statistically pre-Cauchy.

Now suppose that x is statistically pre-Cauchy, and that $\varepsilon > 0$ has been given. Let $\delta > 0$ such that $f(\delta) < \varepsilon/2$. Since f is bounded, there exists an integer B such that $f(x) < B$ for all $x \geq 0$. Note that, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{n^2}\sum_{j,k\leq n} f\left(|x_k - x_j|\right) &= \frac{1}{n^2}\sum_{\substack{j,k\leq n\\ |x_k-x_j|<\delta}} f\left(|x_k - x_j|\right) + \frac{1}{n^2}\sum_{\substack{j,k\leq n\\ |x_k-x_j|\geq\delta}} f\left(|x_k - x_j|\right)\\
&\leq f(\delta) + \frac{1}{n^2}\sum_{\substack{j,k\leq n\\ |x_k-x_j|\geq\delta}} f\left(|x_k - x_j|\right)\\
&\leq \frac{\varepsilon}{2} + B\left(\frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \geq \delta, j,k \leq n\}\right|\right). \qquad (1)
\end{aligned}$$

Since x is statistically pre-Cauchy, there is an N such that the right-hand side of (1) is less than ε for all $n > N$. Hence,

$$\lim_n \frac{1}{n^2}\sum_{j,k\leq n} f\left(|x_k - x_j|\right) = 0.$$

□

A similar argument yields the following result.

Theorem 2. *Let $x = (x_k)$ be a sequence and let f be a bounded modulus. Then x is statistically convergent to L if and only if*

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(|x_k - L|\right) = 0.$$

Proof. If

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(|x_k - L|\right) = 0$$

with an arbitrary modulus f, then x is statistically convergent to L by Theorem 8 of [3]. Now suppose that x is statistically convergent to L. We may prove analogously to Theorem 1 that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(|x_k - L|\right) = 0,$$

using that f is a bounded modulus. □

Corollary 3 (Connor *et al*, see [4, Theorem 3]). *Let $x = (x_k)$ be a bounded sequence. Then x is statistically pre-Cauchy if and only if*

$$\lim_n \frac{1}{n^2} \sum_{j,k \leq n} |x_k - x_j| = 0.$$

Proof. Let $B = \sup_k |x_k|$ and define

$$f(x) = \frac{(1+2B)\,x}{1+x}.$$

Then

$$f(|x_k - x_j|) \leq (1+2B)\,|x_k - x_j|$$

and

$$\begin{aligned} f(|x_k - x_j|) &= (1+2B)\,\frac{|x_k - x_j|}{1+|x_k - x_j|} \\ &\geq (1+2B)\,\frac{|x_k - x_j|}{1+|x_k| + |x_j|} \\ &\geq (1+2B)\,\frac{|x_k - x_j|}{1+2B} = |x_k - x_j|. \end{aligned}$$

Hence,

$$\lim_n \frac{1}{n^2} \sum_{j,k \leq n} |x_k - x_j| = 0 \Leftrightarrow \lim_n \frac{1}{n^2} \sum_{j,k \leq n} f(|x_k - x_j|) = 0,$$

and an immediate application of Theorem 1 completes the proof. □

Corollary 4 (Connor, see [2, Theorem 2.1]). *Let $x = (x_k)$ be a bounded sequence. Then x is statistically convergent to L if and only if*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0.$$

Proof. Let $B = \sup_k |x_k|$ and define

$$f(x) = \frac{(1+B+L)\,x}{1+x}.$$

A similar argument as in the proof of Corollary 3 enables us to apply Theorem 2. □

Acknowledgment

The author is grateful to the referees for their suggestions, which have greatly improved the readability of the paper.

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