ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 7, 2003

Convergence in measure and Weyl multipliers for absolute summability of double function series

SIMSON BARON AND HEINO TÜRNPU

ABSTRACT. In this article a quantitative relation between the partial sums of the double function series $\sum_{m,n} c_{mn} f_{mn}(x)$ and the double sequence (c_{mn}) of its coefficients is established. Estimates of powers of the partial sums of the double function series are given and Weyl multipliers for absolute summability almost everywhere by triangular matrix methods are obtained.

1. Introduction

We denote by Q a d-dimensional interval and by λ the Lebesgue measure of sets $E \subset \mathbf{R}^d$ with $\lambda E < \infty$. Let $\mathbf{f} := \{f_{mn}\}$ be a system of functions f_{mn} which are Lebesgue integrable on Q, in short $\mathbf{f} \subset L^1_\lambda(Q)$.

We consider the double function series

$$\sum_{m,n} c_{mn} f_{mn}(x), \tag{1.1}$$

where the double sequence $c := (c_{mn})$ belongs to the Banach space ℓ^p with finite $p > 1$ and $x := (x_1, \ldots, x_d)$ belongs to Q.

Later on, unless otherwise indicated, the free indices always run through all values 0, 1, 2, ..., and summation is likewise over all 0, 1, 2, ...

Let T be a triangular matrix method of summability. The aim of this paper is to seek under which conditions a non-decreasing double sequence $w := (w_{mn})$ of positive numbers is a Weyl multiplier for $\mathbf{f}$ with respect to

Received November 4, 2002.
2000 *Mathematics Subject Classification.* 28A20, 40B05, 40G05, 42C15, 46E30.
Key words and phrases. Double function series, convergence in measure, powers of partial sums, the space ℓ^p, absolute summability, Weyl multipliers, summability factors, Cesàro method, weighted means method.

$|T|$-summability, i.e., under which conditions the double function series (1.1) is absolutely T-summable almost everywhere on Q whenever

$$\sum_{m,n} |c_{mn}|^p \, w_{mn} < \infty. \tag{1.2}$$

The corresponding results (see Theorem 5.1 and its corollaries) are contained in the last Section 5. They rely on an extension of a Nikishin's inequality (see [10], Theorem 7) established in Section 4 (see Theorem 4.1 and its corollary). The preparatory results are contained in Sections 2 and 3.

Let us fix some more notation. If the matrix method T transforms double series to double sequences, we denote its elements by τ_{mnkl}, and if T transforms double series to double series, its elements are denoted by $\bar{\tau}_{mnkl}$.

It is well known (see, e.g., [9], p. 21) that T conserves the absolute convergence of double series if and only if

$$\sum_{m,n=k,l}^{\infty} |\bar{\tau}_{mnkl}| = O(1). \tag{1.3}$$

In this paper λ-measurable sets and λ-measurable functions will simply be called measurable sets and measurable functions, respectively. Therefore, we write dx instead of $d\lambda(x)$.

2. Lemmas on measurable functions and convergence in measure

The following lemma is a piece of lore; we include the proof for the sake of completeness.

Lemma 2.1. *Let g be a measurable function on Q. Then g is finite almost everywhere on Q if and only if for every $\delta > 0$ there exists a measurable subset $Q_\delta \subset Q$ with $\lambda Q_\delta > \lambda Q - \delta$ such that*

$$\int_{Q_\delta} |\, g(x)|\, dx < \infty. \tag{2.1}$$

Proof. Necessity. Let

$$D = Q(|\, g(x)| = +\infty), \qquad E_n = Q(|\, g(x)| > n).$$

Since $D = \bigcap_n E_n$, it follows that $\lim \lambda E_n = \lambda D = 0$. Therefore, for any $\delta > 0$, an integer N exists such that $\lambda E_N < \delta$. Taking $Q_\delta = Q \setminus E_N$, we obtain $\lambda Q_\delta = \lambda Q - \lambda E_N > \lambda Q - \delta$ and also

$$\int_{Q_\delta} |\, g(x)|\, dx \le N \lambda Q_\delta \le N \lambda Q < \infty.$$

Sufficiency. Assume that a measurable subset $B \subset Q$ exists with $\delta = \lambda B > 0$ on which g is not finite. Let a measurable subset $Q_\delta \subset Q$ be such that $\lambda Q_\delta > \lambda Q - \delta$ and (2.1) is valid. Then $\lambda(B \bigcap Q_\delta) > 0$, because $\delta - \lambda(B \bigcap Q_\delta) = \lambda(B) - \lambda(B \bigcap Q_\delta) = \lambda(B \bigcup Q_\delta) - \lambda(Q_\delta) \leq \lambda(Q) - \lambda(Q_\delta) < \delta$. Moreover, the function g is Lebesgue integrable and therefore finite almost everywhere on Q_δ. This contradicts the infiniteness of g on $B \bigcap Q_\delta$. □

The one-dimensional case (with $Q = [0,1]$) of the next lemma is due to Nikishin [10], p. 137, Lemma 1. We include the proof for the sake of completeness.

Lemma 2.2. *Let $l \geq 2$ be an integer and let $\eta > 0$ be a real number. If measurable sets $E_i \subset Q$ and $\Phi_i \subset Q$, for each $i = 1, \dots, l^3$ and $k = 2, \dots, l^3$, satisfy the conditions*

$$E_i \subset \Phi_i, \tag{2.2}$$

$$\lambda \Phi_i \leq \eta, \tag{2.3}$$

$$E_k \bigcap \bigcup_{j=1}^{k-1} \Phi_j = \varnothing, \tag{2.4}$$

then there exists an l-tuple $[i_1, \dots, i_l]$ of positive integers with

$$i_1 < i_2 < \cdots < i_l \tag{2.5}$$

such that

$$\lambda[(\bigcup_{\varkappa=1}^{l} E_{i_\varkappa}) \bigcap \bigcup_{k=1}^{l} (\Phi_{i_k} \setminus E_{i_k})] \leq \eta / l. \tag{2.6}$$

Proof. We fix an l-tuple $[i_1, \dots, i_l]$ of positive integers. We denote $L := l^3$, $\delta(\varkappa, k) := \lambda(\Phi_\varkappa \cap E_k)$, and

$$\Psi_l := (\bigcup_{\varkappa=1}^{l} E_{i_\varkappa}) \bigcap \bigcup_{k=1}^{l} (\Phi_{i_k} \setminus E_{i_k}).$$

Since (2.2) and (2.4) imply that

$$E_i \cap E_j = \varnothing \quad (\forall i \neq j), \tag{2.7}$$

we have

$$\lambda \Psi_l = \sum_{\varkappa=1}^{l} \lambda[E_{i_\varkappa} \bigcap \bigcup_{k=1}^{l} (\Phi_{i_k} \setminus E_{i_k})].$$

Hence by (2.4) we obtain

$$\lambda\Psi_l = \sum_{\varkappa=1}^{l} \lambda[E_{i_\varkappa} \bigcap \bigcup_{k=\varkappa}^{l} (\Phi_{i_k} \setminus E_{i_k})].$$

Since $E_{i_\varkappa} \bigcap(\Phi_{i_k} \setminus E_{i_k}) = \varnothing$ for $k = \varkappa$, we get

$$\lambda\Psi_l = \sum_{\varkappa=1}^{l-1} \lambda[E_{i_\varkappa} \bigcap \bigcup_{k=\varkappa+1}^{l} (\Phi_{i_k} \setminus E_{i_k})] \leq$$

$$\leq \sum_{\varkappa=1}^{l-1} \lambda(E_{i_\varkappa} \bigcap \bigcup_{k=\varkappa+1}^{l} \Phi_{i_k}) \leq$$

$$\leq \sum_{\varkappa=1}^{l-1} [\delta(i_{\varkappa+1}, i_\varkappa) + \delta(i_{\varkappa+2}, i_\varkappa) + \dots + \delta(i_l, i_\varkappa)].$$

By (2.4) we obtain that $\delta(i_k, i_\varkappa) = 0$ if $k < \varkappa$, and therefore

$$\lambda\Psi_l \leq \sum_{k,\varkappa=1,\, k\neq\varkappa}^{l} \delta(i_k, i_\varkappa). \tag{2.8}$$

Let J be the set of all possible l-tuples with (2.5), i. e.,

$$J := \{[i_1, ..., i_l] : 1 \leq i_1 < i_2 < \dots < i_l \leq L\}.$$

The set J contains C_L^l elements. Let K_L be the arithmetical means of $\lambda\Psi_l$, that is

$$K_L := \frac{1}{C_L^l} \sum_J \lambda\Psi_l.$$

For the estimate of the means K_L we observe that the number of l-tuples like $[1, 2, i_3, ..., i_l]$ is C_{L-2}^{l-2}, since two indices are fixed. Therefore, the addend $\delta(1, 2)$ is present C_{L-2}^{l-2} times in all possible sums of (2.8). In the same way we verify that the addend $\delta(k, \varkappa)$ with $\varkappa > k$ is also present C_{L-2}^{l-2} times in all possible sums of (2.8). Thus

$$K_L \leq \frac{1}{C_L^l} \sum_J \sum_{k,\varkappa=1,k\neq\varkappa}^{l} \delta(i_k, i_\varkappa) =$$

$$= \frac{1}{C_L^l} \sum_{k,\varkappa=1,\, k\neq\varkappa}^{L} C_{L-2}^{l-2} \delta(k, \varkappa) =$$

$$= \frac{l(l-1)}{L(L-1)} \sum_{\varkappa=1}^{L} \sum_{k=1}^{\varkappa-1} \delta(k, \varkappa).$$

In view of (2.7) and (2.3) we get

$$\sum_{k=1}^{\varkappa-1} \delta(k,\varkappa) = \lambda[\Phi_\varkappa \cap (E_1 \cup E_2 \cup \ldots \cup E_{\varkappa-1})] \leq \lambda\Phi_\varkappa \leq \eta,$$

and, consequently,

$$K_L \leq l(l-1)L^{-1}(L-1)^{-1}L\eta < \eta/l. \tag{2.9}$$

Thus the arithmetical means K_L of the numbers $\lambda\Phi_l$ satisfies the inequality (2.9). Therefore, at least one of these numbers $\lambda\Phi_l$ must satisfy the same inequality, meaning that there exists an l-tuple of positive integers satisfying (2.5) such that (2.6) is valid. □

A double function sequence (a_{mn}) is called convergent in measure on the set Q to the limit function a if for any $\varepsilon > 0$

$$\lim_{m,n} \lambda\{x \in Q:\ |a_{mn}(x) - a(x)| \geq \varepsilon\} = 0.$$

A double function series $\sum_{m,n} a_{mn}$ is called convergent in measure on Q to the sum a if the double sequence of its partial sums converges in measure on Q to a.

We denote the partial sums of the double series (1.1) by $S_{mn}c$, that is

$$S_{mn}c := \sum_{k,l=0}^{m,n} c_{kl} f_{kl}. \tag{2.10}$$

The one-dimensional case (with $Q = [0,1]$) of the following lemma is proved by Nikishin [10], p. 154, Lemma 6. We present below a simpler different proof.

Lemma 2.3. *Let the double series* (1.1) *converge in measure on the set* Q *to the sum*

$$Sc := \sum_{k,l} c_{kl} f_{kl} \tag{2.11}$$

for any $c \in \ell^p$. *Then for every* $\varepsilon > 0$ *there exist a real number* $R_\varepsilon > 0$ *such that the inequality*

$$\lambda\{x \in Q:\ |(Sc)(x)| \geq R_\varepsilon\} \leq \varepsilon \tag{2.12}$$

holds on the unit ball $\{c:\ \|c\| \leq 1\}$ *of* ℓ^p.

Proof. We consider the F-space $M := M(Q)$, which consists of all almost everywhere finite measurable functions f on the set Q with the quasi-norm

$$\|f\|_M := \inf_{\alpha>0} \{ \alpha + \lambda\{x \in Q : |f(x)| \geq \alpha\} \}.$$

A sequence (f_n) converges to f in M if and only if $f_n \to f$ in measure on Q (see [4], p. 104, Lemma 7), that is, for every $\varepsilon > 0$,

$$\lim_n \lambda\{x \in Q : |f_n(x) - f(x)| \geq \varepsilon\} = 0.$$

We consider continuous linear operators $S_{mn} : \ell^p \to M$, defined by (2.10). Since $\mathbf{f} \subset L^1_\lambda(Q)$, it follows that $S_{mn}c$ are finite almost everywhere on Q. By the hypothesis the double series (1.1) converges in measure on Q to the function Sc, defined by (2.11), that is, for every $\varepsilon > 0$,

$$\lim_{m,n} \lambda\{x \in Q : |(S_{mn}c)(x) - (Sc)(x)| \geq \varepsilon\} = 0,$$

for any $c \in \ell^p$. Therefore, the subsequence $(S_{kk}c)$ converges in measure to Sc for any $c \in \ell^p$. Hence, the function Sc is almost everywhere finite and measurable on Q. Since $Sc = \lim_k S_{kk}c$ for each $c \in \ell^p$, it follows that S is a continuous linear operator from ℓ^p into M (see, e.g., [4], p. 54, Theorem 17). Hence

$$\lim_{c\to 0} Sc = 0. \tag{2.13}$$

Denote the unit ball of ℓ^p by U, that is $U := \{c \in \ell^p : \|c\| \leq 1\}$. Let $c \in U$; if $\beta \to 0$, then $\beta c \to 0$ uniformly on U, and by (2.13)

$$\lim_{\beta\to 0} \|\beta\, Sc\|_M = 0,$$

that is,

$$\lim_{\beta\to 0} \inf_{\alpha>0} \{\alpha + \lambda\{x \in Q : |\beta(Sc)(x)| \geq \alpha\} \} = 0$$

uniformly on U. Therefore, for any $\varepsilon > 0$, a $\beta_\varepsilon > 0$ exists such that

$$\gamma_\varepsilon := \inf_{\alpha>0} \{\alpha + \lambda\{x \in Q : |\beta_\varepsilon(Sc)(x)| \geq \alpha\}\} < \varepsilon/2$$

for all $c \in U$. Hence, an $\alpha_\varepsilon > 0$ exists such that

$$\lambda\{x \in Q : |\beta_\varepsilon(Sc)(x)| \geq \alpha_\varepsilon\} - \varepsilon/2 < \gamma_\varepsilon - \alpha_\varepsilon < \gamma_\varepsilon < \varepsilon/2$$

or

$$\lambda\{x \in Q : |(Sc)(x)| \geq \alpha_\varepsilon/\beta_\varepsilon\} < \varepsilon$$

for all $c \in U$. Putting $R_\varepsilon = \alpha_\varepsilon/\beta_\varepsilon$ completes the proof. □

3. Main lemma

The following result has been inspired by ideas of proof in [10, pp. 141-145]. Here and in what follows

$$r := \min\{p,\, 2\}.$$

Lemma 3.1. *Let $l > 3$ be a positive integer and let $\varepsilon, R_\varepsilon, D_{pl}$, and C_l be positive real numbers. Let the double series* (1.1) *converge in measure on Q to the sum* (2.11) *for all $c \in \ell^p$. If for a measurable subset $Q_1 \subset Q$ the estimate*

$$\lambda\{x \in Q_1 : |(Sc)(x)|^r \geq D_{pl} R_\varepsilon\} \leq C_l\, \varepsilon \tag{3.1}$$

holds for all c in the unit ball of ℓ^p, then there exists a measurable subset $e \subset Q_1$ with the measure

$$\lambda e \leq l^3\, C_l\, \varepsilon \tag{3.2}$$

so that

$$\lambda\{x \in Q_1 \setminus e : |(Sc)(x)|^r \geq l A_p D_{pl} R_\varepsilon\} \leq 3\, C_l\, \varepsilon / l \tag{3.3}$$

for all c in the unit ball of ℓ^p, where $A_p > 1$ is a constant depending on p.

Proof. Let A_p be determined by the equalities (3.30) and (3.25) below. We find $c_1 \in \ell^p$ such that $\|c_1\| \leq 1$ and

$$\lambda\{x \in Q_1 : |(Sc_1)(x)|^r \geq l A_p D_{pl} R_\varepsilon\} > 3\, C_l\, \varepsilon / l. \tag{3.4}$$

If no such c_1 exists, then (3.3) is true for $e = \varnothing$ and Lemma 3.1 is proved. If (3.4) holds, then denote

$$E_1 = \{x \in Q_1 : |(Sc_1)(x)|^r \geq l A_p\, D_{pl}\, R_\varepsilon\},$$

$$\Phi_1 = \{x \in Q_1 : |(Sc_1)(x)|^r \geq D_{pl}\, R_\varepsilon\}.$$

From (3.4) and (3.1) it follows that

$$E_1 \subset \Phi_1, \quad \lambda E_1 > 3 C_l\, \varepsilon / l, \quad \lambda \Phi_1 \leq C_l\, \varepsilon.$$

We will seek $c_2 \in \ell^p$ such that $\|c_2\| \leq 1$ and such that

$$\lambda\{x \in Q_1 \setminus \Phi_1 : |(Sc_2)(x)|^r \geq l A_p\, D_{pl}\, R_\varepsilon\} > 3\, C_l\, \varepsilon / l. \tag{3.5}$$

If no such c_2 exists, then (3.3) is true for $e = \Phi_1$ and Lemma 3.1 is proved. But if inequality (3.5) is valid for some c_2, then we denote

$$E_2 = \{x \in Q_1 \setminus \Phi_1 : |(Sc_2)(x)|^r \geq l\, A_p\, D_{pl}\, R_\varepsilon\},$$

$$\Phi_2 = \{x \in Q_1 : |(Sc_2)(x)|^r \geq D_{pl} R_\varepsilon\}.$$

From (3.5) and (3.1) we obtain

$$E_2 \subset \Phi_2, \quad E_2 \cap \Phi_1 = \varnothing, \quad \lambda E_2 > 3\, C_l\, \varepsilon / l, \quad \lambda \Phi_2 \leq C_l\, \varepsilon.$$

Continuing this process until some step s we obtain points $c_i \in \ell^p$ with $\|c_i\| \leq 1$ and the sets

$$E_i = \{x \in Q_1 \setminus \bigcup_{j=1}^{i-1} \Phi_j : |(Sc_i)(x)|^r \geq l\, A_p\, D_{pl}\, R_\varepsilon\}, \tag{3.6}$$

$$\Phi_i = \{x \in Q_1 : |(Sc_i)(x)|^r \geq D_{pl}\, R_\varepsilon\}, \tag{3.7}$$

satisfying conditions (2.2), (2.4), and

$$\lambda \Phi_i \leq C_l\, \varepsilon, \tag{3.8}$$

$$\lambda E_i > 3\, C_l\, \varepsilon / l, \tag{3.9}$$

for each $i = 1, ..., s$ and $k = 2, ..., s$. If we show that in the case $s = l^3$ inequality (3.1) leads to a contradiction, then Lemma 3.1 holds, because if the above-described process terminated before $s = l^3$, then Lemma 3.1 would be proved.

For obtaining a contradiction we use Lemma 2.2, taking there $\eta = C_l \varepsilon$. This gives an l-tuple of positive integers satisfying (2.5) such that, with the notation from the proof of Lemma 2.2,

$$\lambda \Psi_l \leq C_l\, \varepsilon / l, \tag{3.10}$$

and also (2.7) being satisfied. We consider the set

$$P = (\bigcup_{\varkappa=1}^{l} E_{i_\varkappa}) \setminus \Psi_l.$$

Its measure, in view of (3.9) and (3.10), is

$$\lambda P = \lambda(\bigcup_{\varkappa=1}^{l} E_{i_\varkappa}) - \lambda \Psi_l \geq 3\, C_l\, \varepsilon - C_l\, \varepsilon / l > 2\, C_l\, \varepsilon.$$

If $x \in P$, then $x \in E_{i_m}$ for some m, but $x \notin \Phi_{i_n}$ for $n \neq m$ by (2.2). Therefore, on the set P, in view of (3.6) and (3.7),

$$|(Sc_{i_m})(x)|^r \geq l\, A_p\, D_{pl}\, R_\varepsilon \tag{3.11}$$

and for $n \neq m$

$$|(Sc_{i_n})(x)|^r < D_{pl}\, R_\varepsilon. \tag{3.12}$$

Now we consider the double sequence $I_\beta := (I_{\mu\nu}\beta) \in \ell^p$ with

$$I_{\mu\nu}\beta := (l\, B_p)^{-1/r} \sum_{m=1}^{l} r_m(\beta)\, c_{\mu\nu}^{i_m}, \quad 0 \leq \beta \leq 1,$$

where r_m are the Rademacher functions (see, e.g., [8], p. 19) and $c_{i_m} = (c_{\mu\nu}^{i_m})$; the numbers B_p will be specified later (see (3.25)). We shall show that there exist $\beta_0 \in [0,1]$ and $P_1 \subset P$ such that

$$\sum_{\mu,\nu} |I_{\mu\nu}\beta_0|^p \leq 1, \tag{3.13}$$

$\lambda P_1 > C_l\, \varepsilon$, and

$$\lambda\{x \in P_1 : |(SI_{\beta_0})(x)|^r \geq D_{pl}\, R_\varepsilon\} > C_l\, \varepsilon, \tag{3.14}$$

which is in contradiction with (3.1) (since $P_1 \subset Q_1$).

By (2.11)

$$(SI_\beta)(x) = (l\, B_p)^{-1/r} \sum_{k=1}^{l} r_k(\beta) \sum_{\mu,\nu} c_{\mu\nu}^{i_k}\, f_{\mu\nu}(x). \tag{3.15}$$

Therefore, denoting

$$Z_\beta(x) := \begin{cases} r_m(\beta) \sum_{\mu,\nu} c_{\mu\nu}^{i_m}\, f_{\mu\nu}(x), & \text{if} x \in P \cap E_{i_m}, \\ 0, & \text{if} x \notin P, \end{cases}$$

and for $k \neq m$

$$\alpha_k(x) := \begin{cases} \sum_{\mu,\nu} c_{\mu\nu}^{i_k}\, f_{\mu\nu}(x), & \text{if} x \in E_{i_m}, \\ 0, & \text{if} x \in E_{i_k}, \end{cases}$$

we obtain from the inequalities (3.11) and (3.12) that

$$|Z_\beta(x)|^r \geq l\, A_p\, D_{pl}\, R_\varepsilon, \tag{3.16}$$

$$|\alpha_k(x)|^r < D_{pl}\, R_\varepsilon. \tag{3.17}$$

But the equality (3.15) yields

$$|(SI_\beta)(x)|^r \geq |(l\, B_p)^{-1/r} |Z_\beta(x)| - (l\, B_p)^{-1/r} |Y_\beta(x)|\,|^r, \tag{3.18}$$

where

$$Y_\beta(x) := \sum_{\varkappa=1}^{l} r_\varkappa(\beta)\, \alpha_\varkappa(x).$$

To estimate $Y_\beta(x)$ we consider the function Υ, putting

$$\Upsilon(\beta) := (l\, B_p)^{-1} \int_P |Y_\beta(x)|^r dx.$$

Since

$$\int_0^1 \Upsilon(\beta) d\beta = (l\, B_p)^{-1} \int_P dx \int_0^1 |Y_\beta(x)|^r d\beta,$$

by Hölder's inequality, in view of $1-r/2>0$ or $r=2$ and the orthonormality of the Rademacher system, we obtain

$$\int_0^1 \Upsilon(\beta) d\beta \le (l\, B_p)^{-1} \int_P dx \quad [\int_0^1 |Y_\beta(x)|^2 d\beta]^{r/2} =$$
$$= (l\, B_p)^{-1} \int_P [\sum_{\varkappa=1}^{l} \alpha_\varkappa^2(x)]^{r/2} dx.$$

Therefore, by (3.17),

$$\int_0^1 \Upsilon(\beta) d\beta \le (l\, B_p)^{-1} \int_P [\sum_{\varkappa=1}^{l} D_{pl}^{2/r}\, R_\varepsilon^{2/r}]^{r/2} dx =$$
$$= (l\, B_p)^{-1} D_{pl}\, R_\varepsilon\, l^{r/2}\, \lambda P,$$

hence

$$\int_0^1 \Upsilon(\beta) d\beta \le (B_p)^{-1} D_{pl}\, R_\varepsilon\, \lambda P. \tag{3.19}$$

Denoting

$$\Omega := \{\beta : \beta \in [0,1], \quad \Upsilon(\beta) \le 2\, B_p^{-1}\, D_{pl}\, R_\varepsilon\, \lambda P\},$$

we obtain $\lambda\Omega > 1/2$, since otherwise the inequality (3.19) is contradicted.

Now we prove that (3.13) holds. We start with the equality

$$\int_\Omega \sum_{\mu,\nu} |I_{\mu\nu}\beta|^p d\beta = (l\, B_p)^{-p/r} \sum_{\mu,\nu} \int_\Omega |\sum_{m=1}^{l} c_{\mu\nu}^{im}\, r_m(\beta)|^p d\beta. \tag{3.20}$$

First let $p > 2$. Then $r = 2$ and using Khinchin's inequality (see, e.g., [8], p. 30, or [6], p. 131) we obtain

$$\int_\Omega \sum_{\mu,\nu} |I_{\mu\nu}\beta|^p d\beta \le K_p\,(l\,B_p)^{-p/2} \sum_{\mu,\nu} (\sum_{m=1}^{l} |c_{\mu\nu}^{i_m}|^2)^{p/2}, \tag{3.21}$$

where K_p is a Khinchin constant. But by Hölder's inequality

$$\sum_{m=1}^{l} |c_{\mu\nu}^{i_m}|^2 \le l^{1-2/p} (\sum_{m=1}^{l} |c_{\mu\nu}^{i_m}|^p)^{2/p},$$

and hence by (3.21)

$$\int_\Omega \sum_{\mu,\nu} |I_{\mu\nu}\beta|^p d\beta \le (l\,B_p)^{-p/2}\,K_p \sum_{\mu,\nu} [\,l^{1-2/p} (\sum_{m=1}^{l} |c_{\mu\nu}^{i_m}|^p)^{2/p}\,]^{p/2} =$$

$$= K_p\,B_p^{-p/2}\,l^{-1} \sum_{m=1}^{l} \|c_{i_m}\|^p \le K_p\,B_p^{-p/2}, \tag{3.22}$$

because $\|c_{i_m}\| \le 1$. Suppose now that $1 < p \le 2$. Then $r = p$ and by (3.20), using Hölder's inequality for integrals if $p < 2$, we conclude that

$$\int_\Omega \sum_{\mu,\nu} |I_{\mu\nu}\beta|^p\,d\beta \le (l\,B_p)^{-1} \sum_{\mu,\nu} [\int_0^1 \sum_{m=1}^{l} |c_{\mu\nu}^{i_m}\,r_m(\beta)|^2\,d\beta]^{p/2}. \tag{3.23}$$

From (3.23), using the monotonicity of the ℓ^p-norms (see [5], p. 28, Theorem 19), we obtain

$$\int_\Omega \sum_{\mu,\nu} |I_{\mu\nu}\beta|^p d\beta \le (l\,B_p)^{-1} \sum_{\mu,\nu} (\sum_{m=1}^{l} |c_{\mu\nu}^{i_m}|^2)^{p/2} \le$$

$$\le (l\,B_p)^{-1} \sum_{m=1}^{l} \sum_{\mu,\nu} |c_{\mu\nu}^{i_m}|^p \le (B_p)^{-1}, \tag{3.24}$$

because $\|c_{i_m}\| \le 1$. We define the numbers B_p as follows (recall that K_p is a Khinchin constant):

$$B_p = \begin{cases} 2 & \text{for } 1 < p \le 2, \\ (2\,K_p)^{2/p} & \text{for } \quad p > 2. \end{cases} \tag{3.25}$$

Then (3.22) and (3.24) yield that for any finite $p > 1$

$$\int_\Omega \sum_{\mu,\nu} |I_{\mu\nu}\beta|^p d\beta \leq 1/2. \tag{3.26}$$

Since $\lambda\Omega > 1/2$, it follows that there exists a number $\beta_0 \in \Omega$ such that inequality (3.13) holds, because otherwise we have a contradiction to (3.26).

Finally we show that (3.14) also holds. Let $\beta_0 \in \Omega$ be such that (3.13) holds. By the definition of Ω

$$(l\,B_p)^{-1} \int_P |Y_{\beta_0}(x)|^r dx = \Upsilon(\beta_0) \leq 2\,B_p^{-1}\,D_{pl}\,R_\varepsilon\,\lambda P. \tag{3.27}$$

We define the set P_1 as follows:

$$P_1 := \{x \in P : \; (l\,B_p)^{-1}|Y_{\beta_0}(x)|^r \leq 4\,B_p^{-1}\,D_{pl}\,R_\varepsilon\}.$$

Let us first prove that

$$\lambda P_1 \geq \lambda P/2. \tag{3.28}$$

In fact, if $\lambda P_1 < \lambda P/2$, then

$$\lambda\{x \in P : \; (l\,B_p)^{-1}|Y_{\beta_0}(x)|^r > 4\,B_p^{-1}\,D_{pl}\,R_\varepsilon\} \geq \lambda P/2$$

and therefore

$$\begin{aligned}(l\,B_p)^{-1} \int_P |Y_{\beta_0}(x)|^r dx > \\ > 4\,B_p^{-1}\,D_{pl}\,R_\varepsilon\,\lambda\{x \in P : \; (l\,B_p)^{-1}|Y_{\beta_0}(x)|^r > 4\,B_p^{-1}\,D_{pl}\,R_\varepsilon\} \geq \\ \geq 4\,B_p^{-1}\,D_{pl}\,R_\varepsilon\,\lambda P/2 = 2\,B_p^{-1}\,D_{pl}\,R_\varepsilon\,\lambda P,\end{aligned}$$

which contradicts (3.27). Since, as we calculated above, $\lambda P > 2\,C_l\,\varepsilon$, we obtain by (3.28) that

$$\lambda P_1 > C_l\,\varepsilon. \tag{3.29}$$

If $x \in P_1$, then, because of (3.18) and (3.16), we have

$$\begin{aligned}|(SI_{\beta_0})(x)|^r &\geq |(l\,B_p)^{-1/r}(l\,A_p\,D_{pl}\,R_\varepsilon)^{1/r} - (4\,B_p^{-1}\,D_{pl}\,R_\varepsilon)^{1/r}|^r = \\ &= D_{pl}\,R_\varepsilon\,|(A_p/B_p)^{1/r} - (4/B_p)^{1/r}|^r.\end{aligned}$$

We choose the numbers A_p as follows:

$$A_p := 4\,(B_p^{1/r} + 1)^r > 1. \tag{3.30}$$

Then

$$\begin{aligned}(A_p/B_p)^{1/r} - (4/B_p)^{1/r} &= (B_p^{1/r} + 1)(4/B_p)^{1/r} - (4/B_p)^{1/r} = \\ &= B_p^{1/r}(4/B_p)^{1/r} = 4^{1/r}.\end{aligned}$$

Hence, if $x \in P_1$, then $|(SI_{\beta_0})(x)|^r \geq 4\,D_{pl}\,R_\varepsilon > D_{pl}\,R_\varepsilon$, and by (3.29) the inequality (3.14) holds. □

4. A relation between the partial sums of a double function series and its coefficients

We recall that $p > 1$ and $r = \min\{p, 2\}$.

Theorem 4.1. *Let the double series* (1.1) *converge in measure on* Q *for each* $c \in \ell^p$. *Then for every* $\varrho \in [1, r)$ *and every* $\eta > 0$ *there exist a measurable subset* $E_{\eta,\varrho} \subset Q$ *with* $\lambda E_{\eta,\varrho} > \lambda Q - \eta$ *and a constant* $K_{\varrho\eta} > 0$ *such that*

$$[\int_{E_{\eta,\varrho}} | \sum_{\mu,\nu=0}^{\infty} c_{\mu\nu} f_{\mu\nu}(x)|^{\varrho} \, dx]^{1/\varrho} \leq K_{\varrho\eta} [\sum_{\mu,\nu=0}^{\infty} |c_{\mu\nu}|^p]^{1/p} \tag{4.1}$$

for each $c \in \ell^p$.

Proof. The proof will develop Nikishin's arguments from [10, pp. 139-141]. From Lemma 2.3 it follows that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\lambda\{x \in Q : \quad |(Sc)(x)|^r \geq R_\varepsilon\} \leq \varepsilon$$

for all c from the unit ball U of ℓ^p. Let us fix an integer $l > 3$. We apply Lemma 3.1 taking $Q_1 = Q$ and $D_{pl} = C_l = 1$. By Lemma 3.1 one can find a measurable set $e_1 \subset Q$ with $\lambda e_1 \leq l^3 \varepsilon$ such that

$$\lambda\{x \in Q \setminus e_1 : \quad |(Sc)(x)|^r \geq l A_p R_\varepsilon\} \leq 3\varepsilon/l$$

for all $c \in U$. Now we use Lemma 3.1 again taking $Q_1 = Q \setminus e_1$, $D_{pl} = lA_p$ and $C_l = 3/l$. By Lemma 3.1 there exists a measurable set $e_2 \subset Q_1$ with $\lambda e_2 \leq 3l^3\varepsilon/l$ such that

$$\lambda\{x \in Q_1 \setminus e_2 : \quad |(Sc)(x)|^r \geq (lA_p)^2 R_\varepsilon\} \leq (3/l)^2 \varepsilon$$

for all $c \in U$. Continuing the above process unlimitedly, we obtain a sequence of disjoint measurable sets (e_k) with $\lambda e_k \leq l^3 (3/l)^{k-1} \varepsilon$ such that

$$\lambda\{x \in Q \setminus \bigcup_{\varkappa=1}^{k} e_\varkappa : \quad |(Sc)(x)|^r \geq (lA_p)^k R_\varepsilon\} \leq (3/l)^k \varepsilon \tag{4.2}$$

for all $c \in U$ and for all $k = 1, 2, \ldots$.

Denote

$$H := \bigcup_{k=1}^{\infty} e_k.$$

Since $l \geq 4$, the measure of H satisfies the condition

$$\lambda H \leq l^3 \varepsilon \sum_{k=1}^{\infty} (3/l)^{k-1} \leq 4 l^3 \varepsilon.$$

We now choose a number $\xi > R_\varepsilon$. There is a number $j \in \{0, 1, \dots\}$ such that

$$R_\varepsilon (l A_p)^j \leq \xi < R_\varepsilon (l A_p)^{j+1}.$$

Using inequality (4.2) we obtain

$$\lambda\{x \in Q \setminus H : \quad |(Sc)(x)|^r \geq \xi\} \leq (3/l)^{j+1} (l/3) \varepsilon \tag{4.3}$$

for all $c \in U$. Since $\xi / R_\varepsilon < (l A_p)^{j+1}$ and $l A_p > 1$, it follows that

$$\ln(\xi / R_\varepsilon) < (j+1) \ln(l A_p).$$

Hence

$$(3/l)^{j+1} \leq (3/l)^{\ln(\xi/R_\varepsilon)/\ln(lA_p)} = e^{h(l) \ln(R_\varepsilon/\xi)} = R_\varepsilon^{h(l)} \xi^{-h(l)}, \tag{4.4}$$

where

$$h(l) := \ln(l/3) / \ln(l A_p).$$

Let $\varrho < r$. Then $r/\varrho > 1$ and we can find a $\delta = \delta(\varrho)$ such that

$$(1 - \delta) r/\varrho =: 1 + \alpha, \quad \alpha > 0.$$

Since $h(l) \to 1$ as $l \to \infty$, there exists $l = l(\delta)$ such that $h(l) > 1 - \delta$.

We now fix a number $\eta > 0$ and find $\varepsilon = \varepsilon(l, \eta) > 0$ such that

$$4 l^3 \varepsilon < \eta, \quad (l/3)\varepsilon \leq 1.$$

Then $\lambda H < \eta$ and, denoting

$$E_{\eta,\varrho} := Q \setminus H,$$

we have $\lambda E_{\eta,\varrho} > \lambda Q - \eta$ and, by (4.3) and (4.4),

$$\lambda\{x \in E_{\eta,\varrho} : |(Sc)(x)|^r \geq \xi\} \leq R_\varepsilon^{h(l)} \xi^{-h(l)}. \tag{4.5}$$

Since

$$\lambda\{x \in E_{\eta,\varrho} : |(Sc)(x)|^\varrho \geq \xi\} = \lambda\{x \in E_{\eta,\varrho} : |(Sc)(x)|^r \geq \xi^{r/\varrho}\},$$

we obtain from (4.5) that

$$\lambda\{x \in E_{\eta,\varrho} : |(Sc)(x)|^{\varrho} \geq \xi\} \leq R_{\varepsilon}^{h(l)}\, \xi^{-rh(l)/\varrho} \leq R_{\varepsilon}^{h(l)}\, \xi^{-(1+\alpha)}.$$

whenever $\xi \geq 1$. As a result, denoting $h_{\varrho} := h(l)$, we have the inequality

$$\lambda\{x \in E_{\eta,\,\varrho} : |(Sc)(x)|^{\varrho} \geq \xi\} \leq R_{\varepsilon}^{h_{\varrho}}\, \xi^{-(1+\alpha)}, \tag{4.6}$$

for all $c \in U$ and $\xi > \max\{1, R_{\varepsilon}\}$.

Denoting, for $k \in \{0, 1, \dots\}$,

$$E_{\eta,\,\varrho}(k) := \{x \in E_{\eta,\,\varrho} : |(Sc)(x)|^{\varrho} \geq k\}$$

and

$$G_{\eta,\,\varrho}(k) := \{x \in E_{\eta,\,\varrho} : k \leq |(Sc)(x)|^{\varrho} < k+1\},$$

we have

$$G_{\eta,\,\varrho}(k) = E_{\eta,\,\varrho}(k) \setminus E_{\eta,\,\varrho}(k+1)$$

and

$$E_{\eta,\,\varrho} = \bigcup_{k=0}^{\infty} G_{\eta,\,\varrho}(k).$$

Let us fix a natural number $\varkappa > \max\{1,\, R_{\varepsilon}\}$. Denoting

$$v_k := \lambda E_{\eta,\,\varrho}(k),$$

by partial summation (see, e.g., [6], p. 1) and (4.6), we have

$$\begin{aligned}\sum_{k=\varkappa}^{n} (k+1)\,(v_k - v_{k+1}) &= \varkappa\, v_{\varkappa} - (n+1)\, v_{n+1} + \sum_{k=\varkappa}^{n} v_k \leq \\ &\leq R_{\varepsilon}^{h_{\varrho}}\, [\varkappa^{-\alpha} + (n+1)^{-\alpha} + \sum_{k=\varkappa}^{n} k^{-1-\alpha}] \leq \\ &\leq R_{\varepsilon}^{h_{\varrho}}\, [\varkappa^{-\alpha} + (n+1)^{-\alpha} + \alpha^{-1}(\varkappa-1)^{-\alpha}],\end{aligned}$$

from which

$$\sum_{k=\varkappa}^{\infty} (k+1)\,(v_k - v_{k+1}) \leq (1+\alpha^{-1})(\varkappa-1)^{-\alpha}\, R_{\varepsilon}^{h_{\varrho}}. \tag{4.7}$$

Now, by (4.7), we obtain

$$\begin{aligned}\int_{E_{\eta,\varrho}} |(Sc)(x)|^{\varrho} dx &\le \sum_k \int_{G_{\eta,\varrho}(k)} |(Sc)(x)|^{\varrho} dx \le \\ &\le \sum_k (k+1)\lambda G_{\eta,\varrho}(k) = \\ &= \sum_k (k+1)(v_k - v_{k+1}) = \\ &\le \sum_{k<\varkappa} + \sum_{k\ge\varkappa} < \\ &< \varkappa(\varkappa+1)\,\lambda Q + \\ &+ (1+\alpha^{-1})(\varkappa-1)^{-\alpha} R_{\varepsilon}^{h_{\varrho}}\end{aligned}$$

for all $c \in U$. Therefore, a constant $K_{\varrho\eta} > 0$ exists such that

$$\int_{E_{\eta,\varrho}} |(Sc)(x)|^{\varrho} dx \le K_{\varrho\eta}^{\varrho} \tag{4.8}$$

for all $c \in U$. In view of (4.8) we proved that for the linear operator $S : l^p \to L^{\varrho} := L^{\varrho}_{\lambda}(E_{\eta,\varrho})$ the inequality $\|Sc\|_{L^{\varrho}} \le K_{\varrho\eta}$ holds for all $c \in U$. This immediately yields (4.1). □

From Theorem 4.1, assuming $\varrho = 1$, we obtain

Corollary 4.2 (cf. [10], p. 158, Theorem 7). *Let the double series* (1.1) *converge in measure on Q for each $c \in \ell^p$. Then for every $\delta > 0$ there exists a measurable subset $Q_\delta \subset Q$ with $\lambda Q_\delta > \lambda Q - \delta$ and a constant $K_\delta > 0$ such that*

$$\int_{Q_\delta} \Big|\sum_{k,l=0}^{m,n} c_{kl}\, f_{kl}(x)\Big| dx \le K_\delta \Big(\sum_{k,l=0}^{m,n} |c_{kl}|^p\Big)^{1/p}$$

for all numbers c_{kl}.

5. Weyl multipliers for absolute summability of double function series

Let $p > 1$ and let

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Recall that a double sequence (a_{mn}) is said to be non-decreasing (or non-increasing) in m, n, whenever it is non-decreasing (respectively, non-increasing) if either index is fixed.

Theorem 5.1. *Let the double function series* (1.1) *converge in measure on* Q *for all* $c \in \ell^p$. *Let the method* T *conserve the absolute convergence. In addition, let the double sequence* w *be unbounded and satisfy the condition*

$$\sum_{m,n} |\tau_{mnmn}|\, w_{mn}^{-q/p} < \infty \tag{5.1}$$

and let the double sequence

$$(|\bar{\tau}_{mnkl}|^{p-1}\, |\tau_{mnmn}|^{-p/q}\, w_{mn}) \tag{5.2}$$

be non-increasing in m, n *for all* $k, l = 0, 1, \dots$.

Then the double series (1.1) *is* $|T|$*-summable almost everywhere on* Q *whenever* c *satisfies condition* (1.2).

Proof. By the definition of $|T|$-summability we must prove that

$$\sum_{m,n} |\sum_{k,l=0}^{m,n} \bar{\tau}_{mnkl}\, c_{kl}\, f_{kl}(x)| < \infty \tag{5.3}$$

almost everywhere on Q. The function defined by the double series (5.3) is measurable on Q (see [7], p. 101, Theorem 1). By Lemma 2.1 we must show that for every $\delta > 0$ a measurable subset $Q_\delta \subset Q$ exists with $\lambda Q_\delta > \lambda Q - \delta$ such that the integral of (5.3) over the set Q_δ is finite. By the theorem of B. Levi for double series (cf. [1], p. 11, [2], p. 174) it is sufficient to prove that there exists a constant $L_\delta > 0$ such that

$$B_{MN} \le L_\delta, \tag{5.4}$$

where

$$B_{MN} = \sum_{m,n=0}^{M,N} \int_{Q_\delta} |\sum_{k,l=0}^{m,n} \bar{\tau}_{mnkl}\, c_{kl}\, f_{kl}(x)| dx.$$

By Corollary 4.2 there exist a subset $Q_\delta \subset Q$ as needed and a constant $K_\delta > 0$ such that

$$B_{MN} \le K_\delta\, C_{MN}, \tag{5.5}$$

where

$$C_{MN} = \sum_{m,n=0}^{M,N} (\sum_{k,l=0}^{m,n} |\bar{\tau}_{mnkl}\, c_{kl}|^p)^{1/p}.$$

Since

$$C_{MN} = \sum_{m,n=0}^{M,N} |\tau_{mnmn}|^{1/q} w_{mn}^{-1/p} |\tau_{mnmn}|^{-1/q} w_{mn}^{1/p} \Big(\sum_{k,l=0}^{m,n} |\bar{\tau}_{mnkl} c_{kl}|^p\Big)^{1/p},$$

applying Hölder's inequality and condition (5.1), we obtain

$$\begin{aligned} C_{MN} &\leq \Big(\sum_{m,n=0}^{M,N} |\tau_{mnmn}| w_{mn}^{-q/p}\Big)^{1/q} \Big(\sum_{m,n=0}^{M,N} |\tau_{mnmn}|^{-p/q} w_{mn} \sum_{k,l=0}^{m,n} |\bar{\tau}_{mnkl} c_{kl}|^p\Big)^{1/p} \\ &= O(1)\Big(\sum_{k,l=0}^{M,N} |c_{kl}|^p \sum_{m,n=k,l}^{M,N} |\bar{\tau}_{mnkl}|^p |\tau_{mnmn}|^{-p/q}\, w_{mn}\Big)^{1/p} \\ &= O(1)\Big(\sum_{k,l} |c_{kl}|^p \sum_{m,n=k,l}^{\infty} |\bar{\tau}_{mnkl}| |\bar{\tau}_{mnkl}|^{p-1} |\tau_{mnmn}|^{-p/q}\, w_{mn}\Big)^{1/p}. \end{aligned}$$

Now, from conditions (5.2), (1.3), and (1.2) it follows that

$$\begin{aligned} C_{MN}^p &= O(1) \sum_{k,l} |c_{kl}|^p |\tau_{klkl}|^{p-1} |\tau_{klkl}|^{-p/q} w_{kl} \sum_{m,n=k,l}^{\infty} |\bar{\tau}_{mnkl}| \\ &= O(1) \sum_{k,l} |c_{kl}|^p\, w_{kl} = O(1). \end{aligned}$$

Therefore, in view of (5.5), we obtain (5.4). □

Theorem 5.1 is an extension of Theorem 3 in [11] to double function series.

We next apply Theorem 5.1 to the Cesàro method $C^{\alpha,\beta}$ and to the weighted means method (R, d) of Riesz.

For the method $C^{\alpha,\beta}$ we have (see [3], p. 84)

$$\bar{\tau}_{mnkl} = \frac{kl A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1}}{m\, n\, A_m^{\alpha}\, A_n^{\beta}},$$

where $0/0 = 1$. The method $C^{\alpha,\beta}$ satisfies condition (1.3) whenever $\alpha, \beta \geq 0$ or $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$ (see [3], p. 85). Since $A_m^{\alpha} \sim (m+1)^{\alpha}/\Gamma(\alpha+1)$, we deduce from Theorem 5.1

Corollary 5.2. *Let the double function series* (1.1) *converge in measure on* Q *for all* $c \in \ell^p$. *Let* $\alpha, \beta \geq 0$ *or* $\operatorname{Re}\alpha$, $\operatorname{Re}\beta > 0$. *In addition, let the double sequence* w *be unbounded and satisfy the condition*

$$\sum_{m,n} |(m+1)^{-\alpha}(n+1)^{-\beta}|\, w_{mn}^{-q/p} < \infty$$

and the double sequence $(|(m+1)^{(\alpha-2)(p-1)}\,(n+1)^{(\beta-2)(p-1)}|\, w_{mn})$ *be non-increasing in* m, n. *Then the double series* (1.1) *is* $|C^{\alpha,\beta}|$*-summable almost everywhere on* Q *whenever* c *satisfies condition* (1.2).

Let (R, d) be the factored weighted means method of Riesz, defined by a factored double sequence $d = (d_{mn})$ of complex numbers with partial sums

$$D_{mn} := \sum_{k,l=0}^{m,n} d_{kl}.$$

In view of (1.3), the method (R, d) conserves the absolute convergence if and only if the condition

$$D_{k-1,l-1} \sum_{m,n=k,l}^{\infty} \left|\frac{d_{mn}}{D_{m-1,n-1}\, D_{mn}}\right| = O(1) \tag{5.6}$$

is satisfied (see [3], p. 114). Thus from Theorem 5.1 we deduce

Corollary 5.3. *Let the double function series* (1.1) *converge in measure on* Q *for all* $c \in \ell^p$ *and let* (R, d) *be a factored method satisfying condition* (5.6). *In addition, let the double sequence* w *be unbounded and satisfy the condition*

$$\sum_{m,n} |d_{mn}/D_{mn}|\, w_{mn}^{-q/p} < \infty$$

and let the double sequence $(|D_{m-1,n-1}|^{p-1}/w_{mn})$ *be non-decreasing in* m, n. *Then the double series* (1.1) *is* $|R, d|$*-summable almost everywhere on* Q *whenever* c *satisfies condition* (1.2).

For a nonfactored method (R, d) it is necessary to put

$$\bar{\tau}_{mnkl} = \bar{\Delta}_{mn} \frac{D_{k-1,l-1} - D_{m,l-1} - D_{k-1,n}}{D_{mn}}$$

in Theorem 5.1.

For double orthogonal series, if $p = q = 2$ and $d = 2$, Corollary 5.3 is known (see Theorem 11 of [2]).

From Theorem 5.1 and Corollaries 5.2 and 5.3 we conclude the following statements on summability factors for double function series.

Let $\varepsilon = (\varepsilon_{mn})$ be a double sequence of complex numbers such that the double sequence $(|\varepsilon_{mn}|)$ non-increases and tends to 0.

Theorem 5.4. *Let the double function series* (1.1) *converge in measure on Q for all $c \in \ell^p$. Let the method T conserve the absolute convergence. In addition, let the double sequence ε satisfy the condition*

$$\sum_{m,n} |\tau_{mnmn}|\, |\varepsilon_{mn}|^q < \infty$$

and let the double sequence

$$(|\bar{\tau}_{mnkl}|^{1-p} |\tau_{mnmn}|^{p/q}\, |\varepsilon_{mn}|^p)$$

be non-decreasing in m, n for all $k, l = 0, 1, \dots$. Then the double series

$$\sum_{m,n} \varepsilon_{mn}\, c_{mn}\, f_{mn}(x) \tag{5.7}$$

is $|T|$-summable almost everywhere on Q whenever $c \in \ell^p$.

Proof. Putting $w_{mn} = |\varepsilon_{mn}|^{-p}$ we see that all the conditions of Theorem 5.1 are satisfied, in particular, $c \in \ell^p$ yields that (1.2) is satisfied for εc, because

$$\sum_{m,n} |\varepsilon_{mn}\, c_{mn}|^p w_{mn} = \sum_{m,n} |c_{mn}|^p < \infty.$$

□

Corollary 5.5. *Let the double function series* (1.1) *converge in measure on Q for all $c \in \ell^p$. Let $\alpha, \beta \geq 0$ or $\operatorname{Re} \alpha$, $\operatorname{Re} \beta > 0$. In addition, let the double sequence ε satisfy the condition*

$$\sum_{m,n} |(m+1)^{-\alpha}(n+1)^{-\beta}|\, |\varepsilon_{mn}|^q < \infty$$

and let the double sequence $(|(m+1)^{(\alpha-2)(p-1)}\, (n+1)^{(\beta-2)(p-1)}|\, |\varepsilon_{mn}|^{-p})$ be non-increasing in m, n. Then the double series (5.7) *is $|C^{\alpha,\beta}|$-summable almost everywhere on Q whenever $c \in \ell^p$.*

Corollary 5.6. *Let the double function series* (1.1) *converge in measure on Q for all $c \in \ell^p$ and let (R, d) be a factored method satisfying condition* (5.6). *In addition, let the double sequence ε satisfy the condition*

$$\sum_{m,n} |d_{mn}/D_{mn}|\, |\varepsilon_{mn}|^q < \infty$$

and let the double sequence $(|D_{m-1,n-1}|^{p-1} |\varepsilon_{mn}|^p)$ be non-decreasing in m, n. Then the double series (5.7) *is $|R, d|$-summable almost everywhere on Q whenever $c \in \ell^p$.*

For double orthogonal series, if $p = q = 2$ and $d = 2$, Corollary 5.6 is known (see Theorem 12 of [2]).

References

1. G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Press, New York–Oxford–Paris, 1961.
2. S. Baron, *On Weyl-type tests for the absolute summability of orthogonal series*, Tartu Riikl. Ül. Toimetised **150** (1964), 165–181. (Russian)
3. S. Baron, *Introduction to the Theory of Summability of Series*, Valgus, Tallinn, 1977. (Russian)
4. N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, A Wiley-Interscience Publ., New York, 1988.
5. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Fourth Edition, Cambridge Univ. Press, 1964.
6. S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Second Edition, Chelsea Publ. Company, New York, 1951.
7. E. Kamke, *Das Lebesgue-Stieltjes-Integral*, Teubner Verlag, Leipzig, 1956.
8. B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Amer. Math. Soc., Providence, Rhode Island, 1989.
9. I. G. Kull, *Multiplication of summable double series*, Tartu Riikl. Ül. Toimetised **62** (1958), 3–59. (Russian)
10. E. M. Nikishin, *Resonance theorems and overlinear operators*, Uspekhi Mat. Nauk **25** (1970), no. 6, 129–191. (Russian)
11. H. Türnpu, *On absolute P-summability of functional series almost everywhere*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **17** (1974), 97–101. (Russian)

S. Baron, Department of Mathematics and Statistics, Bar-Ilan University, Ramat Gan, 52900, Israel

E-mail address: baron@macs.biu.ac.il

H. Türnpu, Galile str. 17, Kiryat Bialik, 27019, Israel

E-mail address: heino@zahav.net.il