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The extended growth curve model – an overview with special reference to two components model

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ABSTRACT. The paper contains an overview of recent results in the extended growth curve model, both with fixed and random effects. Explicit results are shown mainly for the model with two components/profiles. A variance components estimability criterion is proposed.

1. Introduction

The standard extended growth curve model has the form:

$$Y = \sum_{i=0}^{k-1} X_i B_i Z_i + \varepsilon , \tag{1}$$

where

- $B_0, \ldots, B_{k-1}$ are the first order parameters (fixed or random effects),
- X_i and Z_i are known design matrices,
- ε is the random error matrix, $\mathrm{E}\,\varepsilon = 0$, $\mathrm{var}\,(\mathrm{vec}\,\varepsilon) = \Sigma_k \otimes I$,
- dimensions of Y, ε, X_i, B_i and Z_i are $n \times p, n \times p, n \times m_i, m_i \times r_i$, and $r_i \times p$, respectively.

Notation: column space of a matrix G is denoted by $\mathcal{R}(G)$; P_G is the orthogonal projector on $\mathcal{R}(G)$ and $M_G = I - P_G$ the orthogonal projector on its orthogonal complement. If the corresponding (semi)metrics is given by a positive definite (positive semidefinite) matrix A, these projectors will be denoted by P_G^A and M_G^A. The vec-operator stacks columns of an $(m \times n)$-matrix each under the other, thus making from it an mn-vector, $\otimes$ is the Kronecker (tensor, direct) product of matrices. Moore–Penrose inverse of a matrix G is denoted by G^+.

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Often, vectorized (i.e. univariate) form of the model is used:

$$\operatorname{vec}(Y) = \sum_{i=0}^{k-1} \left(Z_i' \otimes X_i\right) \operatorname{vec}(B_i) + \operatorname{vec}(\varepsilon) \ .$$

This allows directly to carry over all univariate results to the multivariate model. However, this is not always recommended, because we lose the multivariate character of the model. For a review of results on the standard growth curve model see Kshirsagar and Smith (1995).

2. Sum-of-profiles model

When all effects are fixed, we have quite simple variance structure, while the structure of the mean is rather complicated:

$$\mathrm{E}\, Y = \sum_{i=0}^{k-1} X_i B_i Z_i, \qquad \operatorname{var}(\operatorname{vec} Y) = \Sigma \otimes I \, . \tag{2}$$

If $\mathcal{R}(X_{k-1}) \subset \mathcal{R}(X_{k-2}) \subset \cdots \subset \mathcal{R}(X_0)$, then maximum likelihood estimators of B_i and Σ are known. However, they can be presented in a half-explicit form only, using recurrent formula. Moreover, the estimator of Σ is not unbiased, see von Rosen (1989).

For $k = 2$ and $\mathcal{R}(X_1) \subset \mathcal{R}(X_0)$, explicit unbiased estimators are available for estimable B_0, B_1 and Σ:

$$\begin{aligned} \widehat{B}_0 &= \left(X_0'X_0\right)^+ X_0'Y\Sigma^{-1}Z_0'\left(Z_0\Sigma^{-1}Z_0'\right)^+ \\ &\quad - \left(X_0'X_0\right)^+ X_0'P_{X_1}Y\left(P_{Z_1'}^{\Sigma^{-1}M_{Z_0'}^{\Sigma^{-1}}}\right)'\Sigma^{-1}Z_0'\left(Z_0\Sigma^{-1}Z_0'\right)^+ , \end{aligned}$$

$$\widehat{B}_1 = \left(X_1'X_1\right)^+ X_1'Y\Sigma^{-1}M_{Z_0'}^{\Sigma^{-1}}Z_1'\left(Z_1\Sigma^{-1}M_{Z_0'}^{\Sigma^{-1}}Z_1'\right)^+ ,$$

$$\widehat{\Sigma} = \frac{1}{n - r(X_0)} Y'M_{X_0}Y \, .$$

If Σ is unknown, we need to put $\widehat{\widehat{B}}_0 = \widehat{B}_0\left(\widehat{\Sigma}^{-1}\right)$, $\widehat{\widehat{B}}_1 = \widehat{B}_1\left(\widehat{\Sigma}^{-1}\right)$. In such a case the unbiasedness of the estimates is lost, see Žežula (2003a).

If $\Sigma = \sigma^2 G$ with G known in the above model, then we can replace Σ by G in $\widehat{B}_i$ and the unbiased estimator of σ^2 is

$$\widehat{\sigma^2} = \frac{\operatorname{Tr}(A)}{r} \, ,$$

where

$$\begin{aligned} A &= Y'M_{X_0}Y + M_{Z_0'}^{G^{-1}}Y'\left(M_{X_0} - M_{X_1}\right)Y\left(M_{Z_0'}^{G^{-1}}\right)' \\ &\quad + M_{(Z_0',Z_1')}^{G^{-1}}Y'P_{X_1}Y\left(M_{(Z_0',Z_1')}^{G^{-1}}\right)', \\ r &= \left(n - r\left(X_0\right)\right)\mathrm{Tr}(G) + \left(r\left(X_0\right) - r\left(X_1\right)\right)\mathrm{Tr}\left(M_{Z_0'}^{G^{-1}}G\right) \\ &\quad + r\left(X_1\right)\mathrm{Tr}\left(M_{(Z_0',Z_1')}^{G^{-1}}G\right). \end{aligned}$$

Important special case is the model with concomitant variables: here $k = 2$, $Z_1 = I_p$, but usually $\mathcal{R}(X_1) \not\subset \mathcal{R}(X_0)$. Because of the last condition, estimators are known in vectorized form only, see e.g. Wesołowska-Janczarek (1996).

Unbiased estimator of the overall mean (not individual components) and variance matrix, as well as the variance σ^2 in the case of known correlation structure were also derived for the model with arbitrary number of components (profiles). For details see Žežula (2003a).

3. Basic mixed model

The model was introduced by Khatri and Shah (1981). Here we consider one fixed effect and several random effects in the model (1). In this case we have a simple mean structure, but very complicated variance structure. Thus, let B_0 be fixed, and $B_1, \ldots, B_{k-1}$ be independent random variables with

$$\begin{aligned} \mathrm{E}\,B_i = 0\,, \qquad & \mathrm{var}\,(\mathrm{vec}\,B_i) = \Sigma_i \otimes I,\ i = 1, \ldots, k-1\,, \\ \mathrm{E}\,Y = X_0 B_0 Z_0\,, \quad & \mathrm{var}\,(\mathrm{vec}\,Y) = \textstyle\sum_{i=1}^{k-1} Z_i'\Sigma_i Z_i \otimes X_i X_i' + \Sigma_k \otimes I\,. \end{aligned} \tag{3}$$

In order to estimate variance components, let us consider quadratic estimators of the form $Y'AY$, $A_{n\times n}$. According to Ghazal and Neudecker (2000) we have

$$\mathrm{E}\,Y'AY = \sum_{i=1}^{k-1} Z_i'\Sigma_i Z_i \cdot \mathrm{Tr}(AX_iX_i') + \Sigma_k \cdot \mathrm{Tr}(A) + Z_0'B_0'X_0'AX_0B_0Z_0\,.$$

In order to exclude unknown B_0, let us take $A = M_{(X_0,P)}$, where P is some matrix. Thus,

$$\mathrm{E}\,Y'M_{(X_0,P)}Y = \sum_{i=1}^{k-1} Z_i'\Sigma_i Z_i \cdot \mathrm{Tr}(M_{(X_0,P)}X_iX_i') + \Sigma_k \cdot \mathrm{Tr}\left(M_{(X_0,P)}\right).$$

By choosing different matrices P we can obtain potentially different linear combinations of variance components. Estimators of their estimable functions can be obtained by solving the system

$$Y'M_{(X_0,P)}Y = \mathrm{E}\,Y'\widehat{M_{(X_0,P)}}Y\,,$$

with variable P.

In particular, choosing $P = \emptyset$, X_1, (X_1, X_2), ..., $(X_1, X_2, \dots, X_{k-1})$, we get

$$\begin{aligned}
Y'M_{X_0}Y &= \textstyle\sum_{i=1}^{k-1} Z_i'\widehat{\Sigma}_i Z_i \cdot \left(\|X_i\|^2 - \|P_{X_0}X_i\|^2\right) + \widehat{\Sigma}_k\,(n - r(X_0))\,,\\
Y'M_{(X_0,X_1)}Y &= \textstyle\sum_{i=2}^{k-1} Z_i'\widehat{\Sigma}_i Z_i \cdot \left(\|X_i\|^2 - \left\|P_{(X_0,X_1)}X_i\right\|^2\right)\\
&\quad + \widehat{\Sigma}_k\,(n - r\,(X_0, X_1))\,,\\
&\dots\ \dots\ \dots,\\
Y'M_{(X_0,X_1,\dots,X_{k-1})}Y &= 0 + \widehat{\Sigma}_k\,(n - r\,(X_0, X_1, \dots, X_{k-1}))\,.
\end{aligned}$$

This is a triangular system. Notice, that the above projection matrices can also be calculated recursively using the formula

$$M_{(X_0,X_1,\dots,X_{k-1})} = M_{X_0} \cdot M_{X_1}^{M_{X_0}} \cdot M_{X_2}^{M_{(X_0,X_1)}} \cdot \dots \cdot M_{X_{k-1}}^{M_{(X_0,X_1,\dots,X_{k-2})}}.$$

If we denote $A = \{a_{ij}\}_{i,j=1}^k$ and $A^- = \{\alpha_{ij}\}_{i,j=1}^k$, where

$$\begin{aligned}
a_{ij} &= \mathrm{Tr}\left(X_j' M_{(X_0,X_1,\dots,X_{i-1})}X_j\right) = \|X_i\|^2 - \left\|P_{(X_0,X_1,\dots,X_{i-1})}X_i\right\|^2,\\
&\qquad i = 1, \dots, k-1,\ j = i, i+1, \dots, k-1\,,\\
a_{ij} &= 0, \quad i = 1, \dots, k,\ j = 1, \dots, i-1\,,\\
a_{ik} &= \mathrm{Tr}\left(M_{(X_0,X_1,\dots,X_{i-1})}\right) = n - r\,(X_0, X_1, \dots, X_{i-1})\,, \quad i = 1, \dots, k\,,
\end{aligned}$$

and

$$Q_j = Y'M_{(X_0,X_1,\dots,X_{j-1})}Y, \quad j = 1, \dots, k\,,$$

then we can see that all solutions of the (potentially singular) system are

$$Z_i'\widehat{\Sigma}_i Z_i = \sum_{j=1}^{k} \alpha_{ij} Q_j$$

(if A is non-singular, $A^- = A^{-1}$ is triangular and the summation goes from $j = i$ to k).

We see that Σ_k is always estimable, unless $r\,(X_0, \dots, X_{k-1}) = n$ (but this is usually not the case in practically important situations). Matrices Z_i are matrices of regression constants and usually of full rank. Thus, estimability of Σ_i depends on the fact whether $\mathcal{R}\,(X_i) \subset \mathcal{R}\,(X_0, \dots, X_{i-1})$ or not, because

$$\mathcal{R}\,(X_i) \subset \mathcal{R}\,(X_0, \dots, X_{i-1}) \Rightarrow \|X_i\|^2 - \left\|P_{(X_0,\dots,X_{i-1})}X_i\right\|^2 = 0\,.$$

The problem of ordering: researcher can – except X_0 – index matrices in this model in an arbitrary way. There are $(k-1)!$ ways how to index (order) random effects. But the estimability condition depends on the chosen way of ordering. As a result, the estimability condition need not be satisfied for one ordering and can be fulfilled for another ordering of the same matrices. Thus, for a specific ordering (indexing), the estimability condition need not be fulfilled even if all the variance components are in fact estimable (i.e. the condition holds for another ordering). That is why we give

Definition. When the conditions $\mathcal{R}(X_i) \cap \mathcal{R}(X_0, \dots, X_{i-1})^c \neq \{0\}\ \forall i$ and $r(X_0, \dots, X_{k-1}) \neq n$ hold after suitable re-indexing of all matrices X_i, we say that there exists a *natural ordering* of these matrices.

In such a case, A^{-1} exists and all variance components are estimable. Thus, existence of a natural ordering is a sufficient condition for the estimability of all variance components. Note that the modified von Rosen's condition

$$\mathcal{R}(X_0) \subsetneqq \mathcal{R}(X_1) \subsetneqq \cdots \subsetneqq \mathcal{R}(X_{k-1})$$

is a sufficient condition for the existence of a natural ordering. In this case,

$$M_{(X_0,\dots,X_{i-1})} = M_{X_{i-1}}, \quad i = 1, \dots, k,$$

and the whole system becomes substantially simpler.

Also note that if a natural ordering exists and we do not use it, a singular matrix A can arise even if all variance components are estimable. Thus, usage of natural orderings for the whole sequence of X_i and its subsequences is a crucial thing.

To check whether a natural ordering exists or not, we can form a *proper subspace graph*:

- the vertices are grouped in "floors",
- the bottom floor is formed by all individual spaces $R_i = \mathcal{R}(X_i)$,
- the 2$^{\text{nd}}$ floor contains all pair-wise union spaces $R_{ij} = \mathcal{R}(X_i, X_j)$,
- ...,
- the top floor contains single vertex $R_{01..k-1} = \mathcal{R}(X_0, X_1, \dots, X_{k-1})$.

An arc (aiming upwards) is put between two vertices in adjacent floors iff the starting space is a proper subspace of the ending space. We form arcs only between adjacent floors of vertices.

It is easy to see that a natural ordering exists iff $R_{01..k-1} \neq \mathbb{R}^n$ and there is a path from (bottom) vertex R_0 to the top vertex.

Example. Let

$$X_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

First, let us notice that R_{012} is a 5-dimensional space, so that $R_{012} \neq \mathbb{R}^6$. Secondly, it is clear that the proper subspace graph for these matrices is

$$\begin{array}{ccccc} & & R_{012} & & \\ & & \uparrow & \nwarrow & \\ R_{01} & & R_{12} & & R_{02} \\ \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\ R_0 & & R_1 & & R_2\ . \end{array}$$

We see that there are 3 paths from the bottom to the top, which implies that we can have the following 3 orderings satisfying our conditions:

$$X_1 \to X_2 \to X_0, \qquad X_2 \to X_1 \to X_0, \quad \text{and} \quad X_2 \to X_0 \to X_1,$$

but – as none of these paths starts in R_0 – there is no natural ordering.

If the design matrix of fixed effects would have been X_1 or X_2, a natural ordering would exist. □

The problem of the existence of a natural ordering reduces to $r\,(X_0, X_1) < n$ and $\mathcal{R}\,(X_1) \not\subset \mathcal{R}\,(X_0)$ for $k = 2$, which is very simple to check.

If no such natural ordering exists, it is an open question whether changing the ordering of matrices X_i or using another P matrices can change our possibilities to estimate more of Σ_i.

However, the system which one suggests is that existence of a natural ordering of matrices X_i could be not only sufficient but also necessary condition for the unbiased estimability of all variance components.

3.1. Testing variance components. Let us consider another sequence of quadratic forms using projectors

$$P_i = M_{(X_0,\dots,X_{i-1})} P_{X_i}^{M_{(X_0,\dots,X_{i-1})}} \ , \quad i = 1, \dots, k-1,$$

and

$$P_k = M_{(X_0,\dots,X_{k-1})} \ .$$

Let us define

$$SS_i = \frac{1}{r\,(X_0,\dots,X_i) - r\,(X_0,\dots,X_{i-1})} Y' P_i Y \ , \quad i = 1, \dots, k-1\,,$$

and

$$SS_e = SS_k = \frac{1}{n - r\,(X_0,\dots,X_{k-1})} Y' P_k Y \ .$$

It is easy to see that

$$\mathrm{E}\,SS_i = \sum_{j=i}^{k-1} \frac{\mathrm{Tr}\left(X_j' P_i X_j\right)}{Tr\,(P_i)} Z_j' \Sigma_j Z_j + \Sigma_k \ , \quad i = 1, \dots, k-1\,,$$

and

$$\mathrm{E}\, SS_k = \Sigma_k \ ,$$

where

$$Tr\,(P_i) = r\,(X_0, \dots, X_i) - r\,(X_0, \dots, X_{i-1})$$

in the former expression.

We can call SS_i the generalized sums of squares. Since $P_i P_{i+1} = 0$ for $i = 1, \dots, k-1$, we can easily see that all these sums must be independent under normality. If a natural ordering of matrices X_i exists, this can be used for testing of variance components.

3.2. Two variance components. In the simplest case, when $k = 2$, $r\,(X_0, X_1) < n$, and $\mathcal{R}\,(X_1) \not\subset \mathcal{R}\,(X_0)$, both variance components are estimable:

$$\widehat{\Sigma}_2 = \frac{1}{n - r(X_0, X_1)} Y' M_{(X_0, X_1)} Y$$

and

$$Z_1' \widehat{\Sigma}_1 Z_1 = \frac{1}{\mathrm{Tr}\,(X_1' M_{X_0} X_1)} \left[Y' M_{X_0} Y - \frac{n - r(X_0)}{n - r(X_0, X_1)} Y' M_{(X_0, X_1)} Y \right] .$$

Explicit form of the plug-in estimator is not available. We have to use the vectorized form of the estimator

$$\begin{aligned} \mathrm{vec}\, \widehat{\widehat{B}}_0 \ &= \ \left[(Z_0 \otimes X_0') \left(\widehat{\Sigma}_2 \otimes I_n + Z_1' \widehat{\Sigma}_1 Z_1 \otimes X_1 X_1' \right)^{-1} (Z_0' \otimes X_0) \right]^{-1} \\ &\quad \times (Z_0 \otimes X_0') \left(\widehat{\Sigma}_2 \otimes I_n + Z_1' \widehat{\Sigma}_1 Z_1 \otimes X_1 X_1' \right)^{-1} \mathrm{vec}\, Y \, , \end{aligned}$$

and de-vectorize it after the computation.

If we want to test the hypothesis $H_0 : \Sigma_1 = 0$, we can use the fact that

$$\mathrm{E}\, SS_1 = \mathrm{E}\, SS_e$$

under H_0. Also, under the normality assumption, we have

$$Y' M_{X_0} P_{X_1}^{M_{X_0}} Y \overset{H_0}{\sim} \mathcal{W}_p \left(\mathrm{Tr} \left(M_{X_0} P_{X_1}^{M_{X_0}} \right), \Sigma_2 \right)$$

and

$$Y' M_{(X_0, X_1)} Y \sim \mathcal{W}_p \left(\mathrm{Tr}\, \left(M_{(X_0, X_1)} \right), \Sigma_2 \right) \ .$$

Therefore we can use the standard tests for H_0:

Wilks's $\Lambda = \dfrac{|SS_e|}{|SS_e + SS_1|}$,

Hotelling–Lawley's $T = \mathrm{Tr}\,(SS_e^{-1} SS_1)$,

Pillai's $P = \mathrm{Tr}\left[SS_1\,(SS_e + SS_1)^{-1} \right]$,

Roy's $M = \lambda_{max}\,(SS_e^{-1} SS_1)$.

3.3. Other special cases. Some results are also available for special variance structures, namely $\Sigma_i = \sigma_i^2 G_i$, and in the two components model also for the uniform correlation structure

$$\Sigma_1 = \sigma_1^2 \left((1-\rho_1)I_{r_1} + \rho_1 \mathbf{1}\mathbf{1}'\right) , \qquad \Sigma_2 = \sigma_2^2 \left((1-\rho_2)I_p + \rho_2 \mathbf{1}\mathbf{1}'\right) ,$$

and the serial correlation structure

$$\Sigma_1 = \sigma_1^2 \begin{pmatrix} 1 & \rho_1 & \dots & \rho_1^{r_1-1} \\ \rho_1 & 1 & \dots & \rho_1^{r_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{r_1-1} & \rho^{r_1-2} & \dots & 1 \end{pmatrix}, \; \Sigma_2 = \sigma_2^2 \begin{pmatrix} 1 & \rho_2 & \dots & \rho_2^{p-1} \\ \rho_2 & 1 & \dots & \rho_2^{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_2^{p-1} & \rho_2^{p-2} & \dots & 1 \end{pmatrix}.$$

Unbiased estimating equations for the parameters in these models are known, for details see Žežula (2003b). For comparison with the standard growth curve model see Lee and Geisser (1996).

Lindsey (1999) uses in some situations model (3) with

$$X_i = I \quad \forall i = 1, \dots, k-1 .$$

In this model the individual matrices Σ_i are not estimable. But we have

$$\operatorname{var}(\operatorname{vec} Y) = \left(\sum_{i=1}^{k-1} Z_i' \Sigma_i Z_i + \Sigma_k\right) \otimes I_n = \Gamma \otimes I_n ,$$

which is the usual growth curve model. The unbiased parameter estimators (if estimable) are

$$\widehat{B}_0 = \left(X_0' X_0\right)^+ X_0' Y \Gamma^{-1} Z_0' \left(Z_0 \Gamma^{-1} Z_0'\right)^+ ,$$

and

$$\widehat{\Gamma} = \frac{1}{n - r(X_0)} Y' M_{X_0} Y .$$

3.4. Balanced case. If model (3) is a balanced one (i.e. all matrices X_i are Kronecker products of vectors $\mathbf{1}$ and identity matrices I), Brown–Mathew condition is fulfilled: there exist $\gamma_i \geq 0$ such that

$$P_{X_0} X_i X_i' P_{X_0} = \gamma_i P_{X_0} , \quad \forall i = 1, \dots, k-1 .$$

The unbiased estimator of B_0 is then

$$\widehat{B}_0 = \left(X_0' X_0\right)^{-1} X_0' Y \Gamma^{-1} Z_0' \left(Z_0 \Gamma^{-1} Z_0'\right)^{-1} ,$$

where

$$\Gamma = \sum_{j=1}^{k-1} \gamma_j Z_j' \Sigma_j Z_j + \Sigma_k ;$$

the unbiased estimator of Γ is

$$\widehat{\Gamma} = \sum_{i=1}^{k-1} \sum_{j=i}^{k} \gamma_i \alpha_{ij} Y' M_{(X_0, X_1, \ldots, X_{j-1})} Y + \alpha_{kk} Y' M_{(X_0, X_1, \ldots, X_{k-1})} Y$$

(if A^{-1} exists, see the beginning of this section).

In many situations we are interested in the best linear unbiased predictor (BLUP) of a linear function of the fixed effects plus random effects. In our setting, probably most suitable multivariate analogue of such a vector is $L_1 B_0 L_2 + B_1$ (for comparison with the univariate case see Searle, Casella and McCulloch (1992)). For $k = 2$, BLUP of this matrix is available, but in the vectorized form only:

$$\text{BLUP}\left[\operatorname{vec}(L_1 B_0 L_2 + B_1)\right] = \operatorname{vec}\left(L_1 \widehat{\widehat{B}}_0 L_2\right) +$$
$$+ \left(\widehat{\Sigma}_1 Z_1 \otimes X_1'\right)\left(Z_1' \widehat{\Sigma}_1 Z_1 \otimes X_1 X_1' + \widehat{\Sigma}_2 \otimes I\right)^{-1} \operatorname{vec}\left(Y - X_0 \widehat{\widehat{B}}_0 Z_0\right).$$

As it was proved in Žežula (1999), we can write the inverse of the variance matrix in the form

$$\left(Z_1' \widehat{\Sigma}_1 Z_1 \otimes X_1 X_1' + \widehat{\Sigma}_2 \otimes I\right)^{-1} = \Gamma^{-1} \otimes P_{X_0} + \sum_{j=1}^{s} \Delta_j^{-1} \otimes M_j ,$$

where $\Gamma = \gamma_1 Z_1' \Sigma_1 Z_1 + \Sigma_2$ and M_j form mutually perpendicular decomposition of M_{X_0} (s depends on the estimability of individual variance components). Using this result, we can easily see that BLUP of $L_1 B_0 L_2 + B_1$ is

$$L_1 \widehat{\widehat{B}}_0 L_2 + X_1' P_{X_0} \left(Y - X_0 \widehat{\widehat{B}}_0 Z_0\right) \widehat{\Gamma}^{-1} Z_1' \widehat{\Sigma}_1 + \sum_{j=1}^{s} X_1' M_j Y \widehat{\Delta}_j^{-1} Z_1' \widehat{\Sigma}_1 ,$$

where $\widehat{\Gamma} = \gamma_1 Z_1' \widehat{\Sigma}_1 Z_1 + \widehat{\Sigma}_2$ and $\widehat{\Delta}_j$ are also functions of variance components estimators; see Žežula (1999) and Žežula (2000).

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