ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 7, 2003

On the A-statistical core of sequences of complex numbers

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Abstract. In [4], the concept of A-statistical core of a sequence of complex numbers was introduced and necessary and sufficient conditions were given for a summability matrix T to satisfy $K\text{-}core(Tx) \subseteq st_A\text{-}core(x)$ whenever x is a bounded complex sequence. In the present paper, we determine the classes of summability matrices T such that $st_A\text{-}core(Tx) \subseteq K\text{-}core(x)$ and $st_A\text{-}core(Tx) \subseteq st_B\text{-}core(x)$ for all bounded complex sequences x.

1. Introduction

Let K be a subset of $\mathbb{N}$, the set of positive integers. Natural density δ of K is defined by

$$\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|,$$

where vertical bars indicate the number of elements in the enclosed set. A sequence $x = (x_k)$ of complex numbers is said to be statistically convergent to the number l if for every ε, $\delta(\{k : |x_k - l| \geq \varepsilon\}) = 0$ (see [5]). In this case, we write st-lim $x = l$. We shall also write S to denote the set of all statistically convergent sequences.

For a given nonnegative regular matrix A, the number

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk}$$

is said to be the A-density of $K \subseteq \mathbb{N}$ (see [7]). A sequence $x = (x_k)$ is said to be A-statistically convergent to a number s if for every $\varepsilon > 0$ the set $\{k : |x_k - s| \geq \varepsilon\}$ has A-density zero (see [2]). In this case, we write

Received November 14, 2002.
2000 *Mathematics Subject Classification.* 40C05, 40J05, 46A45.
Key words and phrases. A-statistical convergence, A-statistical core, matrix transformations, core theorems.

$st_A\text{-}\lim x = s$. By $st(A)$ and $st(A)_0$ we respectively denote the set of all A-statistically convergent and A-statistically convergent to zero sequences.

Let $x = (x_k)$ be a sequence in $\mathbb{C}$, the set of all complex numbers, and let R_k be the closed convex hull of the set $\{x_k, x_{k+1}, x_{k+2}, \dots\}$. The Knopp core (or K-*core*) of x is defined by the intersection of all R_k $(k \in \mathbb{N})$ ([1], p.137). In [12], it is shown that for any bounded sequence x one has

$$K\text{-}core(x) = \bigcap_{z\in\mathbb{C}} B_x(z),$$

where $B_x(z) = \{w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z|\}$.

In [4], the notion of the statistical core of a complex sequence introduced by Fridy and Orhan [6] has been extended to the A-statistical core (or st_A-*core*) and it has been shown that, for an A-statistically bounded sequence x,

$$st_A\text{-}core(x) = \bigcap_{z\in\mathbb{C}} C_x(z),$$

where $C_x(z) = \{w \in \mathbb{C} : |w - z| \leq st_A\text{-}\limsup |x_k - z|\}$.

Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequences of complex numbers with the usual supremum norm. Let $T = (t_{nk})$ $(n, k = 1, 2, \dots)$ be an infinite matrix of complex numbers and let $x = (x_k)$ be a complex sequence. We write $Tx = (T_n x)$ if the series $T_n x = \sum_k t_{nk} x_k$ converges for each n. Let X and Y be any two sequence spaces. If $x \in X$ implies $Tx \in Y$, then we say that the matrix T maps X into Y. By (X, Y) we denote the class of matrices T which map X into Y. If X and Y are equipped with the limits X-lim and Y-lim, respectively, $T \in (X, Y)$ and $Y\text{-}\lim Tx = X\text{-}\lim x$ for all $x \in X$, then we write $T \in (X, Y)_{reg}$. The matrix $T \in (c,\ c)_{reg}$ is said to be regular and the conditions of the regularity of T are well known (see [1]).

In [9], Kolk has proved that $T \in (c, st(A) \cap \ell_\infty)_{reg}$ if and only if

$$\|T\| = \sup_n \sum_k |t_{nk}| < \infty \tag{1.1}$$

and there exists an index set $N = (n_i)$ with $\delta_A(N) = 1$ such that

$$\lim_i t_{n_i,k} = 0 \quad (k \in \mathbb{N}), \tag{1.2}$$

$$\lim_i \sum_k t_{n_i,k} = 1. \tag{1.3}$$

In [4], Demirci has obtained necessary and sufficient conditions for a matrix T to satisfy K-$core(Tx) \subseteq st_A$-$core(x)$ on ℓ_∞. In the present paper, we determine subclasses of the classes $(c, st(A) \cap \ell_\infty)_{reg}$ and $(st(B) \cap \ell_\infty, st(A) \cap \ell_\infty)_{reg}$ such that st_A-$core(Tx) \subseteq K$-$core(x)$ and st_A-$core(Tx) \subseteq st_B$-$core(x)$ for all $x \in \ell_\infty$.

2. The main results

We begin with some lemmas which can easily be derived from Theorem 4.1 of [10].

Lemma 2.1. *One has* $T \in (st(B) \cap \ell_\infty, st(A) \cap \ell_\infty)_{reg}$ *if and only if* $T \in (c, st(A) \cap \ell_\infty)_{reg}$ *and*

$$st_A\text{-}\lim_n \sum_{k \in E} |t_{nk}| = 0 \tag{2.1}$$

for every $E \subseteq \mathbb{N}$ *with* $\delta_B(E) = 0$.

Lemma 2.2. *One has* $T \in (st(A) \cap \ell_\infty, st(A) \cap \ell_\infty)_{reg}$ *if and only if* $T \in (c, st(A) \cap \ell_\infty)_{reg}$ *and* (2.1) *holds for every* $E \subseteq \mathbb{N}$ *with* $\delta_A(E) = 0$.

In the special case $A = B = C_1$, the Cesàro matrix, Theorem 2.1 reduces to the characterization of the class $(S \cap \ell_\infty, S \cap \ell_\infty)_{reg}$ studied in [3] and also to the following results.

Lemma 2.3. *One has* $T \in (st(A) \cap \ell_\infty, S \cap \ell_\infty)_{reg}$ *if and only if* $T \in (c, S \cap \ell_\infty)_{reg}$ *and*

$$st\text{-}\lim_n \sum_{k \in E} |t_{nk}| = 0$$

for every $E \subseteq \mathbb{N}$ *with* $\delta_A(E) = 0$.

Lemma 2.4. *One has* $T \in (S \cap \ell_\infty, st(A) \cap \ell_\infty)_{reg}$ *if and only if* $T \in (c, st(A) \cap \ell_\infty)_{reg}$ *and* (2.1) *holds for every* $E \subseteq \mathbb{N}$ *with* $\delta(E) = 0$.

Theorem 2.5. *The inclusion* st_A*-core*$(Tx) \subseteq K$*-core*(x) *holds for all* $x \in \ell_\infty$ *if and only if* $T \in (c, st(A) \cap \ell_\infty)_{reg}$ *and*

$$st_A\text{-}\lim_n \sum_k |t_{nk}| = 1. \tag{2.2}$$

Proof. Let st_A-*core*$(Tx) \subseteq K$-*core*(x) for all $x \in \ell_\infty$. If $x \in c$ and $\lim x = l$, then K-*core*$(x) = \{l\}$ and so, by hypothesis, st_A-*core*$(Tx) \subseteq \{l\}$ which implies that $st_A\text{-}\lim Tx = l$. Hence, we have $T \in (c, st(A) \cap \ell_\infty)_{reg}$.

Next, suppose that the condition (2.2) does not hold. Then

$$st_A\text{-}\lim_n \sum_k |t_{nk}| > 1.$$

The conditions (1.1)–(1.3) allow us to choose two strictly increasing sequences $\{n_i\}$ and $\{k(n_i)\}$ $(i = 1, 2, \ldots)$ of positive integers such that

$$\sum_{k=0}^{k(n_{i-1})} |t_{n_i,k}| < \frac{1}{4}, \qquad \sum_{k=k(n_{i-1})+1}^{k(n_i)} |t_{n_i,k}| > 1 + \frac{1}{2}$$

and

$$\sum_{k=k(n_i)+1}^{\infty} |t_{n_i,k}| < \frac{1}{4}.$$

Now, let us define a sequence $x = (x_k)$ by

$$x_k = \operatorname{sign}\{t_{n_i,k}\}, \ k(n_{i-1}) + 1 \leq k < k(n_i).$$

Then, it is clear that K-$core(x) \subseteq B_x(0)$. Also,

$$\begin{aligned} |(Tx)_{n_i}| &\geq \sum_{k=k(n_{i-1})+1}^{k(n_i)} |t_{n_i,k}| - \sum_{k=0}^{k(n_{i-1})} |t_{n_i,k}| - \sum_{k=k(n_i)+1}^{\infty} |t_{n_i,k}| \\ &> 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = 1. \end{aligned}$$

Since $T \in (c, st(A) \cap \ell_\infty)_{reg}$, it follows that $\{(Tx)_{n_i}\}$ is bounded, and hence (Tx) has an A-statistically convergent subsequence whose st_A-limit cannot be in $B_x(0)$. This is a contradiction with the fact that

$$st_A\text{-}core(Tx) \subseteq K\text{-}core(x).$$

Thus, (2.2) must hold.

Conversely, suppose that $T \in (c, st(A) \cap \ell_\infty)_{reg}$ and (2.2) holds. Let $w \in st_A$-$core(Tx)$. Then, by (2.2), we can write, for any $z \in \mathbb{C}$,

$$\begin{aligned} |w - z| &\leq st_A\text{-} \limsup_n |z - Tx| \\ &= st_A\text{-} \limsup_n \left| z - \sum_k t_{nk} x_k \right| \\ &\leq st_A\text{-} \limsup_n \left| \sum_k t_{nk}(z - x_k) \right| + st_A\text{-} \limsup_n |z| \left| 1 - \sum_k t_{nk} \right| \\ &= st_A\text{-} \limsup_n \left| \sum_k t_{nk}(z - x_k) \right|. \end{aligned}$$

Now, put $\limsup_k |x_k - z| = l$. Then, for any $\varepsilon > 0$, $|x_k - z| \leq l + \varepsilon$ whenever $k \geq k_0$. Hence, since

$$\begin{aligned} \left| \sum_k t_{nk}(z - x_k) \right| &= \left| \sum_{k<k_0} t_{nk}(z - x_k) + \sum_{k \geq k_0} t_{nk}(z - x_k) \right| \\ &\leq \sup_k |z - x_k| \sum_{k<k_0} |t_{nk}| + (l + \varepsilon) \sum_{k \geq k_0} |t_{nk}| \\ &\leq \sup_k |z - x_k| \sum_{k<k_0} |t_{nk}| + (l + \varepsilon) \sum_k |t_{nk}|, \end{aligned}$$

using (1.2) and (2.2) we get

$$|w - z| \leq st_A\text{-}\limsup_n \Big|\sum_k t_{nk}(z - x_k)\Big| \leq l + \varepsilon.$$

Since ε is arbitrary, we conclude that $|w - z| \leq \limsup_k |x_k - z|$ which means that $w \in K\text{-}core(x)$. □

For $A = C_1$, Theorem 2.5 reduces to the Theorem 5.1 of [11].

Theorem 2.6. *One has* $st_A\text{-}core(Tx) \subseteq st_B\text{-}core(x)$ *for all* $x \in \ell_\infty$ *if and only if* $T \in (st(B) \cap \ell_\infty, st(A) \cap \ell_\infty)_{reg}$ *and* (2.2) *holds.*

Proof. Let $st_A\text{-}core(Tx) \subseteq st_B\text{-}core(x)$ for all $x \in \ell_\infty$. Then, the necessity of (2.2) follows from Theorem 2.5, since $st_B\text{-}core(x) \subseteq K\text{-}core(x)$. To show that $T \in (st(B)\cap\ell_\infty, st(A)\cap\ell_\infty)_{reg}$, it is sufficient to take any $x \in st(B)\cap\ell_\infty$ with $st_B\text{-}\lim x = l$.

Conversely, assume that the conditions hold and $w \in st_A\text{-}core(Tx)$. Then, as in Theorem 2.5, we have

$$|w - z| \leq st_A\text{-}\limsup_n \Big|\sum_k t_{nk}(z - x_k)\Big|$$

for any $z \in \mathbb{C}$.

Further, if we put $st_A\text{-}\limsup_k |x_k - z| = s$, then, since for any $\varepsilon > 0$, $\delta_A(E) = \delta_A(\{k : |x_k - z| > s + \varepsilon\}) = 0$, and

$$\begin{aligned} \Big|\sum_k t_{nk}(z - x_k)\Big| &= \Big|\sum_{k\in E} t_{nk}(z - x_k) + \sum_{k\notin E} t_{nk}(z - x_k)\Big| \\ &\leq \sup_k |z - x_k| \sum_{k\in E} |t_{nk}| + (s + \varepsilon) \sum_{k\notin E} |t_{nk}| \\ &\leq \sup_k |z - x_k| \sum_{k\in E} |t_{nk}| + (s + \varepsilon) \sum_k |t_{nk}|, \end{aligned}$$

the conditions (2.1) and (2.2) imply that

$$st_A\text{-}\limsup_n \Big|\sum_k t_{nk}(z - x_k)\Big| \leq s + \varepsilon.$$

Hence, since ε is arbitrary, it is obtained that $|w - z| \leq s$ which means $w \in st_A\text{-}core(x)$. □

Finally, we note that our Theorem 2.6 for $A = B = C_1$ gives a simple proof of Theorem 3 of Lie and Fridy [8] and also the following results.

Corollary 2.7. *One has* $st_A\text{-}core(Tx) \subseteq st\text{-}core(x)$ *for all* $x \in \ell_\infty$ *if and only if* $T \in (S \cap \ell_\infty, st(A) \cap \ell_\infty)_{reg}$ *and* (2.2) *holds.*

Corollary 2.8. *One has* $st\text{-}core(Tx) \subseteq st_A\text{-}core(x)$ *for all* $x \in \ell_\infty$ *if and only if* $T \in (st(A) \cap \ell_\infty, S \cap \ell_\infty)_{reg}$ *and*

$$st\text{-}\lim_n \sum_k |t_{nk}| = 1.$$

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