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Strong summability defined by p-convex modulus functions and Kuttner's theorem

VIRGE SOOMER

ABSTRACT. The purpose of this paper is to extend Thorpe's generalization of Kuttner's theorem (cf. Theorem K) to strong summability with respect to a p-convex modulus function.

1. Introduction and preliminaries

A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* (or simply a *modulus*), if f is strictly increasing, continuous on $[0, \infty)$, $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$ and $f(0) = 0$.

Let $0 < p \leq 1$. A function $f : [0, \infty) \to [0, \infty)$ is called *p-convex* if

$$f(\alpha t + \beta u) \leq \alpha^p f(t) + \beta^p f(u)$$

for all $t, u \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha^p + \beta^p = 1$.

In this paper we consider p-convex $(0 < p < 1)$ modulus functions. Note that the notion of 1-convex functions coincides with the notion of convex functions.

Example 1. The function $f(t) = t^p$, $0 < p < 1$ is p-convex and it is not r-convex if $r > p$.

Let E be a sequence space and let f be a modulus function. The space $E(f)$ is defined as

$$E(f) = \{x = (\xi_k) : \mathcal{F}(x) = (f(|\xi_k|)) \in E\}. \tag{1}$$

A real functional g on a linear space E is called an *F-norm* if

(i) $g(x) = 0$ if and only if $x = 0$,
(ii) $|\alpha| \leq 1$ $(\alpha \in \mathbb{K}) \Rightarrow g(\alpha x) \leq g(x)$ for all $x \in E$,
(iii) $g(x + y) \leq g(x) + g(y)$ for all $x, y \in E$,
(iv) $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$, $x \in E \Rightarrow \lim_n g(\alpha_n x) = 0$.

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An *F-space* is defined as a complete *F-normed* space. If a sequence space E is an F-space on which the coordinate functionals $\pi_k(x) = \xi_k$ are continuous, then E is called an *FK-space*. An FK-space with normable topology is called a *BK-space*. Some authors include local convexity in the definition of a Fréchet space and of an FK-space. We do not and we follow the definition used by Maddox and by Wilansky (cf. [14]).

Let ϕ be the space of all finite sequences. An F-space E containing ϕ is called an *AK-space*, if $\lim_n \sum_{k=1}^n \xi_k e_k = x$ for all $x = (\xi_k) \in E$.

An F-norm g in a sequence space E is called *absolutely monotone* if $|\xi_k| \le |\eta_k|$, $k \in \mathbb{N}$, implies $g(x) \le g(y)$ for all $x = (\xi_k)$, $y = (\eta_k)$ in E.

Let $g_f(x) = g(\mathcal{F}(x))$. The topologization of the space $E(f)$ was studied by E. Kolk and by the author. According to these results we get

Theorem 1 ([11]). *Let f be a modulus function and let g be an absolutely monotone F-norm on a solid sequence space E. The functional g_f defines an absolutely monotone F-norm on $E(f)$ if the following condition holds:*

(F) *There exists a function ν such that $f(ut) \le \nu(u)f(t)$, $0 < u \le 1$, $t \ge 0$ and $\lim_{u\to 0+} \nu(u) = 0$.*

Remark 1. It is easy to check that condition (F) holds for each p-convex $(0 < p \le 1)$ modulus function.

A sequence space E is called *solid* (or *normal*) if $(\eta_k) \in E$ and $|\xi_k| \le |\eta_k|$ imply $(\xi_k) \in E$.

Theorem 2 ([4]). *If E is a solid AK-FK-space with an absolutely monotone F-norm g, then $E(f)$ is a solid AK-FK-space with an absolutely monotone F-norm g_f.*

Let now $A = (a_{nk})$ be an infinite matrix with $a_{nk} \ge 0$ and let c_A be the summability field of matrix method A, i.e.

$$c_A = \{x = (\xi_k) : A(x) = \lim_n \sum_{k=1}^{\infty} a_{nk}\xi_k \text{ exists}\}.$$

Then, passing to strong summability,

$$[c_A] = \{x = (\xi_k) : \exists \ell, \quad \lim_n \sum_{k=1}^{\infty} a_{nk}|\xi_k - \ell| = 0\}$$

and

$$[c_A]_0 = \{x = (\xi_k) : \lim_n \sum_{k=1}^{\infty} a_{nk}|\xi_k| = 0\}$$

are the spaces of strongly A-summable and strongly A-summable to zero sequences, respectively.

If we put $E = [c_A]_0$ in (1) and f is a modulus, then

$$[c_A]_0(f) = \{x = (\xi_k) : \lim_n \sum_{k=1}^{\infty} a_{nk} f(|\xi_k|) = 0\}.$$

In the case $f(t) = t^p$, $0 < p \leq 1$, we have $[c_A]_0(f) = [c_A]_0^p$, the space of the sequences that are strongly A-summable to zero with index p. By taking $A = (C, 1)$, the Cesàro matrix, and for $0 < p < \infty$ the space $[c_A]_0^p$ is usually denoted by $w_0(p)$, i.e.

$$w_0(p) = \{x = (\xi_k) : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} |\xi_k|^p = 0\}.$$

Let ℓ_∞ denote the space of bounded sequences. Thorpe (cf. [13]) gave the following generalization of Kuttner's theorem.

Theorem K ([13]). *If $0 < p < 1$ and X is a locally convex FK-space, then $X \supset \ell_\infty$ whenever $X \supset w_0(p)$.*

Kuttner [5] proved this result in the case $X = c_A$ where A is a regular matrix method (Kuttner's theorem).

The purpose of this paper is to give some extensions of Theorem K by replacing $w_0(p)$ by $[c_A]_0(f)$.

2. Extension of Kuttner's theorem in the case of $[c_A]_0(f)$

For a sequence space E we denote by E^α and E^β the Köthe-Toeplitz duals of E, i.e.

$$E^\alpha = \{\alpha = (\alpha_k) : \sum_{k=1}^{\infty} |\alpha_k \xi_k| < \infty \text{ for all } (\xi_k) \in E\}$$

and

$$E^\beta = \{\alpha = (\alpha_k) : \sum_{k=1}^{\infty} \alpha_k \xi_k \text{ converges for all } (\xi_k) \in E\}.$$

For an F-normed sequence space E we denote by E' the topological dual of E and in the case $\phi \subset E$, we use the notation

$$E^\varphi = \{(\varphi(e_k)) : \varphi \in E'\}.$$

If the matrix $A = (a_{nk})$ satisfies the condition

(F1) $\sup_n a_{nk} > 0$ for each $k \in \mathbb{N}$,

then $[c_A]_0$ is a solid AK-BK-space with the norm

$$\|x\| = \sup_n \sum_{k=1}^{\infty} a_{nk} |\xi_k|$$

(cf. [1]). Then, by Theorem 2, the space $[c_A]_0(f)$ is a solid AK-FK-space with the F-norm

$$g_f(x) = \sup_n \sum_{k=1}^{\infty} a_{nk} f(|\xi_k|).$$

Since for every solid AK-FK-space E we have

$$E^\alpha = E^\beta = E^\varphi, \tag{2}$$

this is also true for $E = [c_A]_0(f)$.

For a positive matrix method $A = (a_{nk})$ we define

$$B(A,p) = \{x = (\xi_k) : \lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p} |\xi_k| = 0\}.$$

Theorem 3. *Let f be a modulus and let $A = (a_{nk})$ be a positive regular matrix method with finite rows satisfying the conditions* (F1) *and*

(F2) $\sum\limits_{k=1}^{\infty} a_{nk} = 1$ *for each* $n \in \mathbb{N}$.

Then the following statements hold:

(i) *$B(A,p)$ is a solid AK-BK-space with the norm*

$$q(x) = \sup_n \sum_{k=1}^{\infty} a_{nk}^{1/p} |\xi_k|.$$

(ii) *If f is p-convex, then $[c_A]_0(f) \subset B(A,p)$.*

(iii) *$\ell_\infty \subset B(A,p)$ if and only if* $\lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p} = 0$.

Proof. (i) This is well known.

(ii) We use Jensen's inequality: if f is a p-convex function and $\alpha_k \geq 0$, $\sum\limits_{k=1}^{n} \alpha_k^p = 1$, $t_k \geq 0$, then

$$f(\sum_{k=1}^{n} \alpha_k t_k) \leq \sum_{k=1}^{n} \alpha_k^p f(t_k).$$

Taking $\alpha_k = a_{nk}^{1/p}$ and $t_k = |\xi_k|$ we have (note that the matrix A has finite rows and satisfies (F2))

$$f(\sum_{k=1}^{\infty} a_{nk}^{1/p} |\xi_k|) \leq \sum_{k=1}^{\infty} a_{nk} f(|\xi_k|).$$

Then (ii) follows by the properties of modulus functions.

(iii) It is clear (cf. [2], Theorem 2.4.1 (of Schur)) that the matrix method $A_p = (a_{nk}^{1/p})$ sums all bounded sequences if and only if $\lim_n \sum_{k=1}^{n} a_{nk}^{1/p} = 0$. □

The following theorem gives an extension principle of Kuttner's theorem.

Theorem 4. *Let X be a locally convex FK-space. If the matrix method A and the modulus f satisfy conditions of Theorem 3 and $([c_A]_0(f))^{\varphi} \subset (B(A,p))^{\varphi}$, then the condition $\lim_n \sum_{k=1}^{\infty} a_{nk}^{1/p} = 0$ is sufficient for the implication*

$$X \supset [c_A]_0(f) \Rightarrow X \supset \ell_\infty.$$

Proof. Suppose that $X \supset [c_A]_0(f)$, then $X^{\varphi} \subset ([c_A]_0(f))^{\varphi}$ and by the respective assumption of the present theorem also $X^{\varphi} \subset (B(A,p))^{\varphi}$. Since by (i) of Theorem 3 the BK-space $B(A,p)$ is an AK-space and hence also an AD-space (i.e. ϕ is dense in $B(A,p)$), $X \supset B(A,p)$ follows from Theorem 4 of [10]. Thus, by Theorem 3 (iii), we have $X \supset \ell_\infty$. □

Remark 2. If $[c_A]_0(f)$ is a closed subspace of $B(A,p)$, then $([c_A]_0(f))^{\varphi} \subset (B(A,p))^{\varphi}$ (cf. [15], 7.2.7).

We see that for extending of Kuttner's theorem it is essential to know the spaces $([c_A]_0(f))^{\varphi}$ and $(B(A,p))^{\varphi}$. But we have not much information about these spaces. In the following part of this paper we give an extension of Theorem K for a certain class of matrix methods.

3. Extension of Kuttner's theorem in the special case $N_\Theta^0(f)$

An increasing sequence $\Theta = (k_r)$ of non-negative integers is called a *lacunary sequence* if $k_0 = 0$ and $\lim_r (k_{r+1} - k_r) = \infty$. We use the notation

$$h_r = (k_{r+1} - k_r), \quad \sum_{(r)} = \sum_{k=k_r}^{k_{r+1}-1}, \quad \max_{(r)} = \max_{k_r \le k \le k_{r+1}-1}.$$

The space N_Θ^0 is defined as

$$N_\Theta^0 = \{x = (\xi_k) : \lim_r \frac{1}{h_r} \sum_{(r)} |\xi_k| = 0\}.$$

Then N_Θ^0 is the strong null-summability field of the matrix method $A_\Theta = (a_{rk}^\Theta)$ where

$$a_{rk}^\Theta = \begin{cases} 1/h_r & \text{for } k_r \le k \le k_{r+1} - 1, \\ 0 & \text{otherwise} \end{cases} \qquad (r, k \in \mathbb{N}).$$

Since A_Θ fulfils (F1), N_Θ^0 is a solid AK-BK-space with the norm

$$\|x\|_\Theta = \sup_r \frac{1}{h_r} \sum_{(r)} |\xi_k|.$$

In the special case $\Theta = (2^r)$, we have $N_\Theta^0 = w_0(1)$ and the norm $\|x\|_\Theta$ is equivalent to the usual norm $\|x\| = \sup_n (n+1)^{-1} \sum_{k=0}^{n} |\xi_k|$ in $w_0(1)$

(cf. [6], Chapter 7). By Theorem 2 we get that $N^0_\Theta(f)$ is a solid *AK-FK*-space with the F-norm

$$g_f(x) = \sup_r \frac{1}{h_r} \sum_{(r)} f(|\xi_k|).$$

The space N^0_Θ was first studied in [3] and the spaces like $N^0_\Theta(f)$ are considered, for instance, in [9].

We define

$$M_\Theta(p) = \{\alpha = (\alpha_k) : \sum_{r=0}^{\infty} h_r^{1/p} \max_{(r)} |\alpha_k| < \infty\}. \tag{3}$$

Theorem 5. *Let f be an unbounded p-convex modulus function satisfying the condition*

(F3) $f(t^{1/p}) = O(t), \quad t \to \infty.$

Then

$$(N^0_\Theta(f))^\alpha = M_\Theta(p).$$

Proof. 1) Let $x = (\xi_k) \in N^0_\Theta(f)$, $\alpha = (\alpha_k) \in M_\Theta(p)$ and let f^{-1} be the inverse function of f. Let $A_{rk} = |\alpha_k| h_r^{1/p}$ $(r, k \in \mathbb{N})$. Then

$$\sum_{(r)} |\alpha_k \xi_k| \le \max_{(r)} A_{rk} \frac{1}{h_r^{1/p}} \sum_{(r)} |\xi_k| = \max_{(r)} A_{rk} f^{-1}[f(\frac{1}{h_r^{1/p}} \sum_{(r)} |\xi_k|)].$$

Applying Jensen's inequality we have

$$\sum_{(r)} |\alpha_k \xi_k| \le \max_{(r)} A_{rk} f^{-1}[\frac{1}{h_r} \sum_{(r)} f(|\xi_k|)] = \max_{(r)} A_{rk} f^{-1}[g_f(x)]$$

and

$$\sum_{r=0}^{\infty} |\alpha_r \xi_r| = \sum_{r=0}^{\infty} \sum_{(r)} |\alpha_k \xi_k| \le f^{-1}[g_f(x)] \sum_{r=0}^{\infty} h_r^{1/p} \max_{(r)} |\alpha_k| < \infty.$$

Hence $\alpha = (\alpha_k) \in (N^0_\Theta(f))^\alpha$ and thus $M_\Theta(p) \subset (N^0_\Theta(f))^\alpha$.

2) Suppose that $\alpha = (\alpha_k) \notin M_\Theta(p)$. Then the series in (3) is divergent, and therefore there exists a sequence (b_r), $0 < b_r \to 0$, $r \to \infty$ such that

$$\sum_{r=0}^{\infty} b_r h_r^{1/p} \max_{(r)} |\alpha_k| = \infty. \tag{4}$$

Let $\max_{(r)} |\alpha_k| = |\alpha_{k_r}|$ and let $\tilde{x} = (\tilde{\xi}_k)$ be defined by

$$\tilde{\xi}_k = \begin{cases} b_r h_r^{1/p} & \text{for } k = k_r, \\ 0 & \text{for } k \ne k_r \end{cases} \quad (r, k \in \mathbb{N}).$$

Since $b_r \to 0$, $r \to \infty$, we have $b_r < 1$ for sufficiently large r. Now by p-convexity of f, by the condition $f(0) = 0$ and by (F3) we have

$$\frac{1}{h_r}\sum_{(r)} f(|\xi_k|) = \frac{1}{h_r} f(b_r h_r^{1/p}) \leq \frac{b_r^p f(h_r^{1/p})}{h_r} = o(1), \quad r \to \infty.$$

Hence $x \in N_\Theta^0(f)$. But $\sum_{(r)} |\alpha_k \tilde{\xi}_k| = |\alpha_{k_r}| b_r h_r^{1/p}$ so that by (4) the series $\sum_{k=0}^{\infty} |\alpha_k \tilde{\xi}_k|$ diverges and therefore $(\alpha_k) \notin (N_\Theta^0(f))^\alpha$. This completes the proof. □

Theorem 6. *Let X be a locally convex FK-space and let f be a p-convex unbounded modulus satisfying the condition* (F3). *Then the following statements hold:*
(i) $(N_\Theta^0(f))^\varphi \subset (B(A_\Theta, p))^\varphi$,
(ii) $X \supset N_\Theta^0(f) \Longrightarrow X \supset l_\infty$.

Proof. (i) Since $N_\Theta^0(f)$ and $B(A_\Theta, p)$ are solid AK-FK-spaces, their α-duals and φ-duals are equal and so it is sufficient to prove $(N_\Theta^0(f))^\alpha \subset (B(A_\Theta, p))^\alpha$. By Theorem 5 it is sufficient to show $M_\Theta(p) \subset (B(A_\Theta, p))^\alpha$. Let $\alpha = (\alpha_k) \in M_\Theta(p)$, then for each $x = (\xi_k) \in B(A_\Theta, p)$ we have

$$\sum_{r=0}^{\infty} |\alpha_r \xi_r| = \sum_{r=0}^{\infty}\sum_{(r)} |\alpha_k \xi_k| \leq \sum_{r=0}^{\infty} h_r^{1/p} \max_{(r)} |\alpha_k| \frac{1}{h_r^{1/p}} \sum_{(r)} |\xi_k|$$
$$\leq q(x) \sum_{r=0}^{\infty} h_r^{1/p} \max_{(r)} |\alpha_k| < \infty$$

which implies that $(\alpha_k) \in (B(A_\Theta, p))^\varphi$.

(ii) The matrix $A_\Theta = (a_{nk}^\Theta)$ satisfies conditions of Theorem 3, $(N_\Theta^0(f))^\varphi \subset (B(A_\Theta, p))^\varphi$ by (i) and $\lim_r \sum_{(r)} (a_{rk}^\Theta)^{1/p} = \lim_r h_r^{1-1/p} = 0$. Consequently the statement (ii) of the present theorem follows immediately by Theorem 4. □

A generalization of the space $N_\Theta^0(f)$, namely the space

$$N_\Theta^0(\mathcal{F}) = \{x = (\xi_k) : \mathcal{F}(x) = (f_k(|\xi_k|)) \in N_\Theta^0\},$$

where $\mathcal{F} = (f_k)$ is a sequence of modulus functions, was studied in [12]. According to the assumptions which are different from the assumptions of Theorem 6 in the present paper, there have been proved some extensions of Kuttner's theorem in the case of $N_\Theta^0(\mathcal{F})$. Note that in the case $f_k(t) = t^p$ for each $k \in \mathbb{N}$, $0 < p < 1$, Theorem 6 in the present paper and Theorems 7 and 8 in [12] give the same result.

A generalization of Theorem K for non-constant $p = (p_k)$ was given by Maddox [7]. In the case $\omega_0(1)(f)$ the extension of Theorem K was proved by Maddox in [8].

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Institute of Pure Mathematics, University of Tartu, J. Liivi 2-603, 50409 Tartu, Estonia

E-mail address: soomer@math.ut.ee