ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 7, 2003

S-nuclearity and n-diameters of infinite Cartesian products of bounded subsets in Banach spaces

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Abstract. In this paper we classify compact subsets of a normed space according to the rate of convergence to zero of its sequence $\{\delta_n(B)\}$ of Kolmogorov diameters. We introduce σ-compact sets to satisfy that $\{\delta_n(B)\} \in \sigma$ where σ is an ideal of convergent to zero sequences. Examples of sequence ideals are the ideals of rapidly decreasing sequences $\{\lambda_n\}$ satisfying $\lim_{n\to\infty} \lambda_n\, n^{\alpha} = 0$ for any $\alpha > 0$, or radically decreasing sequences satisfying $\lim_{n\to\infty} \sqrt[n]{|\lambda_n|} = 0$. In case σ is the ideal of rapidly decreasing sequences, this notion is identical to the S-nuclearity introduced by K. Astala and M. S. Ramanujan in 1987. We show that the infinite Cartesian product $\prod_{i=1}^{\infty} B_i$ of compact sets B_i is ℓ^p-compact in $\ell^p(X_i)$, for all $p > 0$, if $(\delta_0(B_i)) \in S$. In this case we give upper estimates of n-th diameters of $\prod_{i=1}^{\infty} B_i$ in $\ell^p(X_i)$ for any $p > 0$.

1. Introduction

Astala and Ramanujan have suggested to call a subset B of a normed space S-nuclear if and only if its sequence of Kolmogorov diameters (see Definition 6.1) $\{\delta_n(B)\} \in S$ where S is the space of rapidly decreasing sequences (see Section 2). This happens if and only if $\sum_{n=0}^{\infty} \delta_n^p(B) < \infty$ for all $p > 0$. However it is known [3] that a bounded subset B of a normed space X is precompact if and only if its sequence of Kolmogorov diameters $\{\delta_n(B)\}$ converges to zero. This motivated us to call a subset B of a normed space σ-compact if and only if $\{\delta_n(B)\} \in \sigma$, where σ is any sequence ideal. In fact, we make a slight modification of the definition of the sequence ideal mentioned in [4]. We only require that the ideal σ contains the dilated sequence $\{\lambda_{[n/2]}\}$ together with any sequence $\{\lambda_n\} \in \sigma$. This requirement

Received June 1, 2003.
2000 *Mathematics Subject Classification.* 47B06.
Key words and phrases. S-nuclearity, Kolmogorov diameters, rapidly decreasing sequences, radically decreasing sequences, n-diameters of Cartesian products.

is weaker than the symmetricity condition included in the definition of the sequence ideal mentioned in [4].

2. Notation and basic definitions

1. By ℓ^∞ we denote the space of all bounded sequences of real numbers and by c_0 its closed subspace of all convergent to zero sequences.

2. By S we denote the space of all rapidly decreasing sequences of real numbers defined by

$$S = \left\{ \{\lambda_n\}_{n=1}^\infty : \sup_n n^\alpha \, |\lambda_n| < \infty \quad \forall \ \alpha > 0 \right\}.$$

Example for a rapidly decreasing sequence is $\{\lambda_n\} = \{2^{-n}\}$.

3. For a sequence $\alpha = \{\alpha_i\}_{i=1}^\infty$, $0 < \alpha_1 < \alpha_2 < \dots$, satisfying that there exists a constant $c > 0$ such that $\alpha_{2n} \leq c\alpha_n$, $n = 1, 2, \dots$, the space S_α is defined as the following space of sequences:

$$S_\alpha = \left\{ \{\lambda_n\}_{n=1}^\infty : \sup_n x^{\alpha_n} \, |\lambda_n| < \infty \quad \forall \ x \in \mathbb{R}^+ \right\}.$$

4. By $\Re$ we denote the space of all radically decreasing sequences of real numbers defined by

$$\Re = \left\{ \{\lambda_n\}_{n=1}^\infty : \lim_{n\to\infty} \sqrt[n]{|\lambda_n|} = 0 \right\}.$$

3. Remarks and consequences on definitions

1. One can easily see that a sequence $\{\lambda_n\}_{n=1}^\infty$ is rapidly decreasing, i.e., satisfying the condition $\sup_n \ n^\alpha \, |\lambda_n| < \infty$ for any $\alpha > 0$, if and only if $\lim_{n\to\infty} |\lambda_n| \ n^\alpha = 0$ for each $\alpha > 0$.

2. A sequence $\{\lambda_n\}$ belongs to the space S_α if and only if $\sup_n \ x^{\alpha_n} \, |\lambda_n| < \infty$ for any $x \in \mathbb{R}^+$. This condition is equivalent to $\lim_{n\to\infty} x^{\alpha_n} \, |\lambda_n| = 0$ for any $x \in \mathbb{R}^+$.

3. If $\lim_{n\to\infty} x^{\alpha_n} \, |\lambda_n| = 0$ for any $x \geq 1$ then $\lim_{n\to\infty} x^{\alpha_n} \, |\lambda_n| = 0$ for all $0 \leq x \leq 1$.

4. The space S of rapidly decreasing sequences is a special case of the space S_α. In fact, taking $\alpha_n = \log n$ and $x = e^k$ for any $k > 0$, we get

$$x^{\alpha_n} = (e^k)^{\log n} = n^k.$$

For this choice of α we get

$$S_\alpha = S, \quad \alpha = (\alpha_n)_{n=1}^\infty.$$

Proposition 3.1. (1) *If* $\{\lambda_n\}_1^\infty$ *is a radically decreasing sequence then* $\{\lambda_n\, n^\alpha\}$ *is also radically decreasing, i.e., if* $\{\lambda_n\}_{n=1}^\infty \in \Re$ *then* $\{\lambda_n\, n^\alpha\} \in \Re \subseteq c_0$ *for any* $\alpha > 0$.

(2) *Each radically decreasing sequence is rapidly decreasing and the converse is not necessarily true, i.e.,* $\Re \subseteq S \subseteq c_0$, $\Re \neq S$.

Proof. Let $\{\lambda_n\}_1^\infty \in \Re$; then $\lim \sqrt[n]{|\lambda_n|} = 0$. Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, we get for each $\alpha > 0$

$$\lim_{n\to\infty} \sqrt[n]{|\lambda_n|\, n^\alpha} = \lim_{n\to\infty} \sqrt[n]{|\lambda_n|}\ \lim_{n\to\infty} (\sqrt[n]{n})^\alpha = 0,$$

and hence $\{\lambda_n\, n^\alpha\} \in \Re$. Therefore $\lim_{n\to\infty} |\lambda_n|\, n^\alpha = 0$ for all $\alpha > 0$, and $\sup_n |\lambda_n|\, n^\alpha < \infty$. Hence $\Re \subseteq S \subseteq c_0$.

Moreover, since $\lim_{n\to\infty} n^\alpha/2^n = 0$ for all $\alpha > 0$, we have $\{1/2^n\} \in S$. However $\{1/2^n\} \notin \Re$ since $\lim_{n\to\infty} \sqrt[n]{1/2^n} = 1/2$. Therefore $\{1/2^n\} \in S \setminus \Re$ and so $\Re \subsetneqq S$. □

Remark 3.2. There is enough variety of sequence spaces with different rates of convergence to zero of its general terms λ_n. For example, we can take $\left\{\sqrt[n]{|\lambda_n|}\right\} \in S$ or $\left\{\sqrt[n]{|\lambda_n|}\right\} \in \Re$, and so on,

$$\Re = \left\{\{\lambda_n\} : \left\{\sqrt[n]{|\lambda_n|}\right\} \in c_0\right\},$$

$$\Re^{(2)} = \left\{\{\lambda_n\} : \left\{\sqrt[n]{|\lambda_n|}\right\} \in \Re\right\},$$

$$\Re^{(k+1)} = \left\{\{\lambda_n\} : \left\{\sqrt[n]{|\lambda_n|}\right\} \in \Re^{(k)}\right\}, \quad k = 1, 2, \ldots .$$

4. Sequence ideals

A sequence ideal σ on the scaler field is a subset of the space ℓ^∞ satisfying the following conditions :

(1) $e_i \in \sigma$, where $e_i = (0,\ 0, \ldots,\ 1,\ 0, \ldots)$, the one in the i-th place.

(2) If $\lambda_1,\ \lambda_2 \in \sigma$ then $\lambda_1 + \lambda_2 \in \sigma$.

(3) If $\lambda = \{\lambda_i\}_{i=0}^\infty \in \ell^\infty$ and $\mu = \{\mu_i\}_{i=0}^\infty \in \sigma$ then $\lambda\, \mu = \{\lambda_i\, \mu_i\}_{i=0}^\infty \in \sigma$.

(4) If $\lambda = \{\lambda_0, \lambda_1, \ldots\} \in \sigma$ then the dilated sequence $\{\lambda_0, \lambda_0, \lambda_1, \lambda_1, \ldots\} = \{\lambda_{[n/2]}\}_{n=0}^\infty \in \sigma$, where $[x]$ is the integer part of the real number x.

For a sequence ideal σ, we call the operator $D : \sigma \to \sigma$, defined by

$$D(\{\lambda_n\}) = \{\lambda_{[n/2]}\}_{n=0}^\infty \in \sigma,$$

the dilatation operator, and condition (4) the dilatation property.

Examples of sequence ideals. The sequence spaces S, $\Re$, and S_α are examples of sequence ideals. We will verify the satisfaction of the dilatation property of the above-mentioned examples.

(1) The space S : taking $\{\lambda_n\} \in S$, we get

$$\lim_{n\to\infty} n^\alpha \lambda_n = 0 \ \forall \alpha > 0.$$

Therefore,

$$\lim_{n\to\infty} n^\alpha \lambda_{[n/2]} \leq 2^\alpha \lim_{n\to\infty} ([n/2]+1)^\alpha \lambda_{[n/2]} = 0.$$

Hence $D(\lambda_n) \in S$.

(2) The space $\Re$: for a radical sequence $\{\lambda_n\} \in \Re$ and for large values of n we get $\left|\lambda_{[n/2]}\right| < 1$, and therefore

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|\lambda_{[n/2]}|} &= \lim_{n\to\infty} \left\{ \sqrt[n/2]{|\lambda_{[n/2]}|} \right\}^{1/2} \leq \\ &\leq \lim_{n\to\infty} \left\{ \sqrt[[n/2]]{|\lambda_{[n/2]}|} \right\}^{1/2} \leq \\ &\leq \lim_{n\to\infty} \left\{ \sqrt[[n/2]]{|\lambda_{[n/2]}|} \right\}^{1/4} = 0. \end{aligned}$$

Hence $D(\lambda_n) \in \Re$.

(3) The space S_α : for $\{\lambda_n\} \in S_\alpha$ and $x \geq 1$ we get, using the notation from Section 2,

$$\begin{aligned} \sup_n x^{\alpha_n} |\lambda_{[n/2]}| &\leq \sup_n x^{\alpha_{2[n/2]+1}} |\lambda_{[n/2]}| = \\ &= x \sup_n x^{\alpha_{2[n/2]}} |\lambda_{[n/2]}| \leq \\ &\leq x \sup_n (x^{c\, \alpha_{[n/2]}} |\lambda_{[n/2]}|) = \\ &= x \sup_n (x^c)^{\alpha_n} |\lambda_n| < \infty. \end{aligned}$$

Hence $D(\lambda_n) \in S_\alpha$.

5. Infinite Cartesian products of unit balls

Let $X_1, X_2, \dots$ be a sequence of normed spaces with closed unit balls $U_{X_1}, U_{X_2}, \dots$, respectively. By $\ell^\infty(X_i)$, $\ell^p(X_i)$ we denote the linear subspaces of the Cartesian product $X_1 \times X_2 \times X_3 \dots$ equipped with the norms

$$\|x\|_\infty = \sup_{i\in N} \|x_i\|, \quad \|x\|_p = \sqrt[p]{\sum_{i=1}^{\infty} \|x_i\|^p},$$

respectively. By $c_0(X_i)$ we denote the subspace of $\ell^\infty(X_i)$ of all convergent to zero sequences, i.e.,

$$c_0(X_i) = \{x = \{x_i\} \in \ell^\infty(X_i) : \lim_{i\to\infty} \|x_i\| = 0\}.$$

By $U_{\ell^\infty(X_i)}$, $U_{c_0(X_i)}$ and $U_{\ell^p(X_i)}$ we denote the closed unit balls of the spaces $\ell^\infty(X_i)$, $c_0(X_i)$ and $\ell^p(X_i)$, respectively.

Proposition 5.1. *Let $\{X_i\}$ be a sequence of Banach spaces. Then the following holds.*

(1) *The unit ball of $\ell^\infty(X_i)$ is the Cartesian product of the unit balls U_{X_i}, i.e.,*

$$U_{\ell^\infty(X_i)} = \prod_{i=1}^{\infty} U_{X_i} .$$

(2) *For any absolutely p-summable sequence $\lambda = \{\lambda_i\} \in \ell^p$, $p \geq 1$, such that $\sum_{i=1}^{\infty} |\lambda_i|^p \leq 1$ one has*

$$\prod_{i=1}^{\infty}(\lambda_i\, U_{X_i}) \subseteq U_{\ell^p(X_i)} \subseteq U_{c_0(X_i)} \subseteq U_{\ell^\infty(X_i)}.$$

Proof. This is an easy straightforward verification. □

Lemma 5.2. *Let F_1, $F_2, \ldots$ be a sequence of finite dimensional subspaces of normed spaces X_1, $X_2, \ldots$, respectively. Let $\dim F_i = n_i$ ($\dim F_i = 0$ if and only if $F_i = \{0\}$) and let $\sum_{i=1}^{\infty} n_i = m$. In this case only finite number of spaces F_i are not trivially $\{0\}$ and $F = \prod_{i=1}^{\infty} F_i$ is a finite dimensional subspace of $\prod_{i=1}^{\infty} X_i$ with*

$$\dim F = \sum_{i=1}^{\infty} \dim F_i = \sum_{i=1}^{\infty} n_i = m.$$

Proof. Let $\Im = \{i : \dim F_i = n_i \neq 0\} = \{i_1,\ i_2, \ldots,\ i_k\}$. For any $j = 1,\ 2, \ldots,\ k$, the subspace F_{i_j} has a basis say

$$\{x_1^{i_j}, x_2^{i_j}, \ldots, x_{n_{i_j}}^{i_j}\}.$$

Therefore the elements

$$z_i = \left\{ \begin{array}{ll} (0,\ 0, \ldots,\ x_i^{i_1},\ 0, \ldots) & 1 \leq i \leq n_{i_1} \\ (0,\ 0, \ldots,\ x_i^{i_2},\ 0, \ldots) & n_{i_1} < i \leq n_{i_2} \\ \quad\quad \cdot \quad \cdot \quad \cdot & \\ \quad\quad \cdot \quad \cdot \quad \cdot & \\ (0,\ 0, \ldots,\ x_i^{i_k},\ 0, \ldots) & n_{i_{k-1}} < i \leq n_{i_k}, \end{array} \right\}$$

form a basis for the subspace $F = \prod_{i=1}^{\infty} F_i$ with

$$\dim F = \sum_{i=1}^{\infty} \dim F_i = \sum_{i=1}^{\infty} n_i = m.$$

□

6. n-Diameters of Cartesian products

Definition 6.1 (see [4]). Let B be a bounded subset of a Banach space X with the closed unit ball U. For $n = 0, 1, \ldots$, the n-th diameter $\delta_n(B)$ of a bounded subset B is defined as

$$\delta_n(B) = \inf \delta_n(B, F),$$

where the infimum is taken over all subspaces F with $\dim F \leq n$ and

$$\delta_n(B, F) = \inf\{c > 0 : B \subseteq c\, U + F\}.$$

These diameters, introduced by A. N. Kolmogorov, satisfy the following properties (see, e.g. [6]).

1) $\delta_0(B) \geq \delta_1(B) \geq \cdots \geq 0$.

2) $\delta_n(B) = 0$ if and only if B is contained in a linear subspace F of X of dimension at most n, i.e.,

$$\delta_n(B) = 0 \text{ if and only if } B \subset F, \dim F \leq n.$$

3) B is precompact (its closure is compact) if and only if $\{\delta_n(B)\} \in c_0$.

Astala and Ramanujan [1] have suggested the following definition. The subset B is called S-nuclear if and only if $\{\delta_n(B)\} \in S$. Parallel to their suggestion we suggest the following definition.

Definition 6.2 (see [2]). For a sequence ideal $\sigma \subseteq c_0$ we will call a subset B, in a normed space X, a σ-compact subset if and only if $\{\delta_n(B)\} \in \sigma$.

In the special case where σ is the space $\mathfrak{R}$ of radically decreasing sequences we will call a subset B, in a Banach space X, radically compact if and only if $\{\delta_n(B)\} \in \mathfrak{R}$.

Main Theorem. *Let B_0, $B_1, \ldots$ be bounded subsets of normed spaces X_0, $X_1, \ldots$, respectively. Then the following holds.*

(1) *If $\delta_0(B_i) \to_{i\to\infty} 0$ and B_i are compact subsets, i.e., $\delta_n(B_i) \to_{n\to\infty} 0$ for all $i = 0, 1, \ldots$, then $\prod_{i=0}^{\infty} B_i$ is compact in $\ell^\infty(X_i)$ and*

$$\sup_i \delta_m(B_i) \leq \delta_m^\infty(\prod_{i=0}^{\infty} B_i) \leq \inf_{\sum n_i = m} \sup_i \delta_{n_i}(B_i) \longrightarrow 0.$$

(2) *If B_i are compact subsets and $\{\delta_0(B_i)\} \in S$ then $\prod_{i=0}^{\infty} B_i$ is compact in $\ell^p(X_i)$ for any $p > 0$ and*

$$\delta_m^p(\prod_{i=0}^{\infty} B_i) \leq \inf_{\sum n_i = m} \left(\sum_{i=0}^{\infty} (\delta_{n_i}(B_i))^p\right)^{1/p} \longrightarrow 0.$$

Here $\delta_n^p(C)$ is the n-th diameter of C in the space $\ell^p(X_i)$.

Remark 6.3. The item (2) of the Main Theorem holds in case $0 < p < 1$, only we remark that the space ℓ^p, $0 < p < 1$, is not a normed space, but a p-normed space.

To prove this theorem we need the following lemmas.

Lemma 6.4. *Let* $\{\lambda_n^i\}_{n=0}^{\infty}, i = 0, 1, \ldots,$ *be a sequence of monotonically decreasing sequences of non-negative real numbers,* $\lambda_{n+1}^i \leq \lambda_n^i, n = 0, 1, \ldots.$ *A necessary and sufficient condition for the sequence*

$$\eta_m = \inf_{\sum n_i = m} \sup_i \lambda_{n_i}^i \ , \ m = 1, 2, \ldots,$$

to converge to zero as $m \to \infty$ *is that*

(1) $\lim_{i\to\infty} \lambda_0^i = 0$ *and* (2) $\{\lambda_n^i\}_{n=0}^{\infty} \to 0$ *for all* $i = 0, 1, \ldots.$

Proof. Necessity of condition (1). Suppose at first that $\{\lambda_0^i\}$ does not converge to zero. Then it contains a subsequence $\left\{\lambda_0^{i_k}\right\}_{k=1}^{\infty}$ such that $\lambda_0^{i_k} \geq \varepsilon_0 > 0$, for a certain number ε_0 and for $k = 1, 2, \ldots.$ Since for each m and each representation $\sum_{i=0}^{\infty} n_i = m$, the subset $G = \{i : n_i \neq 0\}$ is a finite subset of indices, we have $i_k \notin G$ for all indices i_k except a finite number of indices k. Hence for some k_0 it is true that

$$\sup_i \lambda_{n_i}^i \geq \sup_{i \notin G} \lambda_0^i \geq \lambda_0^{i_{k_0}} \geq \varepsilon_0.$$

Therefore,

$$\eta_m = \inf_{\sum n_i = m} \sup_i \lambda_{n_i}^i \geq \varepsilon_0 \quad \forall\, m = 1, 2, \ldots.$$

Hence η_m does not converge to zero.

Necessity of condition (2). Suppose that $\{\lambda_n^{i_0}\}_{n=0}^{\infty}$ does not converge to zero for some $i = i_0$. The monotonicity of $\{\lambda_n^{i_0}\}_{n=0}^{\infty}$ shows that there exists $\beta > 0$ such that $\lambda_n^{i_0} \geq \beta$ for all $n = 0, 1, 2, \ldots.$ In fact we take $\beta = \inf_n \lambda_n^{i_0} > 0$. In this case we get $\sup_i \lambda_{n_i}^i \geq \lambda_{n_{i_0}}^{i_0} \geq \beta$. Since this is true for any choice of $\sum n_i = m$, we have $\eta_m = \inf_{\sum n_i = m} \sup_i \lambda_{n_i}^i \geq \beta > 0$.

Sufficiency part. Since $\lambda_0^i \to 0$ as $i \to \infty$, for any $\varepsilon > 0$ there exists i_0 such that $\lambda_0^i < \varepsilon$ for any $i \geq i_0$. From the convergence of $\{\lambda_n^i\}$ to zero then for any $i = 0,\ 1,\ 2, \ldots,\ i_0 - 1$ and for any $\varepsilon > 0$, there exists n_i^0 such that $\lambda_n^i < \varepsilon$, for any $n \geq n_i^0$. Taking $m_0 = \sum_{i=0}^{i_0-1} n_i^0$ then we get

$$\begin{aligned} \eta_m &\leq \eta_{m_0} = \inf_{\sum n_i = m_0} (\sup_i \lambda_{n_i}^i) \leq \\ &\leq \max(\sup_{0 \leq i < i_0} \lambda_{n_i^0}^i \ , \ \sup_{i \geq i_0} \lambda_0^i) < \varepsilon \end{aligned}$$

for any $m \geq m_0$. □

Lemma 6.5. *Let* $\{\lambda_n^i\}_{n=0}^{\infty}$, $i = 0, 1, \ldots,$ *be a sequence of monotonically decreasing sequences of non-negative real numbers. A necessary and sufficient condition for the sequence* $\xi_m = \inf_{\sum n_i = m} \sum_{i=0}^{\infty} \lambda_{n_i}^i$ *to converge to zero as* $m \to \infty$ *is that*

(1) $\sum_{i=0}^{\infty} \lambda_0^i < \infty$ *and* (2) $\{\lambda_n^i\}_{n=0}^{\infty} \to 0$ *for all* $i = 0,\ 1, \ldots.$

Proof. Necessity of condition (1). For each m and each representation $\sum_{i=0}^{\infty} n_i = m$, the subset $G = \{i : n_i \neq 0\}$ is a finite subset of indices. Therefore the two series $\sum_{i=0}^{\infty} \lambda_{n_i}^i$ and $\sum_{i=0}^{\infty} \lambda_0^i$ differ in only finite number of terms, so they converge or diverge together. So condition (1) is necessary for the series $\sum_{i=0}^{\infty} \lambda_{n_i}^i$ to be convergent and for ξ_m to exist.

Necessity of condition (2). If $\{\lambda_n^i\}$ does not converge to zero, for some i_0, then there exists $\delta > 0$ such that $\lambda_n^{i_0} \geq \delta > 0$ for all $n = 0, 1, \ldots.$ Hence,

$$\xi_m = \inf_{\sum n_i = m} \sum_{i=0}^{\infty} \lambda_{n_i}^i \geq \lambda_{n_{i_0}}^{i_0} \geq \delta,$$

and ξ_m does not converge to zero.

Sufficiency part. From condition (1) we get that for any $\varepsilon > 0$ there exists i_0 such that $\sum_{i=i_0}^{\infty} \lambda_0^i < \varepsilon/2$. From condition (2) we get that for any $i = 0, 1, \ldots, i_0$, and any $\varepsilon > 0$ there exists $n_{i_0}^0$ such that $\lambda_n^i < \varepsilon/2i_0$ for all $n > n_i^0$. Hence for any

$$m \geq m_0 = \sum_{i=0}^{i_0-1} n_i^0$$

we get

$$\begin{aligned} \xi_m &= \inf_{\sum n_i = m} \sum_{i=0}^{\infty} \lambda_{n_i}^i \leq \\ &\leq \sum_{i=0}^{i_0-1} \lambda_{n_i^0}^i + \sum_{i=i_0}^{\infty} \lambda_0^i < \\ &< \sum_{i=0}^{i_0-1} \frac{\varepsilon}{2i_0} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

□

Remark 6.6. Excluding monotonicity of λ_n^i in Lemmas 6.4 and 6.5 we get, nearly by the same proof, the following.

1. If $\{\lambda_n^i\}_{n=0}^{\infty} \in c_0$ for all i and $\sup_n \lambda_n^i$ converges to zero when i tends to infinity then $\inf_{\sum n_i = m} \sup_i \lambda_{n_i}^i$ converges to zero.

2. If $\{\lambda_n^i\}_{n=0}^{\infty} \in c_0$ for all i and $\sum_{i=0}^{\infty} \lambda_n^i < \infty$ then $\inf_{\sum n_i = m} \sum_{i=0}^{\infty} \lambda_{n_i}^i$ converges to zero.

Lemma 6.7. *Let B_0, $B_1, \ldots$ be bounded subsets of normed spaces X_0, $X_1, \ldots$ with closed unit balls U_{X_0}, $U_{X_1}, \ldots$, respectively. Then the following holds.*

(1) *If* $\{\delta_0(B_i)\} \in \ell^\infty$ *then*

$$\sup_i \delta_m(B_i) \leq \delta_m^\infty(\prod_{i=0}^{\infty} B_i) \leq \inf_{\sum n_i = m} \sup_i \delta_{n_i}(B_i).$$

(2) *If* $\{\delta_0(B_i)\} \in \ell^p$ *then*

$$\delta_m^p(\prod_{i=0}^{\infty} B_i) \leq \inf_{\sum n_i = m} \left(\sum_{i=0}^{\infty} (\delta_{n_i}(B_i))^p \right)^{1/p}.$$

Here $\delta_n^p(C)$ *is the n-th diameter of* C *in the space* $\ell^p(X_i)$.

Proof. (1) From the definition of n-diameters $\delta_{n_i}(B_i)$ we have

$$B_i \subseteq (1+\varepsilon)\delta_{n_i}(B_i)U_{X_i} + F_i, \quad i = 0, 1, 2, \ldots,$$

for some F_i, $\dim F_i \leq n_i$. Thus

$$\prod_{i=0}^{\infty} B_i \subseteq (1+\varepsilon) \prod_{i=0}^{\infty} \delta_{n_i}(B_i)U_{X_i} + \prod_{i=0}^{\infty} F_i.$$

Taking $\sum n_i = m$, then $F_i \neq \{0\}$ for only finite numbers of indices i. In this case $F = \prod_{i=0}^{\infty} F_i$ is a finite dimensional subspace with $\dim F = \sum n_i = m$ and we get

$$\prod_{i=0}^{\infty} B_i \subseteq (1+\varepsilon) \sup_i \delta_{n_i}(B_i) \prod_{i=0}^{\infty} U_{X_i} + F.$$

From Proposition 5.1(2) we get

$$\prod_{i=0}^{\infty} B_i \subseteq (1+\varepsilon) \sup_i \delta_{n_i}(B_i)\, U_{l^\infty(X_i)} + F.$$

Since $\dim F = m$, we get

$$\delta_m^\infty(\prod_{i=0}^{\infty} B_i) \leq (1+\varepsilon) \sup_i \delta_{n_i}(B_i),$$

and since $\varepsilon > 0$ is arbitrary,

$$\delta_m^\infty(\prod_{i=0}^{\infty} B_i) \leq \sup_i \delta_{n_i}(B_i).$$

This is true for any choice of $m = \sum_{i=0}^{\infty} n_i$. Therefore,

$$\delta_m^\infty(\prod_{i=0}^{\infty} B_i) \leq \inf_{\sum n_i = m} \sup_i \delta_{n_i}(B_i).$$

(2) Taking $\sum_{i=0}^{\infty}(\delta_{n_i}(B_i))^p = \mu^p$, we have

$$\sum_{i=0}^{\infty} \frac{(\delta_{n_i}(B_i))^p}{\mu^p} = 1.$$

Similarly to the proof of (1) we get

$$\prod_{i=0}^{\infty} B_i \subseteq (1+\varepsilon) \prod_{i=0}^{\infty} \delta_{n_i}(B_i) U_{X_i} + F.$$

Then

$$\prod_{i=0}^{\infty} B_i \subseteq (1+\varepsilon)\mu \prod_{i=0}^{\infty} \frac{1}{\mu}\, \delta_{n_i}(B_i) U_{X_i} + F.$$

Using Proposition 5.1(2), we get

$$\prod_{i=0}^{\infty} B_i \subseteq (1+\varepsilon)\mu\, U_{\ell^p(X_i)} + F.$$

Therefore

$$\delta_m^p(\prod_{i=0}^{\infty} B_i) \leq (1+\varepsilon) \left(\sum_{i=0}^{\infty} (\delta_{n_i}(B_i))^p \right)^{1/p}.$$

Since ε is arbitrary, we get

$$\delta_m^p(\prod_{i=0}^{\infty} B_i) \leq \left(\sum_{i=0}^{\infty} (\delta_{n_i}(B_i))^p \right)^{1/p}.$$

Since this is true for any choice of $m = \sum n_i$,

$$\delta_m^p(\prod_{i=0}^{\infty} B_i) \leq \inf_{\sum n_i = m} \sum_{i=0}^{\infty} ((\delta_{n_i}(B_i))^p)^{1/p}.$$

□

Proof of the Main Theorem. Proof of part (1) comes from Lemma 6.4 and Lemma 6.7 and the proof of part (2) comes from Lemma 6.5 and Lemma 6.7.

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