ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 8, 2004

Optimum chemical balance weighing design with diagonal matrix of errors based on balanced block designs

BRONISŁAW CERANKA AND MAŁGORZATA GRACZYK

ABSTRACT. The paper is studying the estimation problem of individual weights of objects using the chemical balance weighing design under the restriction on the number of times in which each object is weighed. Additionally, we assume that the errors are uncorrelated and they have different variances. There are given the lower bound of the variance of each of the estimated weights and the sufficient and necessary conditions under which this lower bound is attained. For the new construction method for optimum chemical balance weighing design, we use the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs.

1. Introduction

The chemical balance weighing design is the design which permits to determine unknown measurements of p objects using n measurement operations according to the experimental model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}, \tag{1}$$

where $\mathbf{X}$ is the design matrix which describes how to place the objects on the pans of the balance, $\mathbf{X} \in \Phi_{n\times p,m}(-1,0,1)$ the class of the $n\times p$ matrices $\mathbf{X}$ with elements equal to -1, 1 or 0 if the object is placed on the left pan, right pan or it is not included in the particular measurement operation, respectively. We denote m as the maximum number of elements equal to -1 and 1 in each column of the matrix $\mathbf{X}$, in other words, the maximum number times in which each object is weighed, $\mathbf{y}$ is the $n\times 1$ random observed vector of the recorded results of weights, $\mathbf{w}$ is a $p\times 1$ vector representing unknown

Received October 5, 2003.
2000 *Mathematics Subject Classification.* 62K15.
Key words and phrases. Balanced bipartite weighing design, balanced incomplete block design, chemical balance weighing design.

weights of objects and $\mathbf{e}$ is the $n\times 1$ random vector of errors. We assume that $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, and the errors are uncorrelated and they have different variances, i.e. $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is the $n \times 1$ vector of zeros, $\mathbf{G}$ is the $n \times n$ positive definite diagonal matrix.

For the estimation of individual unknown weights of the objects we use the normal equations

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\widehat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}, \tag{2}$$

where $\widehat{\mathbf{w}}$ is the vector of the weights estimated by the least squares method. The chemical balance weighing design is singular or nonsingular depending on whether the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. It is obvious that because of the assumption connected with $\mathbf{G}$, the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, i.e. if and only if $\mathbf{X}$ is of full column rank $(= p)$.

If $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, the least squares estimator of $\mathbf{w}$ is given in the form

$$\widehat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y} \tag{3}$$

and the variance-covariance matrix of $\widehat{\mathbf{w}}$ is given by formula

$$\mathrm{Var}(\widehat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}. \tag{4}$$

Some problems connected with the chemical balance weighing designs have been studied by Raghavarao (1971) and Banerjee (1975). In the case when $\mathbf{G} = \mathbf{I}_n$, Hotelling (1944) has shown that for the chemical balance weighing design the minimum attainable variance for each of the estimated weights is σ^2/n. He proved the theorem that variance of the estimated weights attaines the lower bound if and only if $\mathbf{X}'\mathbf{X} = n\mathbf{I}_p$. This design is said to be the optimum chemical balance weighing design. The condition given by Hotelling implies that elements of the matrix $\mathbf{X}$ of the optimum chemical balance weighing design are equal to -1 and 1, only. In this case several methods of the construction of the optimum chemical balance weighing design are available in the literature. In the situation when not all objects are included in each weighing operation (i.e. the elements of the design matrix are equal to -1, 0 or 1) Ceranka and Graczyk (2001) have showed that for a chemical balance weighing design the minimum attainable variance of the estimated weights is σ^2/m. They also proved that each variance of estimated weights attains the lower bound if and only if $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$. This design is said to be also the optimum chemical balance weighing design. Several methods of the construction of the optimum chemical balance weighing design with the design matrix $\mathbf{X}$ with elements equal to -1, 0 or 1 are given in Swamy (1982), Ceranka, Katulska and Mizera (1998), Ceranka and Katulska (1999), Ceranka and Graczyk (2001).

In the case when $\mathbf{G}$ is positive definite diagonal matrix, Katulska (1989) has shown that the minimum attainable variance for each of the estimated weights for the optimum chemical balance weighing design with positive definite diagonal variance-covariance matrix of errors is $\sigma^2/tr(\mathbf{G}^{-1})$. In the same paper she has showed that each variance of the estimated weights attains the minimum if and only if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = tr(\mathbf{G}^{-1})\mathbf{I}_p$. This design is said to be optimum chemical balance weighing design. This condition implies that elements of the matrix $\mathbf{X}$ of the optimum chemical balance weighing design are equal -1 or 1, only. In this case some methods of the construction of the optimum chemical balance weighing design are given in the literature.

In the present paper we investigate the necessary and sufficient conditions under which the minimum variance is attained for estimated weights for the chemical balance weighing design with positive definite diagonal variance-covariance matrix of errors for the case when the design matrix $\mathbf{X}$ has the elements -1, 0 or 1. We give some methods of the construction of such design under the restriction on the number of objects placed on the either pan. It is based on the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs.

2. Variance limit of the estimated weights

Assume, we have got at our disposal two balances, with usual and higher precision, respectively. Let n_1, n_2 be the numbers of times in which the respective precision is used. Then the variance-covariance matrix of errors $\sigma^2\mathbf{G}$ is given by

$$\mathbf{G} = \begin{bmatrix} \frac{1}{a}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & \mathbf{I}_{n_2} \end{bmatrix}, \tag{5}$$

where $n_1 + n_2 = n$, $a > 0$ and $\mathbf{I}_{n_h}$ is the $n_h \times n_h$ identity matrix, $h = 1, 2$. In other words, we determine the optimality in the class $\mathbf{\Phi}_{n\times p,m}(-1, 0, 1)$ with a restriction for a given number of measurement operations n and for a given form of the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in the form (5). Let us suppose further that the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1, 0, 1)$ is likewise partitioned as the matrix $\mathbf{G}$, i.e.

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}. \tag{6}$$

Ceranka and Graczyk (2003) have given the following definition.

Definition 1. A nonsingular chemical balance weighing design is said to be optimal for the estimation of individual weights of objects if the variances of their estimators attain the minimum, i.e.

$$\mathrm{Var}(\widehat{w}_j) = \frac{\sigma^2}{am_1 + m_2}, \qquad j = 1, 2, ..., p,$$

where $m_h = \max(m_{h1}, m_{h2}, ..., m_{hp})$, m_{hj} is the number of elements equal to -1 and 1 in jth column of the matrix $\mathbf{X}_h$, $h = 1, 2$.

In the same paper the following result is proved.

Theorem 1. *For any matrix* $\mathbf{X} \in \Phi_{n\times p,m}(-1, 0, 1)$ *of the nonsingular chemical balance weighing design each variance of the estimated weights attains the minimum if and only if*

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = (am_1 + m_2)\mathbf{I}_p\,. \tag{7}$$

3. Balanced block designs

In this section we remind the definitions of the balanced incomplete block designs given in Raghavarao (1971) and of the balanced bipartite weighing designs given in Huang (1976).

A balanced incomplete block design is an arrangement of ν treatments in b blocks, each of size k in such a way that each treatment occurs at most once in each block, occurs in exactly r blocks and every pair of the treatments occurs together in λ blocks. The integers ν, b, r, k, λ are called the parameters of the balanced incomplete block design. Let $\mathbf{N}_1$ be the incidence matrix of such a design. The parameters satisfy the following relations

$$\begin{aligned} \nu r &= bk, \\ \lambda(\nu - 1) &= r(k - 1), \\ \mathbf{N}_1\mathbf{N}_1' &= (r - \lambda)\,\mathbf{I}_\nu + \lambda\mathbf{1}_\nu'\,. \end{aligned} \tag{8}$$

A balanced bipartite weighing design is an arrangement of ν treatments in b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. Every pair of the treatments from different subblocks appears together in λ_1 blocks and every pair of treatments from the same subblock appears together in λ_2 blocks. The integers ν, b, r, k_1, k_2, λ_1, λ_2 are called the parameters of the balanced bipartite weighing design. Let $\mathbf{N}_2$ be the incidence matrix of such a design. The parameters are not independent and they are related by the following identities

$$\begin{aligned} \nu r &= bk, \\ b &= \frac{\lambda_1\nu(\nu - 1)}{2k_1k_2}, \\ \lambda_2 &= \frac{\lambda_1\left[k_1(k_1 - 1) + k_2(k_2 - 1)\right]}{2k_1k_2}, \\ r &= \frac{\lambda_1 k(\nu - 1)}{2k_1k_2}, \\ \mathbf{N}_2\mathbf{N}_2' &= (r - \lambda_1 - \lambda_2)\,\mathbf{I}_\nu + (\lambda_1 + \lambda_2)\,\mathbf{1}_\nu\mathbf{1}_\nu'. \end{aligned} \tag{9}$$

If in the balanced bipartite weighing design the number of objects in the first subblock k_1 is not equal to the number of objects in the second subblock k_2 then each object exists in r_1 blocks in the first subblock and in r_2 blocks in the second subblock, $r_1 + r_2 = r$. Thus

$$r_1 = \frac{\lambda_1(\nu - 1)}{2k_2}, \qquad r_2 = \frac{\lambda_1(\nu - 1)}{2k_1}. \tag{10}$$

4. The design matrices

In this section we will present a method of the construction of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ of the optimum chemical balance weighing design. It is based on the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs.

Let $\mathbf{N}_1$ be the incidence matrix of the balanced incomplete block design with parameters ν, b_1, r_1, k_1, λ_1 and let $\mathbf{N}_2$ be the incidence matrix of the balanced bipartite weighing design with parameters ν, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22}. Using $\mathbf{N}_2$ we form the matrix $\mathbf{N}_2^*$ by replacing k_{12} elements equal to $+1$ of each column, which correspond to the elements belonging to the first subblock, by -1. Now we construct the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ in the form (6). Thus

$$\mathbf{X} = \begin{bmatrix} 2\mathbf{N}_1' - \mathbf{1}_{b_1}\mathbf{1}_\nu' \\ \mathbf{N}_2^{*\prime} \end{bmatrix}. \tag{11}$$

In this design we determine unknown measurements of $p = \nu$ objects. Each object is weighed $m = b_1 + r_2$ times in $n = b_1 + b_2$ measurement operations. Notice that when $k_{12} \neq k_{22}$, then each column of this matrix includes $r_1 + r_{22}$ elements equal to 1 and $b_1 - r_1 + r_{12}$ elements equal to -1, where $r_{12} + r_{22} = r_2$.

In the case $\mathbf{G} = \mathbf{I}_n$, Ceranka and Graczyk (2002) have considered the necessary and sufficient conditions under which the minimum variance for estimated weights is attained and have given the methods of construction the design matrix $\mathbf{X}$ based on the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs.

Since $\mathbf{G}$ is a positive definite diagonal matrix, the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, i.e. if and only if $\mathbf{X}$ is of full column rank $(= p)$.

Lemma 1. *Chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in the form* (11) *is nonsingular if and only if*

$$\nu \neq k_1 \qquad or \qquad k_{12} \neq k_{22}.$$

Proof. The statement is a consequence of the equalities

$$\mathbf{X}'\mathbf{X} = (4(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12})\,\mathbf{I}_n + (b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12})\,\mathbf{1}_\nu\mathbf{1}_\nu'$$

and

$$\det(\mathbf{X}'\mathbf{X}) = (4(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12})^{\nu-1} \times \left(\frac{r_1}{k_1}(\nu - 2k_1)^2 + \frac{(\nu-1)\lambda_{12}}{2k_{12}k_{22}}(k_{12} - k_{22})^2 \right).$$

□

Theorem 2. *Any nonsingular chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in the form* (11) *and with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$, *where* $\mathbf{G}$ *is given in* (5), *is optimal for estimation unknown measurements of objects if and only if*

$$a\,(b_1 - 4(r_1 - \lambda_1)) + (\lambda_{22} - \lambda_{12}) = 0. \tag{12}$$

Proof. The statement is a consequence of Theorem 1, Lemma 1 and the equation

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = (4a(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12})\,\mathbf{I}_\nu + (ab_1 - 4a(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12})\,\mathbf{1}_\nu\mathbf{1}_\nu'.$$

□

If the chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ in the form (11) with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (5), is optimal, then

$$\mathrm{Var}(\widehat{w}_j) = \frac{\sigma^2}{ab_1 + r_2}, \qquad j = 1,2,...,p\,.$$

If the parameters of the balanced incomplete block designs belong to the A-family given in Raghavarao (1971) then the condition

$$b_1 - 4(r_1 - \lambda_1) = 0 \tag{13}$$

is fulfilled and if the parameters of the balanced bipartite weighing designs satisfy the equality

$$\lambda_{22} - \lambda_{12} = 0 \tag{14}$$

then we have

Theorem 3. *If the equalities* (13) *and* (14) *hold, then any nonsingular chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in the form* (11) *with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$, *where* $\mathbf{G}$ *is of the form* (5), *is optimal if and only if nonsingular chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in the form* (11) *with the variance-covariance matrix of errors* $\sigma^2\mathbf{I}_n$ *is optimal.*

The methods of construction of the optimum chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ in the form (11) with the variance-covariance matrix of errors $\sigma^2\mathbf{I}_n$, are given in Ceranka and Graczyk

(2002). In other words, if these parameters are connected by equalities (13) and (14), then the chemical balance weighing design is optimal for each form of the positive definite diagonal matrix $\mathbf{G}$ given in (5).

For a given number of objects p and measurement operations n we choose the class of matrices $\mathbf{\Phi}_{n\times p,m}(-1,0,1)$. For a given form of the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, which is defined through the parameter a, in the class $\mathbf{\Phi}_{n\times p,m}(-1,0,1)$ we have to find such a design matrix $\mathbf{X}$, which permits the optimal estimation of unknown measurements of objects. The construction of this design matrix $\mathbf{X}$ is based on the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs. Now we formulate a corollary according to which we are able to form the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ for the given numbers $a > 0$, p, and n.

Corollary 1. *If for a given number of objects v*

1. $a = \dfrac{l^2-(2c+1)l+c(c-1)}{2}$ *and the parameters of the balanced incomplete block designs are equal to*
   1.1. $\nu = b_1 = 4s^2-1, \quad r_1 = k_1 = 2s^2-1, \quad \lambda_1 = s^2-1 \quad s = 2,3,...,$
   1.2. $\nu = b_1 = 4s+3, \quad r_1 = k_1 = 2s+1, \quad \lambda_1 = s, \quad s = 1,2,...,$ $4s+3$ *is prime or prime power,*
   1.3. $\nu = b_1 = 8s+7, \quad r_1 = k_1 = 4s+3, \quad \lambda_1 = 2s+1 \quad s = 1,2,...,$
2. $a = \dfrac{l^2-(2c+1)l+c(c-1)}{4}$ *and the parameters of the balanced incomplete block designs are equal to* $\nu = 4s+1, \quad b_1 = 8s+2,$ $r_1 = 4s, \quad k_1 = 2s, \quad \lambda_1 = 2s-1, \quad s = 2,3,..., \quad 4s+1$ *is prime or prime power,*

and the parameters of the balanced bipartite weighing design are equal to $\nu, \quad b_2 = \frac{\nu(\nu-1)}{2}, \quad r_2 = \frac{(\nu-1)(l+c)}{2}, \quad k_{12} = c, \quad k_{22} = l, \quad \lambda_{12} = cl, \quad \lambda_{22} = \frac{l^2-l+c(c-1)}{2}$, *where* $c = 1$ and $l \geq 4$ or $c = 2,3$ and $l \geq 2c+1$, *then the chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in the form* (11) *and with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$ *is optimal, where* $\mathbf{G}$ *is given in* (5).

Let us replace in the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ the incidence matrices, i.e. $\mathbf{N}_1$ is the incidence matrix of the balanced bipartite weighing design with parameters ν, b_1, r_1, k_{11}, k_{21}, λ_{11}, λ_{21} and $\mathbf{N}_2$ is the incidence matrix of the balanced incomplete block design with parameters

ν, b_2, r_2, k_2, λ_2. Thus we get

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{*'} \\ 2\mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_\nu' \end{bmatrix}. \tag{15}$$

The chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ of the form (15) with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where

$$\mathbf{G} = \begin{bmatrix} \frac{1}{a^*}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}_{n_2}' \\ \mathbf{0}_{n_2}\mathbf{0}_{n_1}' & \mathbf{I}_{n_2} \end{bmatrix} \tag{16}$$

is optimal if and only if

$$a^*[\lambda_{21} - \lambda_{11}] + [b_2 - 4(r_2 - \lambda_2)] = 0, \tag{17}$$

where $a^* = \frac{1}{a}$. It implies

Corollary 2. *The optimum chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *of the form* (11) *with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$, *where* $\mathbf{G}$ *is given in* (5), *exists if and only if there exists the optimum chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *of the form* (15) *with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$, *where* $\mathbf{G}$ *is of the form* (16).

5. The example

Let us consider the estimation problem of $p = 7$ objects using $n = 28$ measurement operations. We assume that each object is weighed at least $m = 22$ times. We have got at our disposal two balances, one of higher precision and the other one of usual level. Hence the variance-covariance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix $\mathbf{G} = \begin{bmatrix} \frac{1}{2}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}_{n_2}' \\ \mathbf{0}_{n_2}\mathbf{0}_{n_1}' & \mathbf{I}_{n_2} \end{bmatrix}$, $a = 2$. For estimation we use the optimum chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ given by formula (11) and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (5). To construct the design matrix we use the incidence matrix of the balanced incomplete block design with parameters $\nu = 7$, $b_1 = 7$, $r_1 = 3$, $k_1 = 3$, $\lambda_1 = 1$

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and the incidence matrix of the balanced bipartite weighing design with parameters $\nu = 7$, $b_2 = 21$, $r_2 = 15$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 4$, $\lambda_{22} = 6$, where 1_1 and 1_2 denote objects belonging to the first and second subblock, respectively,

$$\mathbf{N}_2 =$$

$$\begin{bmatrix}
1_1 & 0 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 0 & 1_2 & 0 & 1_2 & 0 & 1_1 & 1_2 & 1_2 \\
1_2 & 1_1 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 0 & 1_1 & 1_2 \\
1_2 & 1_2 & 1_1 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 0 & 1_1 \\
1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 0 \\
1_2 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 0 & 1_1 & 1_2 & 1_2 & 1_2 & 0 & 1_2 \\
0 & 1_2 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 0 & 1_1 & 1_2 & 1_2 & 1_2 & 0 \\
0 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 0 & 0 & 0 & 1_2 & 0 & 1_1 & 1_2 & 1_2 & 1_2
\end{bmatrix}.$$

As a result we get the design matrix $\mathbf{X} \in \Phi_{28\times 7,22}(-1, 0, 1)$:

$$\mathbf{X} = \begin{bmatrix}
1 & 1 & -1 & 1 & -1 & -1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & -1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & -1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & -1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & -1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & -1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & -1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & -1 \\
-1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & -1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & -1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & -1 \\
-1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & -1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & -1 & 0 & 1 & 0 & 1
\end{bmatrix}.$$

References

Banerjee, K. S. (1975). *Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics.* Marcel Dekker Inc., New York.

Ceranka, B. and Graczyk, M. (2001). Optimum chemical balance weighing designs under the restriction on weighings. *Discuss. Math. Probab. Stat.* **21**, 111–120.

Ceranka, B. and Graczyk, M. (2002). Optimum chemical balance weighing designs based on balanced incomplete block designs and balanced bipartite block designs. *Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math.* **11**, 19–27.

Ceranka, B. and Graczyk, M. (2003). Optimum chemical balance weighing designs. *Tatra Mt. Math. Publ.* **26**, 49–57.

Ceranka, B. and Katulska, K. (1999). Chemical balance weighing designs under the restriction on the number of objects placed on the pans. *Tatra Mt. Math. Publ.* **17**, 141–148.

Ceranka, B., Katulska, K. and Mizera, D. (1998). The applications of ternary balanced block designs to chemical balance weighing designs. *Discuss. Math. Algebra Stochastic Methods* **18**, 179–185.

Hotelling, H. (1944). Some improvements in weighing designs and other experimental techniques. *Ann. Math. Statist.* **15**, 297–306.

Huang, Ch. (1976). Balanced bipartite weighing designs. *J. Combin. Theory Ser. A* **21**, 20–34.

Katulska, K. (1989). Optimum chemical balance weighing designs with nonhomogeneity of the variances of errors. *J. Japan Statist. Soc.* **19**, 95–101.

Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Designs of Experiments.* John Wiley Inc., New York.

Swamy, M. N. (1982). Use of balanced bipartite weighing designs as chemical balance designs. *Comm. Statist. A—Theory Methods* **11**, 769–785.

Department of Mathematical and Statistical Methods, Agricultural University of Poznań, Wojska Polskiego 28, PL-60637 Poznań, Poland

E-mail address: bronicer@owl.au.poznan.pl

E-mail address: magra@owl.au.poznan.pl