Concerning continuity of inverse in quotients of topological algebras

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Abstract. We construct a commutative complete unital locally convex algebra $A$, so that the operation of taking an inverse in $A$, $x \rightarrow x^{-1}$, is continuous on the group $G(A)$ of all invertible elements in $A$, but there is a closed ideal $I \subset A$ such that the operation of taking an inverse is discontinuous in the quotient algebra $A/I$.

By a topological algebra we mean a (real or complex) Hausdorff topological vector space (t.v.s.) equipped with a jointly continuous associative multiplication. For informations about topological algebras the reader is referred to [1], [2] or [3]. In this paper we shall show (contrary to a claim in [1], p. 71) that it can exist a (commutative) topological algebra with a continuous inverse, whose some quotient algebra has such an inverse discontinuous. This result will be based upon the following theorem obtained in [4] (the second part of the theorem is not formulated there, but it easily follows from its proof):

Theorem A. Let $A$ be a countably generated algebra. Then

(i) The algebra $A$ equipped with the maximal locally convex topology $\tau_{\max}^{LC}$ is a complete locally convex topological algebra.

(ii) The topology of any quotient algebra of $A$ (the quotient topology) is again the topology $\tau_{\max}^{LC}$.

The above means that there is a countable subset $S \subset A$, such that $A$ coincides with its smallest subalgebra containing $S$. The topology $\tau_{\max}^{LC}$ on a t.v.s. $X$ is given by means of all seminorms on it (all seminorms are continuous). Under this topology all linear subspaces of $X$ are closed (cf. [4]). If $A$ is any locally convex algebra with the topology given by means

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of a family \( (\| \cdot \|_\alpha) \) of seminorms, and if \( I \) is a closed two-sided ideal in \( A \), then the quotient topology on \( A/I \) is given by all seminorms of the form \( \| x + I \|_\alpha = \inf \{ \| x + z \|_\alpha : z \in I \} \).

We pass now to the construction of our example. Let \( A \) be the (commutative, unital) algebra of all polynomials with scalar (real or complex) coefficients, in the variables \( t, t_0, t_1, t_2, \ldots \), its unit element will be denoted by \( e \). We equip it with the topology \( \tau_{\max} \), so that by the theorem A (i), it becomes a complete locally convex topological algebra. Clearly it has a continuous inverse, since the only invertible elements in \( A \) are non-zero scalar multiples of \( e \). Define in \( A \) an ideal \( I \) by means of the following relations:

\[
vt_0 = e,
\]

and

\[
(t + k^{-1}e)t_k = e, \quad k = 1, 2, \ldots,
\]

i.e. the elements of \( I \) are of the form

\[
x = u_0(vt_0 - e) + \sum_{k=1}^n u_k((t + k^{-1}e)t_k - e), \quad n \in \mathbb{N},
\]

where \( u_k, k = 0, 1, \ldots, n, \) are arbitrary elements in \( A \). The ideal \( I \) is closed, since all linear subspaces of \( A \) are closed. For any element \( u \) in \( A \) denote by \([u]\) its coset modulo \( I \), i.e. the set \( u + I \), so that the quotient algebra \( A = A/I \) consists of all such cosets. Clearly the elements \([t + k^{-1}e], k = 1, 2, \ldots\), are invertible in \( A \) with inverses \([t_k]\) since the coset \([e]\) is the unit element in \( A \). Also \([t]\) is invertible with the inverse \([t_0]\).

First we shall show that the elements \([t_0], [t_1], \ldots\), are linearly independent in \( A \). Or, equivalently, the elements \( t_0, t_1, \ldots \) are linearly independent modulo the ideal \( I \), i.e. the relation

\[
x = \alpha_0t_0 + \alpha_1t_1 + \cdots + \alpha_nt_n \in I,
\]

where \( \alpha_i \) are scalar coefficients, implies \( \alpha_0 = \alpha_1 = \ldots = \alpha_n = 0 \). Since \( x \) is in \( I \), it is of the form (3) and we can write

\[
\alpha_0t_0 + \alpha_1t_1 + \cdots + \alpha_nt_n = u_0(vt_0 - e) + \sum_{k=1}^n u_k((t + k^{-1}e)t_k - e),
\]

for some \( u_0, \ldots, u_m \in A \). Without loss of generality we can assume \( m \geq n \). Take now rational functions \( t(\xi) = \xi, t_0(\xi) = \xi^{-1} \), and \( t_k(\xi) = (\xi + k^{-1})^{-1} \) for \( k \geq 1 \) and substitute them instead of variables \( t, t_0, \ldots, t_m \) into the formula (4), replacing there the unity \( e \) by the constant equal to \( 1 \). Since our rational functions satisfy relations (1) and (2), the right-hand expression in the formula (4) is zero, so that the left-hand is zero too. But the considered rational functions are linearly independent. To see this, observe that \( t_0(\xi) \) has a pole at \( \xi_0 = 0 \), \( t_0(\xi), k \geq 1 \), has a pole at \( \xi_k = -k^{-1} \) and \( t_m(\xi_k) \) is finite.
for $m \neq k$. Consequently $\alpha_0 = \cdots = \alpha_n = 0$ and the considered elements are linearly independent.

Observe now that the sequence $[t + k^{-1}e] = [t] + k^{-1}[e]$ tends to $[t]$ in each seminorm on $A$, as $k \to \infty$, and so, by (ii) of the Theorem A, in the topology of $A$. In order to obtain our conclusion it is sufficient to show that the inverses $[t_k]$ of the elements $[t + k^{-1}e]$ do not tend to the inverse $[t_0]$ of $[t]$. To this end it is sufficient to construct a seminorm $|\cdot|_0$ on $A$ such that

$$[t_k - t_0]|_0 = 1 \text{ for all } k = 1, 2, \ldots.$$ 

Since the elements $[t_k], k \geq 0$, are linearly independent, the elements $[t_k - t_0], k \geq 1$, are linearly independent too, and so they can be included into a Hamel basis $(\eta_i), i = 1, 2, \ldots$, for $A$ (one can easily see that such a countable basis exists). Now, every element $x$ in $A$ can be written as

$$x = \sum_i f_i(x)\eta_i,$$

where $f_i$ are linear functionals on $A$, and for each $x \in A$ only finitely many values $f_i(x)$ are different from zero. We put now

$$|x|_0 = \sum_i |f_i(x)|.$$ 

Clearly it is a seminorm on $A$, and it is continuous, since all seminorms there are continuous. We have also $|\eta_i|_0 = 1$ for all $i$. Consequently, $[t_k - t_0]|_0 = 1$ for all $k$, and we are done.

The algebra of the above example is not metrizable. The author does not know whether a similar construction is possible for an $F$-algebra (a completely metrizable topological algebra). It is also not known, whether it can exist a complete topological algebra without proper topologically invertible elements, such that some of its quotient algebras has such elements. Recall that an element $a$ in a unital topological (not necessarily commutative) algebra $A$ is said topologically invertible, if there are nets $(u_\alpha)$ and $(v_\beta)$ of elements of $A$, such that $\lim \alpha u_\alpha a = \lim \beta v_\beta a = e$, and such an element is said proper, if it is non-invertible. We have shown in [5], that a commutative $F$-algebra with a discontinuous inverse must possess proper topologically invertible elements. The constructed above example shows that such a result fails to be true in case when the algebra in question is non-metrizable. In fact, our algebra $A$ has a discontinuous inverse, and cannot have proper topologically invertible elements, since for such an element $a$ the ideal $I = aA$ would be dense (for an arbitrary $u$ in $A$ the elements $uav_\beta$ are in $I$ and tend to $u$), and $A$ has all ideals closed.
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References


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