ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 8, 2004

On risk processes with double barriers

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Abstract. We consider risk processes with two barriers. The risk process starts with an initial capital $u > 0$ and the two barriers are set at 0 and $v (> u)$. We are interested in finding the probability $\varphi(u, v)$ that the risk process hits the upper barrier v before 0. Both cases of positive and negative relative safety loading are considered. Explicit formulae for $\varphi(u, v)$ are obtained in the case of positive safety loading and in a special case of negative safety loading when the claims are exponentially distributed. For the general case of negative safety loading an integral equation is derived for $\varphi(u, v)$, similar to the classical result for the case of a single barrier at 0.

1. Introduction

The risk process is a type of stochastic processes to model the wealth of an insurance company. It is defined by $X(t) = ct - \sum_{k=1}^{N(t)} Z_k$ where c is interpreted as a gross premium rate (the company receives c units of money per unit time), $N(t)$ is the number of claims on the company during the time interval $(0, t]$. Each time when N grows the company has to pay out random amount of money (Z_k). The value of $X(t)$ is the profit of the company over the time interval $(0, t]$. *Ruin* of the company means that starting with initial capital u its wealth $u + X(t)$ becomes negative at some time point t.

The problem of calculation and estimation of the ruin probability is a central issue in the ruin theory. Several basic results (which we now call "classical") concerning risk processes were obtained by F. Lundberg and H. Cramér (see, e.g., Cramér (1930)). Of good accounts of risk processes we refer to books by Gerber (1979) and Grandell (1991), the latter being devoted to various generalizations of classical results.

Received December 5, 2003.

2000 *Mathematics Subject Classification.* 62P05, 91B28.

Key words and phrases. Risk process, ruin probability, double barriers, exponential claims.

Supported by Estonian Science Foundation Grant 5277 and Target Financed Project 0181776s01.

In this paper we consider the situation when also an *upper* barrier $v(> u)$ exists. We are interested in finding the probability $\varphi(u, v)$ that starting from u the upper barrier v is reached before the lower barrier (0) – a fargoing generalization of the famous gambler's ruin problem. Hitting the level v can be an important goal from the business point of view for it can create interest among investors.

The problem of hitting one barrier without hitting earlier the other barrier is also studied in financial mathematics, where other types of underlying stochastic processes are of interest. For example, pricing double barrier options under jump diffusion processes is recently tackled by Sepp (2004).

The paper is built up as follows. We first fix notation and give a brief review of classical results concerning the one barrier problem (0). Then the case of two barriers is studied. Both cases, positive and negative relative safety loading are considered. Explicit formulae for $\varphi(u, v)$ are obtained in the case of positive safety loading and in a special case of negative safety loading when the claims are exponentially distributed. For the general case of negative safety loading an integral equation is derived for $\varphi(u, v)$, similar to the classical result for the case of a single barrier at 0.

2. Classical risk processes

Classical risk process is a standard model for an insurance company. It is defined by $X(t) = ct - \sum_{k=1}^{N(t)} Z_k$ where $c > 0$ is a constant, $\{Z_k\}_{k=1}^{\infty}$ are i.i.d. random variables having common distribution function $F(z)$ with $F(0) = 0$ and mean value $EZ_k = \mu$, $N(t)$ is a homogeneous Poisson process with intensity α and independent of $\{Z_k\}$. The process $X(t)$ is interpreted as the profit of the company over the time interval $(0, t]$.

An important parameter of the risk process is its relative safety loading. Let us calculate the expected profit over $(0, t]$:

$$EX(t) = ct - E[N(t)]E(Z_k) = ct - \alpha t\mu = (c - \alpha\mu)t.$$

The ratio $\rho = (c-\alpha\mu)/\alpha\mu$ is called the *relative safety loading*. It is positive when the company makes profit in average but it can also be negative. The latter occurs, for example, when the company keeps prices of policies low in order to win new customers.

The *ruin probability* of a company having initial capital u is defined by

$$\Psi(u) = P\{u + X(t) < 0 \text{ for some } t \in (0, \infty)\}. \tag{1}$$

The *non-ruin probability* is denoted by $\Phi(u) = 1 - \Psi(u)$. The calculation of the ruin probability is a challenging task. It has been shown (see, e.g., Grandell (1991), Ch.1) that the function $\Phi(u)$ satisfies the following (Volterra type) integral equation:

$$\Phi(u) = \Phi(0) + \frac{\alpha}{c} \int_0^u \Phi(u-z)[1-F(z)]dz. \tag{2}$$

From this equation so called Cramér–Lundberg asymptotic formula has been derived:

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = C, \tag{3}$$

with constants $R > 0$ (*Lundberg exponent*) and $C > 0$, both depending on μ, α, c, and F.

An exact formula for the ruin probability is available for the case of exponentially distributed claims, $Z_k \sim Exp(1/\mu)$, when we have

$$\Psi(u) = \frac{1}{1+\rho} e^{-Ru} \tag{4}$$

with $R = \frac{\rho}{\mu(1+\rho)}$.

3. Risk processes with two barriers

Let u be the initial capital of the risk process, $u \geq 0$. Besides the lower barrier 0, which is the only barrier in the traditional set-up, we also fix an upper barrier v, $v > u$. When the risk process starts between upper and lower barrier it is almost sure, by the Strong Law of Large Numbers (SLLN), that the process eventually leaves the strip $[0, v]$. In many cases it is interesting to know which barrier (upper or lower) is reached first? What is the probability that we become "rich" (v) before the ruin?

Let T_u be the time of the first ruin, $T_u = \inf\{t : u + X(t) < 0\}$. If ruin ever occurs then $T_u < \infty$, and if ruin never occurs then $T_u = \infty$. Let T_{uv} be the time of the first passage of the upper barrier v, $T_{uv} = \inf\{t : u + X(t) \geq v\}$. Note that for almost all trajectories of $X(t)$ only one of the variables T_u and T_{uv} can take value $+\infty$, since the process achieves at least one of the two barriers a.s. Our aim is to find probabilities $P\{T_{uv} < T_u\}$ and $P\{T_{uv} < T_u < \infty\}$. In the first case we want to know the probability that the trajetory of the risk process leaves the strip $[0, v]$ via the upper barrier v, while in the second case also the ruin must follow.

We consider separately the cases of positive and negative safety loadings. It will be seen that the problem solves easily for $\rho > 0$, while it is not elementary in the case of $\rho < 0$.

4. The case of two barriers, $\rho > 0$

In this section we obtain some results concerning calculation of probabilities $P\{T_{uv} < T_u\}$ and $P\{T_{uv} < T_u < \infty\}$ for the case of positive ρ. We will show that in this case the problem can be reduced to the classical problem of calculation of the ruin probability for the risk processes with one (i.e. lower) barrier. Our result is based on the following simple observations.

1. For $\rho > 0$ the risk process $u + X(t)$ has positive trend and due to the SLLN it is almost certain that the process attains the level v, eventually. Thus $P\{T_{uv} < \infty\} = 1$ which, in turn, implies $P\{T_u < T_{uv}\} = P\{T_u < T_{uv} < \infty\}$.

2. As the equality $T_u = T_{uv}$ is excluded, we have

$$P\{T_{uv} < T_u\} + P\{T_u < T_{uv}\} = 1. \tag{5}$$

3. By dividing the event "ruin" into two parts according to when v is reached, one has

$$P\{T_u < \infty\} = P\{T_{uv} < T_u < \infty\} + P\{T_u < T_{uv}\}. \tag{6}$$

4. Finally, by the multiplication rule
$P\{T_{uv} < T_u < \infty\} = P\{T_{uv} < T_u\}P\{T_u < \infty | T_{uv} < T_u\}$.

Since the ruin from the level u, under the condition that we first reach the upper barrier v, is equivalent to the ruin from the level v, the latter probability can be replaced by $P\{T_v < \infty\}$, and therefore

$$P\{T_{uv} < T_u < \infty\} = P\{T_{uv} < T_u\}P\{T_v < \infty\}. \tag{7}$$

Recall now that the probability of ruin is $\Psi(u) = P\{T_u < \infty\}$ and the probability of non-ruin is $\Phi(u) = 1 - \Psi(u)$. Then, solving (7) for $P\{T_{uv} < T_u\}$ and using (5) and (6) one obtains a surprisingly simple formula

$$P\{T_{uv} < T_u\} = \frac{\Phi(u)}{\Phi(v)}. \tag{8}$$

Thus, we have proved

Proposition 1. *If $\rho > 0$, then the probability of hitting the upper barrier $v(> u)$ without previous ruin is given by* (8).

The formula (8) allows precise calculation of $P\{T_{uv} < T_u\}$ in situations when there is an exact formula available for the ruin probability $\Psi(u)$. For instance, if the claims are exponentially distributed, then for (4) we have

$$P\{T_{uv} < T_u\} = \frac{1 + \rho - e^{-Ru}}{1 + \rho - e^{-Rv}}. \tag{9}$$

At the same time, for large values of u and v, Cramér–Lundberg approximation (3) can be used to evaluate $\Phi(u)$ and $\Phi(v)$. If we require, in addition, the ruin after the first passage of v, then the result is:

$$P\{T_{uv} < T_u < \infty\} = \frac{\Phi(u)}{\Phi(v)}\Psi(v). \tag{10}$$

5. The case of two barriers, $\rho < 0$

We now assume that $\rho < 0$. Then the risk process $u + X(t)$ has negative drift and, again due to the SLLN, we have $\lim_{t\to\infty}[u + X(t)] = -\infty$. Hence, the ruin is inevitable, $\Psi(u) = P\{T_u < \infty\} = 1$, for all $u \geq 0$. Since now $\Phi(u) = 1 - \Psi(u) = 0$ for all u, the formula (8) cannot hold.

5.1. Achieving an upper barrier. Let $v > u \geq 0$. We first find the probability

$$\Psi(u, v) := P\{T_{uv} < \infty\} \equiv P\{\text{starting from } u \text{ the level } v \text{ will be reached}\}. \tag{11}$$

As we clearly have $\Psi(u, v) = \Psi(0, v - u)$, it suffices to study the case $u = 0$ only. There is an important difference between *how* the risk process hits lower and upper barriers. While the lower barrier 0 can only be reached at the moment of a claim (when an amount of money is payed out), any upper barrier v above u can only be reached *between* claims. Furthermore, each trajectory which starts from 0 and hits the level $v > 0$ also hits any level s between 0 and v. Hence, by Markov property, $\Psi(0, v) = \Psi(0, s)\Psi(s, v) = \Psi(0, s)\Psi(0, v - s)$. Therefore the function $\Psi(0, v)$ must be of the form

$$\Psi(0, v) = e^{-\overline{R}v} \tag{12}$$

with some constant $\overline{R} > 0$. We call it *modified Lundberg exponent* and will present an equation to determine $\overline{R}$ below.

5.2. Derivation of integral equation for $P\{T_{uv} < T_u\}$. For further convenience, we denote the probability of leaving the strip $[0, v]$ upwards by

$$\varphi(u, v) := P\{T_{uv} < T_u\}. \tag{13}$$

Note that a simple inequality $\varphi(u, v) \leq \Phi(u, v)$ holds, since the term on the left means restricted path from u to v (keeping $u + X(t)$ on positive side) while the term on the right means unrestricted path from u to v.

We now derive an integral equation for the function $\varphi(u, v)$. As the main tool, the renewal argument is used, similarly to Grandell (1991) where the classical problem with one single barrier is considered. We condition upon the time of the first claim S_1 and its size Z_1 and write $\varphi(u, v) = E[\varphi(u, v)|S_1, Z_1)]$, where $S_1 \sim Exp(\alpha)$ and $Z_1 \sim F$. Note that for values of S_1 greater than $(v - u)/c$ our process attains the level v before the first claim, thus with probability 1. Furthermore, for $S_1 < (v - u)/c$ and for the values of the first claim Z_1 between 0 and $u + cS_1$, the process starts again with initial capital $u + cS_1 - Z_1$, and we may write:

$$\varphi(u, v) = \int_{(v-u)/c}^{\infty} \alpha e^{-\alpha s} ds + \int_0^{(v-u)/c} \int_0^{u+cs} \varphi(u + cs - z, v)\alpha e^{-\alpha s} dF(z) ds. \tag{14}$$

The change of variables $u + cs = x$ leads to

$$\varphi(u,v) = e^{(v-u)/c} + \frac{\alpha}{c} e^{\alpha u/c} \int_u^v e^{\alpha x/c} \int_0^x \varphi(x-z,v)dF(z)ds. \qquad (15)$$

Differentiation by u gives an integro-differential equation:

$$\varphi'(u,v) = \frac{\alpha}{c}\varphi(u,v) - \frac{\alpha}{c}\int_0^u \varphi(u-z,v)dF(z). \qquad (16)$$

Now, integration by u over $[0,t]$ easily leads to

$$\begin{aligned}\varphi(t,v) - \varphi(0,v) =& \frac{\alpha}{c}\int_0^t \varphi(0,v)(1-F(u))du \\ &+ \frac{\alpha}{c}\int_0^t\int_0^u \varphi'(u-z,v)(1-F(z))dzdu.\end{aligned} \qquad (17)$$

After the change of the order of integration we get the following integral equation:

$$\varphi(t,v) - \varphi(0,v) = \frac{\alpha}{c}\int_0^t \varphi(t-z,v)(1-F(z))dz. \qquad (18)$$

The result can be stated as follows.

Proposition 2. *If $\rho < 0$, then the probability $\varphi(u,v)$ of hitting the upper barrier $v(> u)$ without hitting earlier the barrier 0 satisfies the following integral equation*

$$\varphi(u,v) = \varphi(0,v) + \frac{\alpha}{c}\int_0^u \varphi(u-z,v)(1-F(z))dz. \qquad (19)$$

This renewal-type equation is quite similar to the classical equation (2) for the ruin probability. But there is also a big difference between them. Namely, in the classical case it is assumed that $\rho > 0$ and therefore the weight function is *subnormalized* : $\int_0^\infty \frac{\alpha}{c}(1-F(z))dz = \frac{\alpha\mu}{c} = \frac{1}{1+\rho} < 1$. Feller overcame this difficulty by multiplying both sides of the equation (2) by e^{Ru} where R is a properly chosen constant called Lundberg exponent (Feller (1971), Ch. XI, §7a)). In our case $\rho < 0$ and therefore the weight function is *supernormalized*, which makes it impossible to use Feller's trick to solve the equation (19). A further study on how to solve the equation (19) is needed.

5.3. Modified Lundberg exponent. Here we make use of (19) to derive an equation which uniquely determines the modified Lundberg exponent $\overline{R}$.

We first deduce an expression for $\varphi(0,v)$ – the probability that starting with initial capital 0 we first gain the level v and then the ruin follows. For that take $u = v$ in (19):

$$\varphi(v,v) = \varphi(0,v) + \frac{\alpha}{c}\int_0^v \varphi(v-z,v)(1-F(z))dz. \qquad (20)$$

As $\varphi(v,v)=1$, we get

$$\varphi(0,v)=1-\frac{\alpha}{c}\int_0^v \varphi(v-z,v)(1-F(z))dz. \tag{21}$$

Let now $v\to\infty$. On the left hand side clearly $\varphi(0,v)\to 0$. On the right hand side for each fixed $z>0$ and for increasing v the probability of ruin from the level $v-z$ monotonically decreases to zero and therefore the restricted and unrestricted attainings of v from the level $v-z$ become likely probable: $\lim_{v\to\infty}\varphi(v-z,v)=\lim_{v\to\infty}\Psi(v-z,v)=e^{-\overline{R}z}$ where the last equality follows from (12). We see that the result depends only on z. Applying the Monotone Convergence Theorem to the right side of (21) and equating the result to zero, we obtain the relationship

$$\frac{\alpha}{c}\int_0^\infty e^{-\overline{R}z}(1-F(z))dz=1 \tag{22}$$

with $\overline{R}>0$. Note that this equation differs from the definition of Lundberg exponent R only by the sign in the exponent, but as the whole situation is different (wrt ρ), we cannot just take $\overline{R}=-R$. However, for the case of exponentially distributed claims the latter still holds, since solving (22) for $\overline{R}$ gives

$$\overline{R}=\frac{\alpha\mu-c}{\mu c}=-\frac{\rho}{\mu(1+\rho)}=-R, \tag{23}$$

where Lundberg exponent R was evaluated at the end of Section 2.

5.4. Exponentially distributed claims. Assume that $\rho<0$ and let claims be exponentially distributed with expectation $EZ_i=\mu$. We deduce a closed form expression for $\varphi(u,v)$ for this special case. First note that

$$\varphi(u,v)=P\{T_{uv}<\infty\}-P\{T_u<T_{uv}<\infty\}. \tag{24}$$

Due to (12) we already know that $P\{T_{uv}<\infty\}=e^{-\overline{R}(v-u)}$ with $\overline{R}$ defined by (23), and therefore in order to get $\varphi(u,v)$ it suffices to find the probability $P\{T_u<T_{uv}<\infty\}$. The probability that first ruin occurs and then the upper level v is reached can be decomposed as

$$\begin{aligned}P\{T_u<T_{uv}<\infty\} &= P\{T_u<T_{uv}\}\cdot P\{T_{u+X(T_u),v}<\infty\}\\ &= [1-\varphi(u,v)]\cdot P\{T_{u+X(T_u),v}<\infty\}\end{aligned} \tag{25}$$

where $u+X(T_u)<0$ is the wealth of the company immediately after the ruin. The random variable $Z:=-(u+X(T_u))>0$ is called the *overshoot*. From the memoryless property of the exponential distribution the overshoot is also exponentially distributed, $Z\sim Exp(1/\mu)$. Now we condition upon Z:

$$P\{T_{u+X(T_u),v}<\infty\}\equiv P\{T_{-Z,v}<\infty\}=\int_0^\infty P\{T_{-z,v}<\infty\}\cdot\frac{1}{\mu}e^{-z/\mu}dz. \tag{26}$$

As for each fixed z we have $P\{T_{-z,v} < \infty\} = P\{T_{0,z+v} < \infty\} = e^{-\overline{R}(v+z)}$, substitution into (26) leads to

$$P\{T_{u+X(T_u),v} < \infty\} = \frac{e^{-\overline{R}v}}{\overline{R}\mu + 1}. \tag{27}$$

Now, putting together (24), (25) and (27) one easily obtains the formula

$$\varphi(u,v) = \frac{e^{\overline{R}u} - \frac{1}{\overline{R}\mu+1}}{e^{\overline{R}v} - \frac{1}{\overline{R}\mu+1}}. \tag{28}$$

After substitution $\overline{R}$ from (23) we finally get

$$\varphi(u,v) = \frac{1+\rho - e^{-\frac{\rho u}{\mu(1+\rho)}}}{1+\rho - e^{-\frac{\rho v}{\mu(1+\rho)}}}, \tag{29}$$

which coincides (!) with the formula (9) obtained for the case of exponential claims and $\rho > 0$. Therefore, we have proved the following

Proposition 3. *If $\rho \neq 0$ and the claims are exponentially distributed with mean μ, then the probability of reaching an upper barrier $v(> u)$ without previous ruin is given by (29).*

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