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Remarks on transfinite sequences of functions that preserve convergence

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ABSTRACT. In the paper [6] the author investigates sequences of functions $(f_n)_{n=1}^{\infty}$ preserving the convergence of sequences which are alternatively called conservative sequences of functions. In this note we extend this concept to the transfinite sequences of functions and prove that in this case the property of being conservative is equivalent to the property of being poinwise convergent. Then we introduce the concept of p-locally convergence for transfinite sequences of functions and show that if X is a locally Lindelöf first-countable T_1-space and a transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converges p-locally uniformly to a function $f : X \to \mathbb{R}$, then $C(f) = \operatorname{Lim} C(f_\xi)$ where $C(f)$ denotes the set of all continuity points of f.

1. Introduction

The convergence of transfinite sequences of functions was introduced in the paper [18] and studied in several other papers (see e.g. [1], [5], [13], [14], [15], [16], [17]) where, first of all, the closedness of various important classes of functions with respect to pointwise convergence of transfinite sequences was investigated.

We start with some basic definitions and properties of transfinite sequences.

Let X be a non-empty set and Ω be the first uncountable ordinal. The sequence $(x_\xi)_{\xi<\Omega}$ where $x_\xi \in X$ for each $\xi < \Omega$ is called a *transfinite sequence in* X (i.e. it is a function defined on the set of all countable ordinals with values in X).

A transfinite sequence $(x_\xi)_{\xi<\Omega}$ of elements of a metric space (X, ϱ) is said to *converge to* x *in* (X, ϱ) if for any $\varepsilon > 0$ there exists an ordinal μ, $\mu < \Omega$,

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such that for any ordinal ξ, $\mu \le \xi < \Omega$, $\varrho(x_\xi, x) < \varepsilon$. We then say that x is a *limit of* $(x_\xi)_{\xi<\Omega}$ and write $\lim_{\xi\to\Omega} x_\xi = x$ or $x_\xi \to x$.

It is easy to verify (see e.g. [17], [18]) that the following statement holds.

Proposition 1. *If (X, ϱ) is a metric space and a transfinite sequence $(x_\xi)_{\xi<\Omega}$ converges to x in (X, ϱ), then there exists an ordinal μ, $\mu < \Omega$, such that $x_\xi = x$ for each ξ with $\xi \ge \mu$.*

Let X be a set. A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ of functions is said to *converge pointwise (on X) to a function* $f : X \to \mathbb{R}$ provided that for each $x \in X$ $\lim_{\xi\to\Omega} f_\xi(x) = f(x)$. The function f is said to be a *limit* of $(f_\xi)_{\xi<\Omega}$. We write $\lim_{\xi\to\Omega} f_\xi = f$ or $f_\xi \to f$.

We next extend the notion of convergence-preserving (or conservative) sequence of functions investigated in [6] and [19] to the case of transfinite sequences as follows.

Definition 1. Let (X, ϱ) be a metric space. A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ is said to be (transfinite) *convergence-preserving* or *conservative* if for any convergent transfinite sequence $(x_\xi)_{\xi<\Omega}$ in (X, ϱ) the transfinite sequence $(f_\xi(x_\xi))_{\xi<\Omega}$ also converges (in $\mathbb{R}$).

Remark 1. Note that instead of the term "conservative sequence of functions" the term "continuously convergent sequence of functions" is also used (see [19]).

The aim of this paper is to compare the properties of conservative sequences of functions presented in [6] with the properties of analogously defined conservative transfinite sequences of functions. Several related questions are also discussed.

2. Results

It is known (see [6]) that a sequence $(f_n)_{n=1}^{\infty}$, $f_n : [a, b] \to \mathbb{R}$ ($[a, b] \subseteq \mathbb{R}$) is conservative if and only if $(f_n)_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to a continuous function. For transfinite sequences of functions we have

Proposition 2. *Let (X, ϱ) be a metric space. A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ is conservative on (X, ϱ) if and only if it converges pointwise on X.*

Proof. Suppose that $(f_\xi)_{\xi<\Omega}$ is conservative on (X, ϱ) and $t \in X$. The constant transfinite sequence $(t_\xi)_{\xi<\Omega}$ with $t_\xi = t$ for each $\xi < \Omega$ converges in (X, ϱ) and since $f_\xi(t_\xi) = f_\xi(t)$ holds for each $\xi < \Omega$ we obtain that $(f_\xi)_{\xi<\Omega}$ converges in $\mathbb{R}$. Put $f(t) = \lim_{\xi\to\Omega} f_\xi(t)$ for every $t \in X$. Then $(f_\xi)_{\xi<\Omega}$ converges pointwise to f.

Conversely, let $(f_\xi)_{\xi<\Omega}$ converge pointwise to f on X and $(t_\xi)_{\xi<\Omega}$ converge to t in (X, ϱ). Then, by Proposition 1, there exists $\mu < \Omega$ such that $t_\xi = t$

for each $\xi \geq \mu$.Hence, for each $\xi \geq \mu$ we have $f_\xi(t_\xi) = f_\xi(t)$ and therefore $f_\xi(t_\xi) \to f(t)$. □

Remark 2. The result stated in Proposition 2 is mentioned in [7].

Note that in several cases the pointwise convergent transfinite sequences of functions behave similarly to the uniformly convergent sequences of functions. For instance (see [17], [18]), if a transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converges pointwise to $f : X \to \mathbb{R}$, (X, ϱ) is a metric space and for every $\xi < \Omega$ the function f_ξ is continuous, then f is also continuous. If $I \subseteq \mathbb{R}$ is an interval, every $f_\xi : I \to \mathbb{R}$, $\xi < \Omega$, is differentiable at every $t \in I$, $f, g : I \to \mathbb{R}$ are such functions that $f_\xi \to f$ and $f'_\xi \to g$, then $f' = g$ (see [1]). These results can be slightly improved in the following way.

Lemma 1. *Let $M \subseteq \mathbb{R}$ and a be a limit point of M. Let $(f_\xi : M \to \mathbb{R})_{\xi<\Omega}$ be a transfinite sequence of functions which converges pointwise to a function $f : M \to \mathbb{R}$ and $\lim_{x\to a} f_\xi(x) = b_\xi \in \mathbb{R}$ for each $\xi < \Omega$. Then there exists $\lim_{x\to a} f(x) = b \in \mathbb{R}$ and $b_\xi \to b$.*

Proof. Let $(a_n)_{n=1}^\infty$ be an arbitrary sequence in $M \setminus \{a\}$ which converges to a. Clearly, the sequence $(f_\xi(a_n))_{n=1}^\infty$ converges to b_ξ for each $\xi < \Omega$. Since $f_\xi \to f$, for any $n \in \mathbb{N}$ there exists $\mu_n < \Omega$ with $f_\xi(a_n) = f(a_n)$ for $\xi \geq \mu_n$. Obviously, there exists $\mu < \Omega$ such that $\mu_n \leq \mu$ for all $n \in \mathbb{N}$. Then for every $\xi \geq \mu$ and $n \in \mathbb{N}$ we have $f_\xi(a_n) = f(a_n) = f_\mu(a_n)$ and therefore $b_\xi = b_\mu$ and $f(a_n) \to b_\mu$. Hence, $\lim_{x\to a} f(x) = b_\mu$ and, clearly, $b_\xi \to b_\mu$. □

As a consequence of Lemma 1 we obtain

Theorem 1. *Let $(f_\xi : I \to \mathbb{R})_{\xi<\Omega}$ be a transfinite sequence of functions differentiable at every point of an interval $I \subseteq \mathbb{R}$ which converges pointwise to $f : I \to \mathbb{R}$. Then f is also differentiable at every $t \in I$ and $f'_\xi \to f'$.*

Proof. Let $a \in I$ be an arbitrary point. For each $x \in I \setminus \{a\}$ put

$$F(x) = \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad F_\xi(x) = \frac{f_\xi(x) - f_\xi(a)}{x - a} \quad \text{for} \quad \xi < \Omega.$$

It is easy to verify that the transfinite sequence $(F_\xi : I \setminus \{a\} \to \mathbb{R})_{\xi<\Omega}$ converges pointwise to the function $F : I \setminus \{a\} \to \mathbb{R}$ (on $I \setminus \{a\}$). For every $\xi < \Omega$ we have $\lim_{x\to a} F_\xi(x) = f'_\xi(a) \in \mathbb{R}$. According to Lemma 1 there exists

$$\lim_{x\to a} F(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a) \in \mathbb{R}.$$

Moreover, $f'_\xi(a) \to f'(a)$. □

Corollary 1. *If $I \subseteq \mathbb{R}$ is an interval, $(f_\xi : I \to \mathbb{R})_{\xi<\Omega}$ is a transfinite sequence of functions with derivatives of all orders at every $t \in I$ and $f_\xi \to f$ on I, then f has also derivatives of all orders at every $t \in I$ and $f_\xi^{(n)} \to f^{(n)}$ for all $n \in \mathbb{N}$.*

Now consider another property of conservative sequences of functions. Namely: if a sequence $(f_n)_{n=1}^{\infty}$ is conservative on an interval $[a,b] \subseteq \mathbb{R}$, then the limit function $f : [a,b] \to \mathbb{R}$ given by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous. Conservative transfinite sequences of functions fail to have this property as the following simple observation shows. If $(f_\xi : [a,b] \to \mathbb{R})_{\xi<\Omega}$ is a constant transfinite sequence, i.e. $f_\xi = f$ for every $\xi < \Omega$, then $(f_\xi)_{\xi<\Omega}$ converges pointwise to f and therefore it is conservative. Hence, any $f : [a,b] \to \mathbb{R}$ is a limit of a conservative transfinite sequence of functions.

In this connection it seems to be interesting to find some conditions which guarantee the continuity of the limit function of a conservative (i.e. pointwise convergent) transfinite sequence of functions.

In the following investigations we need some stronger types of convergence of transfinite sequences of functions which were originally introduced for the usual sequences of functions (see [2], [4]).

Definition 2. Let (X, ϱ) be a metric space.

(a) A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ is said to *converge uniformly to a function* $f : X \to \mathbb{R}$ (on X) provided that for every $\varepsilon > 0$ there exists an ordinal $\mu < \Omega$ such that $|f_\xi(x) - f(x)| < \varepsilon$ holds for every $\xi \geq \mu$ and every $x \in X$.

(b) A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ is said to *converge locally uniformly to* $f : X \to \mathbb{R}$ (on X) provided that for every $x \in X$ there exists a neighbourhood U of x such that the transfinite sequence $(f_\xi|_U)_{\xi<\Omega}$ converges uniformly to $f|_U$ (on U).

(c) A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ is said to *converge p-locally uniformly to* $f : X \to \mathbb{R}$ (on X) if for each $p \in X$ and $\varepsilon > 0$ there exist a neighbourhood U of p and an ordinal $\mu < \Omega$ such that $|f_\xi(t) - f(t)| < \varepsilon$ holds for each $\xi \geq \mu$ and $t \in U$.

Since the uniform convergence of transfinite sequences of functions is a convergence with respect to the metric σ defined by

$$\sigma(f,g) = \min\{\sup_{x\in X} |f(x) - g(x)|, 1\},$$

from Proposition 1 we obtain

Proposition 3. *A transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converges uniformly to $f : X \to \mathbb{R}$ (on X) if and only if there exists an ordinal $\mu < \Omega$ such that $f_\xi = f$ for all $\xi \geq \mu$.*

It is obvious that if a transfinite sequence $(f_\xi)_{\xi<\Omega}$ converges locally uniformly to a function f on X, then it converges also p-locally uniformly to f on X (and, of course, the uniform convergence implies the locally uniform convergence). The converse is not true as the following example shows.

Example 1. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, B be the set of all countable ordinals and $p \notin A \times B$. Define a metric ϱ on the set $X = (A \times B) \cup \{p\}$ as follows:

$\varrho((\frac{1}{n},\xi),p) = \varrho(p,(\frac{1}{n},\xi)) = \frac{1}{n}$, $\varrho((\frac{1}{n},\xi),(\frac{1}{m},\eta)) = \frac{1}{n} + \frac{1}{m}$ for $(\frac{1}{n},\xi) \neq (\frac{1}{m},\eta)$ and $\varrho(x,x) = 0$ for all $x \in X$. It is easy to see that ϱ is indeed a metric on X. Since $\varrho((\frac{1}{n},\xi),(\frac{1}{m},\eta)) > \frac{1}{n}$ for all $(\frac{1}{m},\eta) \in A \times B$ with $(\frac{1}{m},\eta) \neq (\frac{1}{n},\xi)$ and $\varrho((\frac{1}{n},\xi),p) = \frac{1}{n}$ we obtain that all elements of the set $A \times B$ are isolated points in (X,ϱ). Now, for every $\xi < \Omega$ define $f_\xi : X \to \mathbb{R}$ by $f_\xi(p) = 0$, $f_\xi(\frac{1}{n},\eta) = 0$ for $\eta \leq \xi$ and $n \in \mathbb{N}$, and $f_\xi(\frac{1}{n},\eta) = \frac{1}{n}$ otherwise. The transfinite sequence $(f_\xi)_{\xi<\Omega}$ evidently converges p-locally uniformly to the function $f : X \to \mathbb{R}$ with $f(x) = 0$ for all $x \in X$. In fact, let $x \in X$ and $\varepsilon > 0$. If $x \neq p$, then we can use the neighbourhood $\{x\}$ of x. If $x = p$, then we take the neighbourhood $B_\varepsilon(p) = \{x \in X : \varrho(x,p) < \varepsilon\}$. For every $x \in B_\varepsilon(p)$, $x \neq p$ we have $x = (\frac{1}{n},\eta)$ and $\frac{1}{n} < \varepsilon$. Then $f_\xi(x) = \frac{1}{n}$ or $f_\xi(x) = 0$ for each $\xi < \Omega$ and therefore $|f_\xi(x) - f(x)| \leq \frac{1}{n} < \varepsilon$. On the other hand, let $\varepsilon > 0$, $\xi < \eta < \Omega$ and $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Then $(\frac{1}{n},\eta) \in B_\varepsilon(p)$ and $f_\xi(\frac{1}{n},\eta) = \frac{1}{n} \neq 0 = f_\eta(\frac{1}{n},\eta)$. Hence, for any $\varepsilon > 0$ we obtain that $(f_\xi|_{B_\varepsilon(p)})_{\xi<\Omega}$ does not converge uniformly to $f|_{B_\varepsilon(p)}$ on $B_\varepsilon(p)$, i. e. $(f_\xi)_{\xi<\Omega}$ does not converge locally uniformly to f on X.

It is also easy to see that if a transfinite sequence $(f_\xi)_{\xi<\Omega}$ converges p-locally uniformly to f on X, then it converges pointwise to f on X. The next example shows that the converse is not true.

Example 2. For every $\xi < \Omega$ define $A_\xi \subseteq R$ as follows: $A_0 = \mathbb{Q}$ (the set of all rational numbers). If $0 < \gamma < \Omega$, then $A_\gamma = \mathbb{Q} + a_\gamma = \{r + a_\gamma : r \in \mathbb{Q}\}$ where $a_\gamma \in (\mathbb{R} \setminus \bigcup_{\beta<\gamma} A_\beta)$. It is obvious that for every $\xi < \Omega$ the set A_ξ is a dense subset of $\mathbb{R}$ and $A_\xi \cap A_\zeta = \emptyset$ for every ξ, ζ, $0 \leq \xi < \zeta < \Omega$. Let $\xi < \Omega$ and $f_\xi : \mathbb{R} \to \mathbb{R}$ be the function defined by $f_\xi(t) = 1$ for all $t \in A_\xi$ and $f_\xi(t) = 0$ otherwise. It can be easily verified that $f_\xi \to f$ where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(t) = 0$ for all $t \in \mathbb{R}$ and $(f_\xi)_{\xi<\Omega}$ does not converge p-locally uniformly to f on X.

The usefulness of the notion of p-locally uniform convergence for transfinite sequences of functions is presented in the following assertion, where by ω_g we denote the oscillation of g.

Theorem 2. *Let (X,ϱ) be a metric space, $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converge p-locally uniformly to $f : X \to \mathbb{R}$ and for each $p \in X$, $\varepsilon > 0$ and $\beta < \Omega$ there exist γ, $\beta \leq \gamma < \Omega$, such that $\omega_{f_\gamma}(p) < \varepsilon$. Then f is continuous on (X,ϱ).*

Proof. Let $p \in X$ and $\varepsilon > 0$. Since $(f_\xi)_{\xi<\Omega}$ converges p-locally uniformly to f there exist a neighbourhood U_1 of p and $\beta < \Omega$ such that $|f_\xi(t) - f(t)| < \frac{\varepsilon}{3}$ holds for each $t \in U_1$ and $\xi \geq \beta$. Choose $\gamma > \beta$ with $\omega_{f_\gamma}(p) < \frac{\varepsilon}{3}$. Then there exists a neighbourhood U_2 of p such that $|f_\gamma(t) - f_\gamma(p)| < \frac{\varepsilon}{3}$ for each $t \in U_2$. For each $t \in U_1 \cap U_2$ we have

$$|f(t) - f(p)| \leq |f(t) - f_\gamma(t)| + |f_\gamma(t) - f_\gamma(p)| + |f_\gamma(p) - f(p)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

Hence, f is continuous at p. □

Recall that if $(A_\xi)_{\xi<\Omega}$ is a transfinite sequence of subsets of a set X, then

$$\text{Lim inf } A_\xi = \{x \in X : \exists_{\mu<\Omega}\ \forall_{\mu\le\xi<\Omega}\quad x \in A_\xi\}$$

and

$$\text{Lim sup } A_\xi = \{x \in X : \forall_{\mu<\Omega}\ \exists_{\mu\le\xi<\Omega}\quad x \in A_\xi\}.$$

If, moreover, Lim inf A_ξ = Lim sup $A_\xi = A$, then the set A is called a *transfinite limit of* $(A_\xi)_{\xi<\Omega}$ and we write $A = \text{Lim } A_\xi$. Denote by $C(g)$ the set of all continuity points of the function g. In [9] the following statement is proved.

Theorem 3. *Let X be a locally compact first-countable topological space, Y be a metric space and a transfinite sequence $(f_\xi : X \to Y)_{\xi<\Omega}$ converge locally uniformly to $f : X \to Y$. Then* Lim $C(f_\xi) = C(f)$.

Remark 3. Consider the transfinite sequence $(f_\xi : \mathbb{R} \to \mathbb{R})_{\xi<\Omega}$ and $f : \mathbb{R} \to \mathbb{R}$ from Example 2. Since $C(f_\xi) = \emptyset$ for all $\xi < \Omega$ we obtain Lim $C(f_\xi) = \emptyset$ while $C(f) = \mathbb{R}$. Hence, Lim $C(f_\xi) \neq C(f)$ and $f_\xi \to f$. This shows that in Theorem 3 the locally uniform convergence of $(f_\xi)_{\xi<\Omega}$ cannot be weakened to the pointwise convergence.

In general, the following holds.

Proposition 4. *Let (X, ϱ) be a metric space and $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converge pointwise to $f : X \to \mathbb{R}$. Then* Lim sup $C(f_\xi) \subseteq C(f)$.

Proof. Let $x \in$ Lim sup $C(f_\xi)$. Then for each $\mu < \Omega$ there exists $\gamma \geq \mu$ with $x \in C(f_\gamma)$. We can construct a transfinite sequence of ordinals $\gamma_0 < \gamma_1 < \ldots < \gamma_\alpha < \ldots$ $(\alpha < \Omega)$ cofinal in the set of all countable ordinals such that all functions f_{γ_α} are continuous at x. Since the transfinite sequence $(f_{\gamma_\alpha})_{\alpha<\Omega}$ converges pointwise to f we obtain that f is also continuous at x (see [17]). □

We want to conclude our paper with a slight generalization of Theorem 3. We start with the following observation.

Proposition 5. *Let (X, ϱ) be a separable metric space, $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ be a transfinite sequence and $f : X \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(a) *$(f_\xi)_{\xi<\Omega}$ converges uniformly to f on X.*
(b) *$(f_\xi)_{\xi<\Omega}$ converges locally uniformly to f on X.*
(c) *$(f_\xi)_{\xi<\Omega}$ converges p-locally uniformly on X.*

Proof. It suffices to prove that (c) implies (a). Let $n \in \mathbb{N}$. Then for every $p \in X$ there exist an open neighborhood U_p of p and an ordinal $\mu_n(p) < \Omega$ such that $|f_\xi(t) - f(t)| < \frac{1}{n}$ holds for every $t \in U_p$ and $\xi \geq \mu_n(p)$. The collection $\{U_p : p \in X\}$ is an open cover of (X, ϱ) and since (X, ϱ) is separable there exists a sequence $(p_k)_{k=1}^{\infty}$ in X such that $\bigcup_{k\in\mathbb{N}} U_{p_k} = X$. Choose $\mu_n < \Omega$ with $\mu_n > \mu_n(p_k)$ for all $k \in \mathbb{N}$. Then for every $t \in X$ we have $t \in U_{p_k}$ for a suitable k and $|f_\xi(t) - f(t)| < \frac{1}{n}$ for all $\xi \geq \mu_n$. Finally, let $\mu < \Omega$ and $\mu \geq \mu_n$ for all $n \in \mathbb{N}$. Then for every $t \in X$ and $\xi \geq \mu$ we obtain that $f_\xi(t) = f(t)$, i. e. $f_\xi = f$. Hence, $(f_\xi)_{\xi<\Omega}$ converges uniformly to f on X. □

Observe that in the proof of Proposition 5 the Lindelöf property of the space (X, ϱ) was essential. In the class of metric spaces separability is equivalent to the Lindelöf property but this does not hold for more general topological spaces. For instance, if S denotes the Sorgenfrey line, then the space $S \times S$ is separable without being Lindelöf. The notions of convergence of transfinite sequences of real functions defined on metric spaces studied above can be in an obvious way extended to the more general case of real functions defined on topological spaces. It is easy to verify that Propositions 1, 2 and 4 remain valid after replacing a metric space (X, ϱ) by a first-countable T_1-space X and in Proposition 5 a separable metric space (X, ϱ) can be replaced by a Lindelöf first-countable T_1-space X. Recall that a topological space X is called *locally Lindelöf* provided that for each $x \in X$ there exists a neighbourhood U of x such that the subspace of X determined by the closure $\overline{U}$ of U is a Lindelöf space. It is easy to see that as a consequence of Proposition 5 modified in the above-mentioned sense we obtain

Proposition 6. *If X is a locally Lindelöf first-countable T_1-space and a transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converges p-locally uniformly to $f : X \to \mathbb{R}$ on X, then it converges locally uniformly to f on X.*

Next we prove a theorem which is a generalization of Theorem 3.

Theorem 4. *Let X be a locally Lindelöf first-countable T_1-space and a transfinite sequence $(f_\xi : X \to \mathbb{R})_{\xi<\Omega}$ converge p-locally uniformly to $f : X \to \mathbb{R}$ on X. Then $C(f) = \mathrm{Lim}\, C(f_\xi)$.*

Proof. Since $(f_\xi)_{\xi<\Omega}$ converges poinwise to f, according to (modified as mentioned above) Proposition 4 we obtain that $\mathrm{Lim\,sup}\, C(f_\xi) \subseteq C(f)$. It suffices to verify that $C(f) \subseteq \mathrm{Lim\,inf}\, C(f_\xi)$. Let $x \in C(f)$. According to Proposition 6 $(f_\xi)_{\xi<\Omega}$ converges locally uniformly to f on X and it follows that there exists a neighbourhood U of x for which $(f_\xi|_{\overline{U}})_{\xi<\Omega}$ converges uniformly to $f|_{\overline{U}}$ on $\overline{U}$. Then (see Proposition 3) there exists $\mu < \Omega$ such that $f_\xi|_{\overline{U}} = f|_{\overline{U}}$ for each $\xi \geq \mu$ and therefore $x \in C(f_\xi)$. Hence, $x \in \mathrm{Lim\,inf}\, C(f_\xi)$ and the proof is complete. □

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