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The approximation property in terms of density in operator topologies

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ABSTRACT. Some assertions concerning a Banach space are proven equivalent to the approximation property, improving and complementing results in [3] and [5].

1. Introduction

In the study of the, truly natural, questions in Banach space theory concerning the existence of a Schauder basis and the approximation in norm of compact operators by operators of finite rank the concept of *approximation property* for a Banach space plays a crucial role. Ever since the appearance of Grothendieck's outstanding work [2] it has, together with its many generalized concepts, been subject to major interest. In this article we seek for equivalent formulations of it.

The usual definition of a Banach space X having the approximation property is that for every compact set $K \subset X$ and every $\epsilon > 0$ there is a finite rank operator $T \in \mathcal{F}(X, X)$ such that $\|Tx - x\| < \epsilon$, for all $x \in K$. In this article we study reformulations of the approximation property in terms of density of different sets of operators in the strong, weak and strong adjoint operator topologies. A survey, as well as extensions, of results in this direction obtained in [3], [4] and [5] are given.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators from X to Y, and by $\mathcal{F}(X,Y)$, $\mathcal{K}(X,Y)$ and $\mathcal{W}(X,Y)$ its subspaces of finite rank, compact and weakly compact operators. If $\mathcal{A}$ is $\mathcal{F}$, $\mathcal{K}$, $\mathcal{W}$ or $\mathcal{L}$, then $\mathcal{A}_{w^*}(X^*, Y)$ denotes the subspace of $\mathcal{A}(X^*, Y)$ consisting of those operators which are weak*-weakly continuous.

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The algebraic tensor product $X \otimes Y$ is always canonically identified with a linear subspace of $\mathcal{F}(X^*, Y)$. Let us recall that $X \otimes Y = \mathcal{F}_{w^*}(X^*, Y)$ and $X^* \otimes Y = \mathcal{F}(X, Y)$ and that $T \in \mathcal{L}(X^*, Y)$ is weak*-weakly continuous if and only if $\mathrm{ran} T^* \subset X$.

If $A \subset B$ are any subsets of $\mathcal{L}(X, Y)$ then A is dense in B with respect to the strong (weak) operator topology on $\mathcal{L}(X, Y)$ if and only if for any $T \in B$ there is a net $(T_\alpha) \subset A$ such that $T_\alpha x \to Tx$ (weakly) for every $x \in X$. The weak and strong operator topologies yield the same dual space [1, Theorem VI.1.4], thus the closure of a convex set is the same in the both topologies.

We also consider what could be called the *strong adjoint operator topology* on $\mathcal{L}(X, Y)$. A subbase of this topology consists of all sets

$$V_{(T, y^*, \epsilon)} = \{S \in \mathcal{L}(X, Y) \mid \|S^* y^* - T^* y^*\| < \epsilon\}$$

where $T \in \mathcal{L}(X, Y)$, $y^* \in Y^*$ and $\epsilon > 0$. Thus a net $(T_\alpha) \subset \mathcal{L}(X, Y)$ converges to $T \in \mathcal{L}(X, Y)$ in this topology if and only if $T_\alpha^* y^* \to T^* y^*$ for every $y^* \in Y^*$, i.e. if and only if $T_\alpha^* \to T^*$ in the strong operator topology on $\mathcal{L}(Y^*, X^*)$.

Our notation is rather standard. A Banach space X will always be regarded as a subspace of its bidual X^{**}. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The closed unit ball of X with center at 0 is denoted by B_X and the identity operator on X by I_X.

2. Reformulations of the approximation property

In [2] the following equivalent formulations of the approximation property were proved.

Theorem 1 (Grothendieck). *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*

(b) *For every Banach space Y, one has $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$.*

(c) *For every Banach space Y, one has $\mathcal{K}_{w^*}(X^*, Y) = \overline{X \otimes Y}$.*

The main result of [5] was to translate condition (c) in terms of density with respect to the strong operator topology of the unit ball of spaces of operators in the unit balls of larger spaces as follows.

Theorem 2 (cf. [5]). *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*

(c_i) *For every Banach space Y, $B_{X \otimes Y}$ is dense in $B_{\mathcal{W}_{w^*}(X^*, Y)}$ with respect to the strong operator topology on $\mathcal{L}(X^*, Y)$.*

(c_{ii}) *For every reflexive Banach space Y, $B_{X \otimes Y}$ is dense in $B_{\mathcal{K}_{w^*}(X^*, Y)}$ with respect to the strong operator topology on $\mathcal{L}(X^*, Y)$.*

In these kinds of equivalent reformulations there are numerous possibilities to vary the conditions in terms of the size of the concerned subspaces of $\mathcal{L}(X^*,Y)$ and over which spaces Y should range, and still have equivalence with the approximation property. Strictly speaking only the strongest and the weakest in each of the settings is interesting, i.e. on the one hand find a condition with the largest possible subspace of $\mathcal{L}(X^*,Y)$ in the unit ball of which the unit ball of the underlying space $X \otimes Y$ should be dense when Y ranges over all Banach spaces (in Theorem 2 assertion (c_i)) and on the other hand a condition with the smallest possible subspace of $\mathcal{L}(X^*,Y)$ in the unit ball of which the unit ball of $X \otimes Y$ should be dense when Y ranges over a smallest possible family of Banach spaces (in Theorem 2 assertion (c_{ii})). Intermediate assertions, such as

for every Banach space Y, $B_{X\otimes Y}$ is dense in $B_{\mathcal{K}_{w^*}(X^*,Y)}$ with respect to the strong operator topology on $\mathcal{L}(X^*,Y)$,

for every reflexive Banach space Y, $B_{X\otimes Y}$ is dense in $B_{\mathcal{W}_{w^*}(X^*,Y)}$ with respect to the strong operator topology on $\mathcal{L}(X^*,Y)$

are obviously also equivalent to the approximation property since they trivially are implied by (c_i) and imply (c_{ii}). To avoid lengthy lists of equivalent assertions in theorems and to clarify what results really are improvements of already known ones we will, in the rest of the article, avoid to formulate explicitly this kind of intermediate conditions in theorems.

In [3], [4] and [5] translations of condition (b) in terms of density in the strong operator topology similar to those in Theorem 2 were proved.

Theorem 3 (cf. [3], [4] and [5]). *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*

(b_i) *For every Banach space Y, $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{W}(Y,X)}$ with respect to the strong operator topology on $\mathcal{L}(Y,X)$.*

(b_{ii}) *For every reflexive Banach space Y, $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{K}(Y,X)}$ with respect to the strong operator topology on $\mathcal{L}(Y,X)$.*

In the lists of reformulations in Theorem 2 and Theorem 3 conditions (c_{ii}) and (b_{ii}) are not optimal. Actually one can reduce the spaces over which Y should range to separable reflexive spaces.

Theorem 4. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*

(b_{iii}) *For every separable reflexive Banach space Y, $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{K}(Y,X)}$ with respect to the strong operator topology on $\mathcal{L}(Y,X)$.*

(c_{iii}) *For every separable reflexive Banach space Y, $B_{X\otimes Y}$ is dense in $B_{\mathcal{K}_{w^*}(X^*,Y)}$ with respect to the strong operator topology on $\mathcal{L}(X^*,Y)$.*

Proof. "(a) $\Rightarrow$ (b_{iii})" This is obvious from Theorem 3.

"(b_{iii}) $\Rightarrow$ (c_{iii})" Let Y be any separable reflexive Banach space. If $T \in B_{\mathcal{K}_{w^*}(X^*,Y)}$, then $T^* \in B_{\mathcal{K}(Y^*,X)}$ so there is a net $(S_\alpha) \subset B_{\mathcal{F}(Y^*,X)} = B_{Y^{**}\otimes X} = B_{Y\otimes X}$ such that $S_\alpha y^* \to T^* y^*$ for every $y^* \in Y^*$. Let $T_\alpha = S_\alpha^* \in B_{X\otimes Y}$. Then for every $x^* \in X^*$ and every $y^* \in Y^*$ we have

$$y^*(T_\alpha x^*) = x^*(S_\alpha y^*) \to x^*(T^* y^*) = y^*(Tx^*),$$

i.e. $T_\alpha \to T$ in the weak operator topology. Passing to convex combinations, a net in $B_{X\otimes Y}$ converging to T in the strong operator topology can be obtained.

"(c_{iii}) $\Rightarrow$ (a)" When proving this the only thing we need to know to be able to copy the proof of "(c_3) $\Rightarrow$ (a)" in [5] is that the space X_K, in the Davis-Figiel-Johnson-Pełczyński construction, the space obtained from X and K, where K is the closed absolutely convex hull of a sequence (x_n) that converges in norm to 0, is separable. Lemma 2.1 in [3] contains what is necessary to prove this. Since K is the closed absolutely convex hull of (x_n) it is separable. Lemma 2.1 (iv) [3] tells us then that the unit ball C_K in X_K is separable considered as subset of X, and then by Lemma 2.1 (i) [3], C_K must also be separable considered as subset of X_K, thus X_K is a separable space. An anonymous referee has kindly pointed out that an alternative proof of this step uses (c_{ii}) and the separable 1-complementation property. □

Remark. The density conditions in Theorem 2 and Theorem 3 mean that for any operator T in the larger subspaces there is a bounded net (T_α) in the smaller subspace with $\|T_\alpha\| \leq \|T\|$ for every α converging to T in the strong operator topology. In [5] the formulation of conditions (b_{ii}) and (c_{ii}) is actually more general. It is only demanded that the nets are bounded, not that they should be bounded by $\|T\|$ or have any other bound depending on $\|T\|$. Also the new assertions in Theorem 4 could be phrased in this way and still are equivalent to the approximation property.

Can the boundedness condition on $\|T_\alpha\|$ be removed? By the principle of uniform boundedness any sequence of operators converging in the strong operator topology or in the weak operator topology is automatically bounded, but the same conclusion cannot be drawn for a converging net. Hence it is natural to ask when the approximation property does not only imply density but even sequential density in the strong (weak) operator topology. Recall that if $A \subset B \subset \mathcal{L}(Y,X)$ then A is sequentially dense in B with respect to the strong operator topology if for every $T \in B$ there is a sequence $(T_n) \subset A$

such that $T_n y \to Ty$ for every $y \in Y$. In some cases it is not difficult to prove that sequential density and density are in fact the same.

Proposition 1. *Let X and Y be any two Banach spaces.*

(i) *Let $A \subset B \subset \mathcal{L}(Y,X)$ be bounded convex sets such that A is dense in B with respect to the strong operator topology. If Y is separable then A is sequentially dense in B with respect to the same topology.*

(ii) *Let $A \subset B \subset \mathcal{L}_{w^*}(X^*,Y)$ be bounded convex sets such that A is dense in B with respect to the weak operator topology. If X^* or Y^* is separable then A is sequentially dense in B with respect to the same topology.*

Proof. (i) This holds since the strong operator topology is metrisable on bounded subsets of $\mathcal{L}(Y,X)$ if Y is separable.

(ii) If X^* is separable we have the same situation as in (i). If Y^* is separable let $A^* = \{T^* \mid T \in A\}$ and define B^* analogously. All operators being sets of weak*-weakly continuous operators A^* and B^* can be viewed as subsets of $\mathcal{L}(Y^*,X)$ and as such A^* is dense in B^* with respect to the weak operator topology. By convexity A^* is also dense with respect to the strong operator topology and hence by (i) even sequentially dense in B^*. This trivially implies that A is sequentially dense in B with respect to the weak operator topology. □

The proposition gives immediately the following reformulations of the approximation property in the context of Theorem 2 and Theorem 3. The assertion ($\mathrm{b_{iv}}$) is proven equivalent to the approximation property in [3, Corollary 1.5 (iv)].

Corollary 1. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*

($\mathrm{b_{iv}}$) *For every separable Banach space Y, $B_{\mathcal{F}(Y,X)}$ is sequentially dense in $B_{\mathcal{W}(Y,X)}$ with respect to the strong operator topology on $\mathcal{L}(Y,X)$.*

($\mathrm{c_{iv}}$) *For every Banach space Y with separable dual space Y^*, $B_{X\otimes Y}$ is sequentially dense in $B_{\mathcal{W}_{w^*}(X^*,Y)}$ with respect to the weak operator topology on $\mathcal{L}(X^*,Y)$.*

Proof. In view of Theorem 2, Theorem 3 and Theorem 4 it is obvious that density of $B_{\mathcal{F}(Y,X)}$ (resp. $B_{X\otimes Y}$) in $B_{\mathcal{W}(Y,X)}$ (resp. $B_{\mathcal{W}_{w^*}(X^*,Y)}$) with respect to the strong operator topology whenever Y is a Banach space of the specified type in assertion ($\mathrm{b_{iv}}$) (resp. ($\mathrm{c_{iv}}$)) is equivalent to the approximation property. By Proposition 1 density and sequential density is the same in these cases. □

As noted above sequential density is connected to the question of boundedness of the operators converging in the strong operator topology in such a way that the following conclusion can be drawn.

Corollary 2. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*

($\mathrm{b_v}$) *For every separable reflexive Banach space Y, $\mathcal{F}(Y,X)$ is sequentially dense in $\mathcal{K}(Y,X)$ with respect to the strong operator topology on $\mathcal{L}(Y,X)$.*

($\mathrm{c_v}$) *For every separable reflexive Banach space Y, $X \otimes Y$ is sequentially dense in $\mathcal{K}_{w^*}(X^*,Y)$ with respect to the weak operator topology on $\mathcal{L}(X^*,Y)$.*

Proof. Equivalence with the approximation property will still hold if we replace $\mathcal{W}$ with $\mathcal{K}$ and let Y range only over all separable reflexive Banach spaces in Corollary 1. As remarked, the particular bound $\|T_n\| \leq \|T\|$ for the sequence (T_n) converging to T in the strong (weak) operator topology is not necessary, it suffices to demand only that it has a bound, and since for sequences converging in the strong (weak) operator topology boundedness in norm is automatical we have, in language of density, equivalence to sequential density of the whole underlying subspace in the larger subspace. □

Remark. Notice that ($\mathrm{b_v}$) is stronger than [3, Corollary 1.5 (iv)] where it is not obvious that one can remove the condition $\|T_n\| \leq \|T\|$. It is removed using the results of [5].

3. Reformulations of the approximation property of the dual space

It is possible to find assertions, similar those found in Section 2 concerning density in certain operator topologies, that are equivalent to the approximation property of the dual space of a Banach space X. In [5] the following theorem is proved.

Theorem 5. *Let X be a Banach space. Then the following assertions are equivalent.*

(a^*) *X^* has the approximation property.*

($\mathrm{b}^*_{\mathrm{i}}$) *For every Banach space Y, $B_{\mathcal{F}(X,Y)}$ is dense in $B_{\mathcal{W}(X,Y)}$ with respect to the strong adjoint operator topology on $\mathcal{L}(X,Y)$.*

($\mathrm{b}^*_{\mathrm{ii}}$) *For every reflexive Banach space Y, $B_{\mathcal{F}(X,Y)}$ is dense in $B_{\mathcal{K}(X,Y)}$ with respect to the strong adjoint operator topology on $\mathcal{L}(X,Y)$.*

We can, as in Theorem 1, improve assertion ($\mathrm{b}^*_{\mathrm{ii}}$) to only separable reflexive Y.

Theorem 6. *Let X be a Banach space. Then the following assertions are equivalent.*

(a^*) *X^* has the approximation property.*

($\mathrm{b}^*_{\mathrm{iii}}$) *For every separable reflexive Banach space Y, $B_{\mathcal{F}(X,Y)}$ is dense in $B_{\mathcal{K}(X,Y)}$ with respect to the strong adjoint operator topology on $\mathcal{L}(X,Y)$.*

Proof. It suffices to show that X^* satisfies (c_{iii}) and then apply Theorem 4. This can be done exactly as the proof of "$(b^*_{\mathcal{K}}) \Rightarrow (a^*)$" in [5]. □

We also have a version for sequential density.

Theorem 7. *Let X be a Banach space. Then the following assertions are equivalent.*

(a^*) *X^* has the approximation property.*

(b^*_{iv}) *For every Banach space Y with separable dual space Y^*, $B_{\mathcal{F}(X,Y)}$ is sequentially dense in $B_{\mathcal{W}(X,Y)}$ with respect to the strong adjoint operator topology on $\mathcal{L}(X,Y)$.*

Proof. It is clear from Theorem 5 and Theorem 6 that (b^*_{iv}) implies (a^*) and that (a^*) implies density of $B_{\mathcal{F}(X,Y)}$ in $B_{\mathcal{W}(X,Y)}$ with respect to the strong adjoint operator topology on $\mathcal{L}(X,Y)$. Now let $T \in B_{\mathcal{W}(X,Y)}$ and let (T_α) be a net in $B_{\mathcal{F}(X,Y)}$ that converges to T in the strong adjoint operator topology. Using a standard argument, let (y_n^*) be a dense sequence in B_{Y^*}. From the given net (T_α) pick operators T_{α_n} such that $\|T^*_{\alpha_n} y_1^* - T^* y_1^*\|, \ldots, \|T^*_{\alpha_n} y_n^* - T^* y_n^*\| < \frac{1}{n}$. Then (T_{α_n}) is the desired sequence. □

In [5] Theorem 5 is formulated with the condition that for every reflexive Banach space Y and for every operator $T \in \mathcal{K}(X,Y)$, there exists a bounded net $(T_\alpha) \subset \mathcal{F}(X,Y)$ such that $T_\alpha^* \to T^*$ in the strong operator topology, whereas we have demanded that $\|T_\alpha\| \leq \|T\|$. And we then, as in the previous section, can draw the following conclusion.

Corollary 3. *Let X be a Banach space. Then the following assertions are equivalent.*

(a^*) *X^* has the approximation property.*

(b^*_{v}) *For every separable reflexive Banach space Y, $\mathcal{F}(X,Y)$ is sequentially dense in $\mathcal{K}(X,Y)$ with respect to the strong adjoint operator topology on $\mathcal{L}(X,Y)$.*

Proof. We can use the same proof as that of Corollary 2, since by the principle of uniform boundedness any sequence of operators converging in the strong adjoint operator topology is automatically bounded. □

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