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Concerning continuity of inverse in quotients of topological algebras

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ABSTRACT. We construct a commutative complete unital locally convex algebra A, so that the operation of taking an inverse in A, $x \to x^{-1}$, is continuous on the group $G(A)$ of all invertible elements in A, but there is a closed ideal $I \subset A$ such that the operation of taking an inverse is discontinuous in the quotient algebra A/I.

By a topological algebra we mean a (real or complex) Hausdorff topological vector space (t.v.s.) equipped with a jointly continuous associative multiplication. For informations about topological algebras the reader is referred to [1], [2] or [3]. In this paper we shall show (contrary to a claim in [1], p. 71) that it can exist a (commutative) topological algebra with a continuous inverse, whose some quotient algebra has such an inverse discontinuous. This result will be based upon the following theorem obtained in [4] (the second part of the theorem is not formulated there, but it easily follows from its proof):

Theorem A. *Let A be a countably generated algebra. Then*

(i) *The algebra A equipped with the maximal locally convex topology τ_{max}^{LC} is a complete locally convex topological algebra.*

(ii) *The topology of any quotient algebra of A (the quotient topology) is again the topology τ_{max}^{LC}.*

The above means that there is a countable subset $S \subset A$, such that A coincides with its smallest subalgebra containing S. The topology τ_{max}^{LC} on a t.v.s. X is given by means of all seminorms on it (all seminorms are continuous). Under this topology all linear subspaces of X are closed (cf. [4]). If A is any locally convex algebra with the topology given by means

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of a family $(||\cdot||_\alpha)$ of seminorms, and if I is a closed two-sided ideal in A, then the quotient topology on A/I is given by all seminorms of the form $||x+I||'_\alpha = \inf\{||x+z||_\alpha : z \in I\}$.

We pass now to the construction of our example. Let A be the (commutative, unital) algebra of all polynomials with scalar (real or complex) coefficients, in the variables $t, t_0, t_1, t_2, \ldots$, its unit element will be denoted by e. We equip it with the topology τ^{LC}_{max}, so that by the theorem A (i), it becomes a complete locally convex topological algebra. Clearly it has a continuous inverse, since the only invertible elements in A are non-zero scalar multiples of e. Define in A an ideal I by means of the following relations:

$$tt_0 = e, \tag{1}$$

and

$$(t + k^{-1}e)t_k = e, \quad k = 1, 2, \ldots, \tag{2}$$

i.e. the elements of I are of the form

$$x = u_0(tt_0 - e) + \sum_{k=1}^{n} u_k((t + k^{-1}e)t_k - e), \ n \in \mathbb{N}, \tag{3}$$

where $u_k, k = 0, 1, \ldots, n$, are arbitrary elements in A. The ideal I is closed, since all linear subspaces of A are closed. For any element u in A denote by $[u]$ its coset modulo I, i.e. the set $u+I$, so that the quotient algebra $\mathcal{A} = A/I$ consists of all such cosets. Clearly the elements $[t + k^{-1}e], k = 1, 2, \ldots$, are invertible in $\mathcal{A}$ with inverses $[t_k]$ since the coset $[e]$ is the unit element in $\mathcal{A}$. Also $[t]$ is invertible with the inverse $[t_0]$.

First we shall show that the elements $[t_0], [t_1], \ldots$, are linearly independent in $\mathcal{A}$. Or, equivalently, the elements $t_0, t_1, \ldots$ are linearly independent modulo the ideal I, i.e. the relation

$$x = \alpha_0 t_0 + \alpha_1 t_1 + \cdots + \alpha_n t_n \in I,$$

where α_i are scalar coefficients, implies $\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0$. Since x is in I, it is of the form (3) and we can write

$$\alpha_0 t_0 + \alpha_1 t_1 + \cdots + \alpha_n t_n = u_0(tt_0 - e) + \sum_{k=1}^{m} u_k((t + k^{-1}e)t_k - e), \tag{4}$$

for some $u_0, \ldots, u_m \in A$. Without loss of generality we can assume $m \geq n$. Take now rational functions $t(\xi) = \xi, t_0(\xi) = \xi^{-1}$, and $t_k(\xi) = (\xi + k^{-1})^{-1}$ for $k \geq 1$ and substitute them instead of variables $t, t_0, \ldots, t_m$ into the formula (4), replacing there the unity e by the constant equal to 1. Since our rational functions satisfy relations (1) and (2), the right-hand expression in the formula (4) is zero, so that the left-hand is zero too. But the considered rational functions are linearly independent. To see this, observe that $t_0(\xi)$ has a pole at $\xi_0 = 0$, $t_k(\xi), k \geq 1$, has a pole at $\xi_k = -k^{-1}$ and $t_m(\xi_k)$ is finite for $m \neq k$. Consequently $\alpha_0 = \cdots = \alpha_n = 0$ and the considered elements are linearly independent.

Observe now that the sequence $[t + k^{-1}e] = [t] + k^{-1}[e]$ tends to $[t]$ in each seminorm on $\mathcal{A}$, as $k \to \infty$, and so, by (ii) of the Theorem A, in the topology of $\mathcal{A}$. In order to obtain our conclusion it is sufficient to show that the inverses $[t_k]$ of the elements $[t + k^{-1}e]$ do not tend to the inverse $[t_0]$ of $[t]$. To this end it is sufficient to construct a seminorm $|\cdot|_0$ on $\mathcal{A}$ such that

$$|[t_k - t_0]|_0 = 1 \quad \text{for} \quad \text{all} \quad k = 1, 2, \ldots .$$

Since the elements $[t_k], k \geq 0$, are linearly independent, the elements $[t_k - t_0], k \geq 1$, are linearly independent too, and so they can be included into a Hamel basis $(\eta_i), i = 1, 2, \ldots$, for $\mathcal{A}$ (one can easily see that such a countable basis exists). Now, every element x in $\mathcal{A}$ can be written as

$$x = \sum_i f_i(x)\eta_i,$$

where f_i are linear functionals on $\mathcal{A}$, and for each $x \in \mathcal{A}$ only finitely many values $f_i(x)$ are different from zero. We put now

$$|x|_0 = \sum_i |f_i(x)|.$$

Clearly it is a seminorm on $\mathcal{A}$, and it is continuous, since all seminorms there are continuous. We have also $|\eta_i|_0 = 1$ for all i. Consequently, $|[t_k - t_0]|_0 = 1$ for all k, and we are done.

The algebra of the above example is not metrizable. The author does not know whether a similar construction is possible for an F-algebra (a completely metrizable topological algebra). It is also not known, whether it can exist a complete topological algebra without proper topologically invertible elements, such that some of its quotient algebras has such elements. Recall that an element a in a unital topological (not necessarily commutative) algebra A is said topologically invertible, if there are nets (u_α) and (v_β) of elements of A, such that $\lim_\alpha u_\alpha a = \lim_\beta a v_\beta = e$, and such an element is said proper, if it is non-invertible. We have shown in [5], that a commutative F-algebra with a discontinuous inverse must possess proper topologically invertible elements. The constructed above example shows that such a result fails to be true in case when the algebra in question is non-metrizable. In fact, our algebra $\mathcal{A}$ has a discontinuous inverse, and cannot have proper topologically invertible elements, since for such an element a the ideal $I = a\mathcal{A}$ would be dense (for an arbitrary u in $\mathcal{A}$ the elements uav_β are in I and tend to u), and $\mathcal{A}$ has all ideals closed.

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