

Pricing European-style options under jump diffusion processes with stochastic volatility: applications of Fourier transform

RAUL KANGRO, KALEV PÄRNA AND ARTUR SEPP

ABSTRACT. We develop a general methodology for pricing European-style options under various stochastic processes via the Fourier transform. We generalize previous work in this field and present two approaches for solving the pricing problem: the characteristic formula which is an extension of Lewis (2001) work, and the Black–Scholes-style formula which is an extension and generalization of previous work by Heston (1993) and Bates (1996). We show how to apply our formulas for two types of asset price dynamics: 1) stochastic volatility models with price jumps at a stochastic jump intensity rate, 2) stochastic volatility models with price and volatility jumps. Convergence properties of Fourier integrals arising from both approaches are studied.

1. Introduction

The ubiquitous Black–Scholes (1973) model assumes that the option underlying asset follows a geometric Brownian motion with drift and diffusion parameters. In theory, the diffusion parameter, which is usually called volatility, is constant. In practice, all option markets exhibit a volatility smile phenomenon which means that options with different maturities and strikes have different implied Black–Scholes volatility. As a result, accurate pricing and hedging of options is hard to achieve within the standard Black–Scholes model. Accordingly, it is necessary to generalize asset price dynamics and develop appropriate pricing methods.

Here, we apply Fourier inversion methods to work out “closed-form” formulas for option pricing under jump-diffusions with stochastic volatility. We

Received November 11, 2003.

2000 *Mathematics Subject Classification.* 60J35; 60J60; 60J70.

Key words and phrases. Stochastic volatility, jump-diffusion processes, option pricing, characteristic function, Fourier transform.

Supported by Estonian Science Foundation Grants 5022 and 5277 and Target Financed Project 0181776s01.

can employ the Fourier transform in two basic ways. Firstly, we can represent option value as a Fourier integral and invert it. This method was introduced by Lewis (2001). We refer to this approach as the characteristic formula. Secondly, we can invert the characteristic functions to compute the risk-neutral probabilities associated with the option value. This method is known due to Heston (1993). We generalize this approach and call it the Black-Scholes-style formula.

2. Problem formulation

To cope with market imperfections, a number of alternative asset price dynamics have been proposed in the literature. Heston (1993) in his influential work employed the square root process for modeling the variance of the asset price, and showed how to apply the Fourier transform to solve the pricing problem for vanilla options. Bates (1996) added to the Heston's stochastic volatility model log-normal price jumps governed by a Poisson process with a constant intensity rate. Fang (2000) extended the Bates' model by introducing a stochastic intensity rate. Duffie (2000) added to the Heston's model price and volatility jumps.

It makes sense to generalize all these models and to develop a general pricing methodology for all above models. We propose the following model for the asset price dynamics, which is a generalization of many previous models, for this purpose.

Asset price dynamics. We assume that the asset price dynamics are given under the risk-neutral measure \mathbb{Q} by

$$\begin{cases} dS(t)/S(t) &= (r - d - \lambda(t)m)dt + \sqrt{V(t)}dW^s(t) + (e^J - 1)dN(t), \\ &S(0) = S; \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t) + J^v dN^v(t), \\ &V(0) = V; \\ d\lambda(t) &= \kappa_\lambda(\theta_\lambda - \lambda(t))dt + \varepsilon_\lambda\sqrt{\lambda(t)}dW^\lambda(t), \\ &\lambda(0) = \lambda \end{cases} \quad (1)$$

where for the asset price dynamics we have: $S(t)$ is the asset price, $V(t)$ is the variance of asset return, r is the risk-free (domestic) interest rate, d is the continuous dividend yield (foreign interest rate), $W^s(t)$ and $W^v(t)$ are correlated Wiener processes with constant correlation ρ , $N(t)$ is a Poisson process with a stochastic intensity $\lambda(t)$, J is a random jump size in the logarithm of the asset price with the probability density function (PDF) $\varpi(J)$. We assume that $\mathbb{E}(f(J)) < \infty$ for a smooth function f . We set $m = \mathbb{E}(e^J - 1)$ to make the discounted asset process a martingale.

For the variance dynamics we have: κ is the mean-reverting rate, θ is the long-term mean variance, ε is the volatility of volatility, $N^v(t)$ is a Poisson process with a constant intensity λ^v and J^v ($J^v > 0$) is random jump size in variance with the PDF $\bar{\psi}(J^v)$.

w
Twh
cal3.1
PII
give

For the jump intensity dynamics we have: κ_λ is the mean-reverting rate, θ_λ is the long-term mean rate, ε_λ is the volatility of jump rate intensity, a Wiener process $W^\lambda(t)$ is independent of $W^s(t)$ and $W^v(t)$. We also assume that jump processes are independent of Wiener processes.

In our work, we will solve the pricing problem for European-style claims under general dynamics (1). Therefore, solutions for particular models can be obtained from our solution as particular cases. Our results are valid under the assumption of the constant variance V , the constant intensity λ , and the absence of the jump processes.

Using the martingale pricing, we can represent option value as the integral of a discounted probability density times the payoff function and employ Feynman-Kac theorem [for example, Duffie *et al* (2000)] to derive the partial integro-differential equation (PIDE) satisfied by the value of an option.

Now we make a change of variables from S to $x = \ln S$ and from t to $\tau = T - t$, where T is the option expiration time. Applying Feynman-Kac theorem for the prototype price dynamics (1), we obtain that the value of a European-style claim $f(x, V, \lambda, \tau)$ satisfies the following PIDE

$$\begin{aligned} -f_\tau + (r - d - \frac{1}{2}V - \lambda m)f_x + \frac{1}{2}V f_{xx} + \kappa(\theta - V)f_V + \frac{1}{2}\varepsilon^2 V f_{VV} + \rho\varepsilon V f_{xV} \\ + \kappa_\lambda(\theta_\lambda - \lambda)f_\lambda + \frac{1}{2}\varepsilon_\lambda^2 \lambda f_{\lambda\lambda} + \lambda \int_{-\infty}^{\infty} [f(x + J) - f(x)]\varpi(J)dJ \\ + \lambda^v \int_0^{\infty} [f(V + J^v) - f(V)]\bar{\psi}(J^v)dJ^v = rf, \\ f(x, V, \lambda, 0) = g(e^x, K) \end{aligned} \quad (2)$$

where subscripts indicate the partial derivatives, and g is the payoff function.

The most widely used derivative contracts are European calls and puts. The payoff for a call or put option at time $\tau = 0$ is given by

$$g(e^x, K) = \max\{\varphi[e^x - K], 0\} \quad (3)$$

where the contract parameters are strike K and type φ , with $\varphi = +1$ for a call and $\varphi = -1$ for a put.

3. Solution of the pricing problem

3.1. Preliminaries. In our research we apply Fourier transform to solve PIDE (2) and similar problems. The forward Fourier transform of $f(x)$ is given by

$$\hat{f}(z) = \mathcal{F}[f(x)](z) = v.p. \int_{-\infty}^{\infty} e^{izx} f(x)dx, \quad (4)$$

where v.p. stands for the principal value of integral, and the inverse Fourier transform is given by

$$f(x) = \mathcal{F}^{-1}[\hat{f}(z)](x) = \frac{1}{2\pi} \text{v.p.} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izx} \hat{f}(z) dz, \quad (5)$$

where $i = \sqrt{-1}$ and $z \in \mathbb{C}$, $z = u + vi$ with real part $u = \Re z \in \mathbb{R}$ and imaginary part $v = \Im z \in \mathbb{R}$, is the transform variable. For typical option payoffs integral (4) generally exists only if $\Im z$ is restricted to a strip $\alpha < \Im z < \beta$. We will refer to this strip as the strip of regularity.

To implement the Fourier transform, we need

- 1) to find an analytic representation of the Fourier transform;
- 2) to invert the result with the z -plain integration (5) keeping $\Im z$ in an appropriate strip of regularity.

Integral (5) can be computed using the standard methods of numerical integration.

It makes sense to derive a unified formula for pricing both calls and puts. We consider variable $x(t) = \ln S(t)$. Under the martingale measure \mathbb{Q} , $x(t)$ satisfies

$$\mathbb{E}^{\mathbb{Q}}[x(T)] = x(t) + (r - d)(T - t).$$

It is more convenient to consider the option with the bounded payoff function $g(x) = \min\{e^x, K\}$. Let $f(x, V, \lambda, t)$ denote the value function of this option.

Using the martingale pricing, we can represent the value of a call or put $F(x, t)$ as

$$\begin{aligned} F(x(t), t) &= \mathbb{E}^{\mathbb{Q}}[e^{-(T-t)r} \max\{\varphi[e^{x(T)} - K], 0\}] \\ &= e^{-(T-t)r} \left[\frac{1 + \varphi}{2} \mathbb{E}^{\mathbb{Q}}[e^{x(T)}] + \frac{1 - \varphi}{2} \mathbb{E}^{\mathbb{Q}}[K] - \mathbb{E}^{\mathbb{Q}}[\min\{e^{x(T)}, K\}] \right] \\ &= \frac{1 + \varphi}{2} e^{x(t) - (T-t)d} + \frac{1 - \varphi}{2} e^{-(T-t)r} K - f(x, V, \lambda, t). \end{aligned} \quad (6)$$

Feynman-Kac theorem implies that the option value function $f(x, V, \lambda, \tau)$ satisfies PIDE (2) with the initial condition

$$g(e^x, K) = \min\{e^x, K\}.$$

As a result, we need to solve PIDE (2) for the function $f(x, V, \lambda, \tau)$ and price calls and puts using formula (6).

For our subsequent analysis, we need to find the transformed initial condition $\tilde{g}(z) = \mathcal{F}[\min\{e^x, K\}](z)$. A direct calculation yields

$$\tilde{g}(z) = \frac{K^{iz+1}}{z^2 - iz}, \quad (7)$$

provided that $0 < \Im z < 1$. We will denote this strip of regularity as the payoff strip S_f .

3.2. The characteristic formula. Now we consider a powerful approach for solving a general pricing problem. We assume that the characteristic function corresponding to the price dynamics is given in closed-form. The characteristic function of $x(T) = \ln S(T)$ is defined by

$$\phi_T(z) = \mathbb{E}^{\mathbb{Q}}[e^{izx(T)}] = \int_{-\infty}^{\infty} e^{izx} \omega_T(x) dx \quad (8)$$

where $\omega_T(x)$ is the risk-neutral density of the logarithmic price $x(T)$.

Now we state a modified version of Theorem 3.2 in Léwis (2001), which is very important for option pricing under general stochastic processes. Lewis proposed this formula for option pricing under Lévy processes; we apply his result for general stochastic processes.

Theorem 3.1 (The characteristic formula). *We assume that $x(T)$ has analytic characteristic function $\phi_T(z)$ with the strip of regularity $S_z = \{z : \alpha < \Im z < \beta\}$. Next we assume that $e^{-vx} f(x) \in L^1(\mathbb{R})$ where v is located in the payoff strip S_f with transform $\hat{f}(z)$, $\Im z \in S_f$.*

Then, if $S_F = S_f \cap S_z$ is not empty, the option value is given by

$$f(x(t)) = \frac{e^{-r(T-t)}}{2\pi} \int_{iv-\infty}^{iv+\infty} \phi_T(-z) \hat{f}(z) dz \quad (9)$$

where $z \in S_F = S_f \cap S_z$.

Proof. Using risk-neutral pricing, we have

$$\begin{aligned} f(x(t)) &= \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} f(x(T)) \right] \stackrel{(5)}{=} e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} e^{-izx(T)} \hat{f}(z) dz \right] \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{iv-\infty}^{iv+\infty} \mathbb{E}^{\mathbb{Q}} [e^{-izx(T)}] \hat{f}(z) dz \\ &\stackrel{(8)}{=} \frac{e^{-r(T-t)}}{2\pi} \int_{iv-\infty}^{iv+\infty} \phi_T(-z) \hat{f}(z) dz. \end{aligned}$$

The exchange of integration order is allowed by the Fubini theorem. By assumption $\phi_T(-z)$ exists if $z \in S_z$. In our case, z is already restricted to $z \in S_z$. Accordingly, the whole integrand exists if $z \in S_F = S_f \cap S_z$. \square

3.3. The Black-Scholes-style formula. Now we present another pricing formula which we call the Black-Scholes-style formula. This formula is a generalization of previous work by Heston (1993), Bates (1996) and others. A similar result was first considered by Lewis (2001) for pricing under Lévy processes. Here, we consider it in more details and apply for pricing under general processes.

Theorem 3.2 (The Black-Scholes-style formula). *We assume that the characteristic function $\phi_T(z) = \mathbb{E}^{\mathbb{Q}}[e^{izx(T)}]$ corresponding to the market model*

is analytic and bounded in the strip $0 \leq \Im z \leq 1$. Two characteristics, $\phi_j(u)$ ($j = 1, 2$), $u \in \mathbb{R}$, are given by $\phi_1(u) = e^{-\ln S(t) - (r-d)(T-t)} \phi_T(u - i)$ and $\phi_2(u) = \phi_T(u)$. The cumulative distribution functions (CDF), Π_j , in the variable $y = \ln K$ of the log-spot price $x(T) = \ln S(T)$ are given by

$$\Pi_j(y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{\phi_j(u) e^{-iuy}}{iu} \right] du \tag{10}$$

and variables $P_j(\varphi)$ are defined by

$$P_j(\varphi) = \frac{1 - \varphi}{2} + \varphi \Pi_j(y). \tag{11}$$

Then the current value of a European-style contingent claim, $F(S, t)$, that pays off $\max\{\varphi[S_T - K], 0\}$, where φ is the binary variable ($\varphi = +1$ for a call and $\varphi = -1$ for a put), at time of the expiration date T has the form

$$F(S, t) = \varphi \left[e^{-d(T-t)} S P_1(\varphi) - e^{-r(T-t)} K P_2(\varphi) \right]. \tag{12}$$

Proof. We assume that $\phi_T(-z)$ has the strip of regularity $0 \leq \Im z \leq 1$. First, we re-write the transform-based integral (9) as

$$\begin{aligned} f_{\min(S,K)}(S(t)) &= \frac{e^{-r(T-t)}}{2\pi} \int_{iv-\infty}^{iv+\infty} \phi_T(-z) \frac{K^{1+iz}}{z^2 - iz} dz \\ &= \frac{e^{-r(T-t)} K}{2\pi} \left(\int_{iv-\infty}^{iv+\infty} \phi_T(-z) \frac{iK^{iz}}{z} dz - \int_{iv-\infty}^{iv+\infty} \phi_T(-z) \frac{iK^{iz}}{z-i} dz \right) \\ &= \frac{e^{-r(T-t)} K}{2\pi} (\Re(I_1) + \Re(I_2)). \end{aligned} \tag{13}$$

As usually in complex analysis, in order to evaluate I_1 we employ a contour integral over the contour given by 6 parametric curves: $\Gamma_1 : z = u, u \in (q, R)$ with $q, R > 0$; $\Gamma_2 : z = R + ib, b \in (0, v)$; $\Gamma_3 : z = u + iv, u \in (R, -R)$; $\Gamma_4 : z = -R + ib, b \in (v, 0)$; $\Gamma_5 : z = u, u \in (-R, -q)$; $\Gamma_6 : z = qe^{i\theta}, \theta \in (\pi, 0)$. As the integrand is analytic on this contour, Cauchy's theorem implies that

$$\sum_{k=1}^6 \int_{\Gamma_k} F(z) dz = 0.$$

Evaluating corresponding integrals, taking limits $q \rightarrow 0$ and $R \rightarrow \infty$, and changing variable $u \rightarrow -u$, we obtain that

$$\Re(I_1) = -\pi - \int_{-\infty}^\infty \Re \left[\phi_T(u) \frac{iK^{-iu}}{u} \right] du. \tag{14}$$

Again, to evaluate I_2 we employ a contour integral over the contour given by 6 parametric curves: $\Gamma_1 : z = u + i, u \in (q, R)$ with $q, R > 0$; $\Gamma_2 : z = R + ib, b \in (1, 1+v)$; $\Gamma_3 : z = u + i(1+v), u \in (R, -R)$; $\Gamma_4 : z = -R + ib, b \in (v, 1)$; $\Gamma_5 : z = u + i, u \in (-R, -q)$; $\Gamma_6 : z = i + qe^{i\theta}, \theta \in (0, \pi)$. Evaluating

i
R

Q
int

tic
(
cal
din
(17

corresponding integrals, taking limits $q \rightarrow 0$ and $R \rightarrow \infty$, changing variable $u \rightarrow u - i$, and noting that $\phi_T(-i) = Se^{(r-d)(T-t)}$ we obtain that

$$\Re(I_2) = \frac{Se^{(r-d)(T-t)}}{K} \left(-\pi - \int_{-\infty}^{\infty} \Re \left[\phi_T(u - i) \frac{iK^{-iu}}{u} \right] du \right). \quad (15)$$

Substituting (14) and (15) into (13), using relationship (6) and introducing terms Π_j , we obtain the Black-Scholes-style formula (12). \square

3.4. Analysis of pricing formulas. Thus, the pricing problem for European calls and puts is reduced to the evaluation of Fourier integrals, which has to be done numerically.

We consider the proposed formulas in more details. The Black-Scholes formula (12) includes integrals which are already expressed in terms of the real-valued transform parameter k .

Now, we consider the characteristic approach. Given an explicit expression for the moment generating function (MGF) $G(\Phi, x, V, \lambda, \tau)$, the complex-valued characteristic function is given by $\phi_T(z) = G(iz)$ and, similarly, $\phi_T(-z) = G(-iz)$.

For an option with payoff function $g(x) = \min\{e^x, K\}$, the integral (9) can be represented as

$$f(x, V, \lambda, \tau) = \frac{K}{2\pi} \int_{iv-\infty}^{iv+\infty} e^{iz \ln K} \frac{G(-iz, x, V, \lambda, \tau)}{z^2 - iz} dz. \quad (16)$$

It is more convenient to evaluate integral (16) along a straight line $v = 1/2$ in the complex z -plane parallel to the real axis. Substituting $z = u + i/2$, $u \in \mathbb{R}$, into (16), we obtain that

$$f(x, V, \lambda, \tau) = \frac{K}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(-iu+1/2) \ln K} G(-iu + \frac{1}{2}, x, V, \lambda, \tau)}{u^2 + 1/4} du. \quad (17)$$

For brevity, we introduce

$$Q(u, x, V, \lambda, \tau) = e^{(-iu+1/2) \ln K} G(-iu + \frac{1}{2}, x, V, \lambda, \tau). \quad (18)$$

It can be shown that integrand in (17) is a symmetric function, i.e. $Q(-u) = \overline{Q(u)}$, so that for option pricing we need to evaluate the following integral

$$f(x, V, \lambda, \tau) = \frac{K}{\pi} \int_0^{\infty} \Re \left[\frac{Q(u, x, V, \lambda, \tau)}{u^2 + 1/4} \right] du. \quad (19)$$

Explicit expressions of characteristic functions and $Q(u, x, V, \lambda, \tau)$ for particular market dynamics can be found in Sepp (2003).

Given the value of $f(x, V, \lambda, \tau)$, values of European calls and puts are calculated using formula (6). Thus, the pricing problem is reduced to a one-dimensional integration along the real axis. It can be shown that integrals (17) and (19) are uniformly convergent so that the partial derivatives of the

option value can be computed by differentiating the integrand and inverting the Fourier transform. The same is true for the Black–Scholes-style formula.

As a result, we have obtained two distinct formulas (12) and (19) for solving the pricing problem. The Fourier integral can be computed by means of standard procedures of numerical integration (see, for example, Press *et al* (1992)). We adopted the ten-point Gaussian quadrature routine for this purpose.

We have extensively studied convergence properties of the integrals arising from both approaches. We found that using the Black–Scholes-style formula one can compute the option price about three times faster than using the transform-based approach. This property does not depend on a specific market model. The advantage of characteristic integrals is that they can be computed by virtue of the Fast Fourier Transform (FFT), which can only be implemented for non-singular integrands. The FFT allows for simultaneous calculation of option values for a given level of strikes. This can be very useful for calibrating a model to the implied volatility surface. Implementation of the FFT for option pricing is presented by Carr and Madan (1999).

3.5. Solution to the moment generating function. For implementing the pricing formulas, we need to find explicit expression for the characteristic function. Here we illustrate our approach using the square root processes given by the dynamics (1). Other stochastic processes can be analyzed in a similar manner. Let us recall the following relationship between the characteristic function $\phi(u)$ and MGF $G(\Phi)$: $\phi(u) = G(iu)$. We apply the standard considerations employed by Heston (1993) and consider the MGF $G(\Phi, x, V, \lambda, \tau)$ associated with the log of the terminal asset price $x(\tau) = \ln S(\tau)$ under the measure \mathbb{Q} :

$$G(\Phi, x, V, \lambda, \tau) = \mathbb{E}^{\mathbb{Q}}[e^{\Phi x(\tau)}] = e^{-r\tau} \mathbb{E}^{\mathbb{Q}}[e^{r\tau} e^{\Phi x(\tau)}]. \quad (20)$$

Accordingly, MGF $G(\Phi, x, V, \lambda, \tau)$ can be interpreted as a contingent claim that pays off $e^{r\tau + \Phi x}$ at time τ .

For the process (1), Feynman–Kac theorem implies that $G(\Phi, x, V, \lambda, \tau)$ solves

$$\begin{aligned} & -G_{\tau} + (r - d - \frac{1}{2}V - \lambda m)G_x + \frac{1}{2}VG_{xx} + \kappa(\theta - V)G_V + \frac{1}{2}\varepsilon^2VG_{VV} \\ & + \rho\varepsilon VG_{xV} + \kappa_{\lambda}(\theta_{\lambda} - \lambda)G_{\lambda} + \frac{1}{2}\varepsilon_{\lambda}^2\lambda G_{\lambda\lambda} + \lambda \int_{-\infty}^{\infty} [G(x + J) - G] \varpi(J) dJ = 0, \\ & G(\Phi, x, V, \lambda, 0) = e^{\Phi x}. \end{aligned} \quad (21)$$

We can solve PIDE (21) in closed-form by the method of indetermined coefficients using a guess $G = e^{A(\tau) + B(\tau)V + C(\tau)\lambda}$. Thus, the solution has an

affine-form and, accordingly, these models are called affine (jump-)diffusions. The full solution is specified by

Proposition 3.1. *The solution to PIDE (21) is given by*

$$G(\Phi, x, V, \lambda, \tau) = \exp\{x\Phi + (r - d)\tau\Phi + A(\Phi, \tau) + B(\Phi, \tau)V + C(\Phi, \tau) + D(\Phi, \tau)\lambda\} \quad (22)$$

where

$$A(\Phi, \tau) = -\frac{\kappa\theta}{\varepsilon^2} \left[\psi_{+\tau} + 2 \ln \left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta} \right) \right],$$

$$B(\Phi, \tau) = -(\Phi - \Phi^2) \frac{1 - e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}},$$

$$C(\Phi, \tau) = -\frac{\kappa_\lambda\theta_\lambda}{\varepsilon_\lambda^2} \left[\chi_{+\tau} + 2 \ln \left(\frac{\chi_- + \chi_+ e^{-\xi\tau}}{2\xi} \right) \right],$$

$$D(\Phi, \tau) = 2\Lambda(\Phi) \frac{1 - e^{-\xi\tau}}{\chi_- + \chi_+ e^{-\xi\tau}},$$

$$\psi_\pm = \mp(\kappa - \rho\varepsilon\Phi) + \zeta, \quad \zeta = \sqrt{(\kappa - \rho\varepsilon\Phi)^2 + \varepsilon^2(\Phi - \Phi^2)},$$

$$\chi_\pm = \mp\kappa_\lambda + \xi, \quad \xi = \sqrt{\kappa_\lambda^2 - 2\varepsilon_\lambda^2\Lambda(\Phi)},$$

$$\Lambda(\Phi) = \int_{-\infty}^{\infty} e^{J\Phi} \varpi(J) dJ - 1 - m\Phi, \quad m = \int_{-\infty}^{\infty} e^J \varpi(J) dJ - 1.$$

A complete derivation of the above formula can be found in Sepp (2003).

In the above formula, $\Lambda(\Phi)$ is the so-called jump transform. A few distributions have been proposed for modeling price jumps.

Merton (1976) proposed jump-diffusions where the logarithm of jump size is normally distributed with mean ν and variance δ^2 :

$$\varpi(J) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(J-\nu)^2}{2\delta^2}}.$$

A simple calculation yields that for log-normal price-jumps we have

$$\Lambda(\Phi) = e^{\nu\Phi + \delta^2\Phi^2/2} - 1 - \Phi(e^{\nu + \delta^2/2} - 1). \quad (23)$$

Kou (2002) proposed an asymmetric double exponential distribution for price-jumps:

$$\varpi(J) = p \frac{1}{\eta_u} e^{-\frac{1}{\eta_u} J} \mathbf{1}_{\{J \geq 0\}} + q \frac{1}{\eta_d} e^{\frac{1}{\eta_d} J} \mathbf{1}_{\{J < 0\}}$$

where $1 > \eta_u > 0$, $\eta_d > 0$ are means of positive and negative jumps, respectively; $p, q \geq 0$, $p + q = 1$. p and q represent the probabilities of positive and

negative jumps. The requirement $\eta_u < 1$ is needed to ensure that $\mathbb{E}[e^J] < \infty$ and $\mathbb{E}[S] < \infty$. For double-exponential jumps we have

$$\Lambda(\Phi) = \frac{p}{1 - \Phi\eta_u} + \frac{q}{1 + \Phi\eta_d} - 1 - \Phi\left(\frac{p}{1 - \eta_u} + \frac{q}{1 + \eta_d} - 1\right) \quad (24)$$

provided that $-1/\eta_d < \Im z < 1/\eta_u$.

Once the explicit expression for MGF has been obtained, we can use it to implement the pricing formulas. Explicit expressions for the characteristic formula and the Black–Scholes-style formula are given in Sepp (2003).

Now we consider the simultaneous jumps model (SVSJ) proposed by Duffie, Pan and Singleton (2000) with simultaneous correlated jumps in price and variance. The marginal distribution of the jump size in variance is exponential with mean η . Conditional distribution on a realization J^v of a jump size in variance, a jump in logarithmic asset price is normally distributed with mean $\nu + \rho_J J^v$ and variance δ^2 :

$$J^v \sim \exp\left(\frac{1}{\eta}\right), \quad J^s | J^v \sim N(\nu + \rho_J J^v, \delta^2).$$

The expression for the MGF corresponding jump-diffusion with price and volatility jumps is derived in a similar manner. We omit details to obtain

Proposition 3.2. *The MGF corresponding to the SVSJ process is given by*

$$G(\Phi, x, V, \lambda, \tau) = \exp\{\Phi x + (r - d)\tau\Phi + A(\Phi, \tau) + B(\Phi, \tau)V + \Delta(\Phi, \tau)\} \quad (25)$$

where

$$\begin{aligned} \Delta(\Phi, \tau) = & -\lambda \left(\frac{\Phi e^{\nu + \frac{1}{2}\delta^2}}{1 - \rho_J \eta} + 1 \right) \tau + \lambda e^{\nu\Phi + \frac{1}{2}\delta^2\Phi^2} \left(\frac{\psi_-}{\psi_- L + \eta U} \tau \right. \\ & \left. - \frac{2\eta U}{(\zeta L)^2 - (ML + \eta U)^2} \ln \left[1 - \frac{\psi_+ L - \eta U}{2\zeta L} (1 - e^{-\zeta\tau}) \right] \right), \end{aligned}$$

$U = \Phi - \Phi^2$, $M = \kappa - \rho\varepsilon\Phi$, $L = 1 - \rho_J\eta\Phi$, and $A(\Phi, \tau)$, $B(\Phi, \tau)$, ψ_{\pm} , ζ are defined in formula (22).

4. Conclusions

In this research, we examined a number of jump-diffusion processes for the asset price dynamics. We considered pricing problem of European-style options and derived explicit formulas for European call and put option values using two approaches: the characteristic formula and the Black–Scholes-style formula. We found that the Black–Scholes-style formula yields the option value considerably faster than the characteristic formula. However, the characteristic formula can be computed using the Fast Fourier Transform. Our empirical results with DAX options data show that, to achieve a close fit to

the vanilla market, one needs to incorporate both components into pricing models – the stochastic volatility and jumps.

References

- Bates, D. (1996). Jumps and stochastic volatility: exchange rate processes implicit in Deutsche mark options. *Review of Financial Studies* **9**, 69–107.
- Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637–659.
- Carr, P. and Madan, D. (1999). Option valuation using the Fast Fourier Transform. *Journal of Computational Finance* **2**(4), 61–73.
- Duffie, D., Pan, J. and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusion. *Econometrica* **68**(6), 1343–1376.
- Fang, H. (2000). Option pricing implications of a stochastic jump rate. University of Virginia, unpublished manuscript.
- Press, W., Teukolsky, S., Vetterling, W. and Flannery, B. (1992). *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, available at www.nr.com.
- Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6**, 327–343.
- Kou, S. (2002). A jump diffusion model for option pricings. *Management Science* **48**, 1086–1101.
- Lewis, A. (2001). A simple option formula for general jump-diffusion and other exponential Lévy processes. Unpublished manuscript, available at www.optioncity.net.
- Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3**, 125–144.
- Sepp, A. (2003). Fourier transform for option pricing under affine jump-diffusions: an overview. University of Tartu, unpublished manuscript, available at <http://math.ut.ee/~spartak/>.

INSTITUTE OF MATHEMATICAL STATISTICS, J. LIIVI 2, 50409 TARTU, ESTONIA
E-mail address: rkangro@tartu.audentes.ee
E-mail address: kalev.parna@ut.ee
E-mail address: artur.sepp@ut.ee