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On curvature tensors of non-symmetric affine connection

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ABSTRACT. M. Prvanović [1], using polylinear mappings, has obtained four curvature tensors of non-symmetric affine connection on a differentiable manifold. In the present work we obtain eight new curvature tensors of this connection (Theorem 1) and prove that among these twelve curvature tensors only five are independent, and the others can be expressed in terms of them (Theorem 2).

0. Introduction

Let $\mathcal{M}_N$ be an N-dimensional differentiable manifold, on which a non-symmetric affine connection $\overset{1}{\nabla}$ is introduced. Let $X, Y \in \mathcal{X}(\mathcal{M}_N)$, where $\mathcal{X}(\mathcal{M}_N)$ is the Lie algebra of smooth vector fields on $\mathcal{M}_N$. The mapping (see [1]) $\overset{2}{\nabla} : \mathcal{X}(\mathcal{M}_N) \times \mathcal{X}(\mathcal{M}_N) \to \mathcal{X}(\mathcal{M}_N)$, given by

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X, Y], \tag{0.1}$$

defines another non-symmetric affine connection. This means that

$$\begin{aligned} &a) \ \overset{\theta}{\nabla}_{X+V} Y = \overset{\theta}{\nabla}_X Y + \overset{\theta}{\nabla}_V Y, \qquad &&b) \ \overset{\theta}{\nabla}_{fX} Y = f \overset{\theta}{\nabla}_X Y, \\ &c) \ \overset{\theta}{\nabla}_X (Y+V) = \overset{\theta}{\nabla}_X Y + \overset{\theta}{\nabla}_X V, \qquad &&d) \ \overset{\theta}{\nabla}_X (fY) = Xf \cdot Y + f \overset{\theta}{\nabla}_X Y, \end{aligned} \tag{0.2}$$

for $\theta = 1, 2;\ X, Y, V \in \mathcal{X}(\mathcal{M}_N);\ f \in \mathcal{F}(\mathcal{M}_N)$, where $\mathcal{F}(\mathcal{M}_N)$ is the algebra of smooth real functions on $\mathcal{M}_N$.

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It is proved in [1] that equations

$$
\begin{aligned}
\overset{\theta}{R}(X,Y)Z &= \overset{\theta}{\nabla}_X \overset{\theta}{\nabla}_Y Z - \overset{\theta}{\nabla}_Y \overset{\theta}{\nabla}_X Z - \overset{\theta}{\nabla}_{[X,Y]} Z, \quad \theta = 1,2, \\
\overset{3}{R}(X,Y)Z &= \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{2}{\nabla}_X Y} Z, \qquad (0.3) \\
\overset{4}{R}(X,Y)Z &= \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{2}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{1}{\nabla}_X Y} Z
\end{aligned}
$$

define four curvature tensor fields on $\mathcal{M}_{\mathcal{N}}$.

If we introduce local coordinates $x^1, x^2, ..., x^N$ in a coordinate neighborhood U and put

$$X_i = \partial/\partial x^i, \qquad (0.4)$$

then for $X = X_i$, $Y = X_j$ from (0.1) it follows that

$$\overset{2}{\nabla}_{X_i} X_j = \overset{1}{\nabla}_{X_j} X_i, \qquad (0.5)$$

because $[X_i, X_j] = 0$.

Denoting by L^i_{jk} the connection coefficients in the base $X_1, X_2, ..., X_N$, we can write

$$\overset{1}{\nabla}_{X_k} X_j = L^p_{jk} X_p, \qquad (0.6)$$

and then from (0.5)

$$\overset{2}{\nabla}_{X_k} X_j = L^p_{kj} X_p. \qquad (0.7)$$

1. New curvature tensors of non-symmetric connection

We shall prove that, besides above-mentioned curvature tensors (0.3), there are also other such tensors for the non-symmetric affine connection.

Theorem 1. *By mappings*

$$\overset{5}{R}(X,Y)Z = \nabla^{1\ 2}_{X\,Y} Z + \tilde{\nabla}^{1\ 2}_{Y\,X} Z - \overset{1}{\nabla}_{[X,Y]} Z, \qquad (1.1)$$

$$\overset{6}{R}(X,Y)Z = \nabla^{2\ 1}_{X\,Y} Z - \tilde{\nabla}^{1\ 2}_{Y\,X} Z - \overset{2}{\nabla}_{[X,Y]} Z, \qquad (1.2)$$

$$\overset{7}{R}(X,Y)Z = \frac{1}{2}\{\tilde{\tilde{\nabla}}^{1\ 2}_{X\,Y} Z - \tilde{\nabla}^{1\ 2}_{Y\,X} Z - \overset{1}{\nabla}_{[X,Y]} Z - \overset{2}{\nabla}_{[X,Y]} Z\}, \qquad (1.3)$$

$$\overset{8}{R}(X,Y)Z = \frac{1}{2}\{\tilde{\nabla}^{1\,2}_{X\,Y} Z - \tilde{\nabla}^{1\,2}_{Y\,X} Z - \overset{1}{\nabla}_{[X,Y]} Z - \overset{2}{\nabla}_{[X,Y]} Z\}, \tag{1.4}$$

$$\overset{9}{R}(X,Y)Z = \frac{1}{2}\{\tilde{\nabla}^{1\,2}_{X\,Y} Z - \tilde{\tilde{\nabla}}^{1\,2}_{Y\,X} Z - \overset{1}{\nabla}_{[X,Y]} Z - \overset{2}{\nabla}_{[X,Y]} Z\}, \tag{1.5}$$

$$\begin{aligned}\overset{10}{R}(X,Y)Z = \frac{1}{6}\{ & 2(\tilde{\nabla}^{1\,2}_{X\,Y} Z - \tilde{\nabla}^{1\,2}_{Y\,X} Z + \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{2}{\nabla}_Y \overset{2}{\nabla}_X Z \\ & - \overset{1}{\nabla}_{[X,Y]} Z - 2\overset{2}{\nabla}_{[X,Y]} Z) + \overset{1}{\nabla}_{\overset{1}{\nabla}_X Y} Z \\ & + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{2}{\nabla}_X Y} Z - \overset{2}{\nabla}_{\overset{2}{\nabla}_Y X} Z\},\end{aligned} \tag{1.6}$$

$$\overset{11}{R}(X,Y)Z = \tilde{\nabla}^{1\,2}_{X\,Y} Z - \nabla^{1\,2}_{X\,Y} Z - \overset{2}{\nabla}_{[X,Y]} Z, \tag{1.7}$$

$$\overset{12}{R}(X,Y)Z = \tilde{\nabla}^{1\,2}_{X\,Y} Z - \nabla^{2\,1}_{X\,Y} Z - \overset{1}{\nabla}_{[X,Y]} Z, \tag{1.8}$$

where $(\theta, \omega \in \{1,2\})$

$$\nabla^{\theta\,\omega}_{X\,Y} Z = \overset{\theta}{\nabla}_X \overset{\theta}{\nabla}_Y Z + \overset{\omega}{\nabla}_Y \overset{\omega}{\nabla}_X Z, \tag{1.9}$$

$$\tilde{\nabla}^{\theta\,\omega}_{X\,Y} Z = \overset{\theta}{\nabla}_X \overset{\omega}{\nabla}_Y Z + \overset{\omega}{\nabla}_X \overset{\theta}{\nabla}_Y Z, \tag{1.10}$$

$$\tilde{\tilde{\nabla}}^{\theta\,\omega}_{X\,Y} Z = \overset{\theta}{\nabla}_X \overset{\theta}{\nabla}_Y Z + \overset{\omega}{\nabla}_Y \overset{\omega}{\nabla}_X Z, \tag{1.11}$$

curvature tensor fields of non-symmetric affine connection are defined.

Proof. We have to prove the $\mathcal{F}(\mathcal{M}_N)$-linearity with respect to each of arguments X, Y, Z. Using (0.2) and properties of the commutator $[\,\cdot\,,\,\cdot\,]$, we obtain from (1.1), (1.9) and (1.10) that

$$\overset{5}{R}(X+V,Y)Z = \overset{5}{R}(X,Y)Z + \overset{5}{R}(V,Y)Z,$$

$$\begin{aligned}\overset{5}{R}(fX,Y)Z &= \overset{1}{\nabla}_{fX} \overset{1}{\nabla}_Y Z + \overset{2}{\nabla}_Y \overset{2}{\nabla}_{fX} Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_{fX} Z \\ &\quad - \overset{2}{\nabla}_Y \overset{1}{\nabla}_{fX} Z - \overset{1}{\nabla}_{[fX,Y]} Z \\ &= f\overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z + \overset{2}{\nabla}_Y (f\overset{2}{\nabla}_X Z) - \overset{1}{\nabla}_Y (f\overset{2}{\nabla}_X Z) \\ &\quad - \overset{2}{\nabla}_Y (f\overset{1}{\nabla}_X Z) + \overset{1}{\nabla}_{Yf\cdot X + f[Y,X]} Z \\ &= f\overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z + Yf\overset{2}{\nabla}_X Z + f\overset{2}{\nabla}_Y \overset{2}{\nabla}_X Z - Yf\overset{2}{\nabla}_X Z - f\overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z \\ &\quad - Yf\overset{1}{\nabla}_X Z - f\overset{2}{\nabla}_Y \overset{1}{\nabla}_X Z + Yf\overset{1}{\nabla}_X Z + f\overset{1}{\nabla}_{[Y,X]} Z \\ &= f\overset{5}{R}(X,Y)Z.\end{aligned}$$

In this manner the linearity with respect to X is proved. In the same way we prove the linearity with respect to Y.

From (1.1) one concludes that

$$\overset{5}{R}(X,Y)(Z+V) = \overset{5}{R}(X,Y)Z + \overset{5}{R}(X,Y)V.$$

Further, based on (1.1) and (0.2), we have

$$\begin{aligned}
\overset{5}{R}(X,Y)(fZ) &= \overset{1}{\nabla}_X \overset{1}{\nabla}_Y (fZ) + \overset{2}{\nabla}_Y \overset{2}{\nabla}_X (fZ) \\
&\quad - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X (fZ) - \overset{2}{\nabla}_Y \overset{1}{\nabla}_X (fZ) - \overset{1}{\nabla}_{[X,Y]}(fZ) \\
&= \overset{1}{\nabla}_X (Yf\cdot Z + f\overset{1}{\nabla}_Y Z) + \overset{2}{\nabla}_Y (Xf\cdot Z + f\overset{2}{\nabla}_X Z) \\
&\quad - \overset{1}{\nabla}_Y (Xf\cdot Z + f\overset{2}{\nabla}_X Z) - -\overset{2}{\nabla}_Y (Xf\cdot Z + f\overset{1}{\nabla}_X Z) \\
&\quad - ([X,Y]f\cdot Z + f\overset{1}{\nabla}_{[X,Y]} Z) \\
&= X(Yf)\cdot Z + Yf\overset{1}{\nabla}_X Z + Xf\overset{1}{\nabla}_Y Z + f\overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z \\
&\quad + Y(Xf)\cdot Z + Xf\overset{2}{\nabla}_Y Z + Yf\overset{2}{\nabla}_X Z + f\overset{2}{\nabla}_Y \overset{2}{\nabla}_X Z \\
&\quad - Y(Xf)\cdot Z - Xf\overset{1}{\nabla}_Y Z - Yf\overset{2}{\nabla}_X Z - f\overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z \\
&\quad - Y(Xf)\cdot Z - Xf\overset{2}{\nabla}_Y Z - Yf\overset{1}{\nabla}_X Z - f\overset{2}{\nabla}_Y \overset{1}{\nabla}_X Z \\
&\quad - X(Yf)\cdot Z + Y(Xf)\cdot Z - f\overset{1}{\nabla}_{[X,Y]} Z \\
&= f\overset{5}{R}(X,Y)Z,
\end{aligned}$$

i.e. the linearity with respect to Z is also proved. So, by (1.1) a curvature tensor field is defined. The proofs for $\overset{6}{R} - \overset{12}{R}$ are obtained by the same procedure. □

Substituting in (1.1) Z, Y, X with X_j, X_k, X_l, respectively, by virtue of (0.4), using (0.6), (0.7) and (0.2), we get the components for $\overset{5}{R}$:

$$\begin{aligned}
\overset{5}{R}{}^i{}_{jkl} X_i &= \overset{5}{R}(X_l, X_k) X_j \\
&= X_l(L^i_{jk})\cdot X_i + L^i_{jk} L^p_{il} X_p + X_k(L^i_{lj})\cdot X_i + L^i_{lj} L^p_{ki} X_p \\
&\quad - X_k(L^i_{lj})\cdot X_i - L^i_{lj} L^p_{ik} X_p - X_k(L^i_{jl})\cdot X_i - L^i_{jl} L^p_{ki} X_p \\
&= (L^i_{jk,l} - L^i_{jl,k} + L^p_{jk} L^i_{pl} + L^p_{lj} L^i_{kp} - L^p_{jl} L^i_{kp} - L^p_{lj} L^i_{pk}) X_i.
\end{aligned}$$

2. Independent curvature tensor

From the above, it follows that we have twelve curvature tensors of non-symmetric affine connection. The aim is to find the number of independent among these tensors.

Theorem 2. *Among curvature tensor fields* $\overset{\theta}{R}(X,Y)Z \quad (\theta = 1, ..., 12)$ *(given by* (0.3) *and* (1.1) – (1.8)*) of non-symmetric affine connection, there are five independent, while the rest can be expressed as linear combinations of these five fields and the curvature tensor field* $\overset{0}{R}(X,Y)Z$ *of the corresponding symmetric connection.*

Proof. Introducing the notation

$$\overset{0}{\nabla}_Y Z = \frac{1}{2}(\overset{1}{\nabla}_Y Z + \overset{2}{\nabla}_Y Z), \quad \tau(Y,Z) = \frac{1}{2}(\overset{1}{\nabla}_Y Z - \overset{2}{\nabla}_Y Z), \tag{2.1a,b}$$

we get

$$\overset{1}{\nabla}_Y Z = \overset{0}{\nabla}_Y Z + \tau(Y,Z), \quad \overset{2}{\nabla}_Y Z = \overset{0}{\nabla}_Y Z - \tau(Y,Z). \tag{2.2a,b}$$

Since

$$T(Y,Z) = \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Z Y - [Y,Z] \tag{2.3}$$

is the torsion field of non-symmetric connection $\overset{1}{\nabla}$, from (0.1) and (2.1b) we conclude that

$$\tau(Y,Z) = \frac{1}{2} T(Y,Z), \tag{2.4}$$

and $\overset{0}{\nabla}$ is a symmetric connection.

If L^i_{jk} are components of the connection $\overset{1}{\nabla}$ in local coordinates, putting $Y = X_k$, $Z = X_j$ in (2.1), we get

$$\overset{0}{\nabla}_{X_k} X_j = \frac{1}{2}(\overset{1}{\nabla}_{X_k} X_j + \overset{2}{\nabla}_{X_k} X_j) = \overset{0}{L}{}^i_{jk} X_i, \tag{2.5a}$$

$$\tau(X_k, X_j) = \frac{1}{2}(\overset{1}{\nabla}_{X_k} X_j - \overset{2}{\nabla}_{X_k} X_j) = \tau^i_{jk} X_i, \tag{2.5b}$$

where

$$\overset{0}{L}{}^i_{jk} = \frac{1}{2}(L^i_{jk} + L^i_{kj}), \quad \tau^i_{jk} = \frac{1}{2}(L^i_{jk} - L^i_{kj}). \tag{2.6a,b}$$

By virtue of (0.3) and (2.2) it follows that

$$\begin{aligned}\overset{1}{R}(X,Y)Z &= \overset{0}{\nabla}_X \overset{0}{\nabla}_Y Z + \tau(X, \overset{0}{\nabla}_Y Z) + \overset{0}{\nabla}_X \tau(Y,Z) + \tau(X, \tau(Y,Z)) \\ &\quad - \overset{0}{\nabla}_Y \overset{0}{\nabla}_X Z - \tau(Y, \overset{0}{\nabla}_X Z) - \overset{0}{\nabla}_Y \tau(X,Z) \\ &\quad - \tau(Y, \tau(X,Z)) - \overset{0}{\nabla}_{[X,Y]} Z - \tau([X,Y],Z).\end{aligned} \tag{2.7}$$

If $\overset{0}{R}(X,Y)Z$ is the curvature tensor field of the symmetric connection $\overset{0}{\nabla}$, putting $X = X_l$, $Y = X_k$, $Z = X_j$, the equation (2.7), based on (2.5) and (2.6), gives

$$\begin{aligned}\overset{1}{R}(X_l, X_k)X_j &= \overset{1}{R}{}^i{}_{jkl} X_i \\ &= \overset{0}{R}{}^i{}_{jkl} X_i + \overset{0}{L}{}^i_{jk} \tau^p_{il} X_p + \tau^i_{jk,l} X_i + \tau^i_{jk} \overset{0}{L}{}^p_{il} X_p + \tau^i_{jk} \tau^p_{il} X_p \\ &\quad - \overset{0}{L}{}^i_{jl} \tau^p_{ik} X_p - \tau^i_{jl,k} X_i - \tau^i_{jl} \overset{0}{L}{}^p_{ik} X_p - \tau^i_{jl} \tau^p_{ik} X_p,\end{aligned}$$

from which

$$\overset{1}{R}{}^i{}_{jkl} = \overset{0}{R}{}^i{}_{jkl} + \tau^i_{jk;l} - \tau^i_{jl;k} + \tau^p_{jk}\tau^i_{pl} - \tau^p_{jl}\tau^i_{pk}, \tag{2.8}$$

where

$$\tau^i_{jk;l} = \tau^i_{jk,l} + \overset{0}{L}{}^i_{pl}\tau^p_{jk} - \overset{0}{L}{}^p_{jl}\tau^i_{pk} - \overset{0}{L}{}^p_{kl}\tau^i_{jp}. \tag{2.9}$$

Introducing notation

$$\mathcal{A} = \tau^i_{jk;l}, \quad \mathcal{B} = \tau^p_{jk}\tau^i_{pl}, \quad \mathcal{C} = \tau^i_{pj}\tau^p_{kl}, \tag{2.10a-c}$$

$$\mathcal{A}' = \tau^i_{jl;k}, \quad \mathcal{B}' = \tau^p_{jl}\tau^i_{pk}, \tag{2.11a,b}$$

and omitting the indices of the curvature tensors, from (2.8) we have

$$\overset{1}{R} = \overset{0}{R} + \mathcal{A} - \mathcal{A}' + \mathcal{B} - \mathcal{B}'. \tag{2.12}$$

In the same way, using (0.3) and (1.1) – (1.8), we get

$$\overset{2}{R} = \overset{0}{R} - \mathcal{A} + \mathcal{A}' + \mathcal{B} - \mathcal{B}'. \tag{2.13}$$

$$\overset{3}{R} = \overset{0}{R} + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' - 2\mathcal{C}, \tag{2.14}$$

$$\overset{4}{R} = \overset{0}{R} + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' + 2\mathcal{C}, \tag{2.15}$$

$$\overset{5}{R} = \overset{0}{R} + \mathcal{A} - \mathcal{A}' + \mathcal{B} + 3\mathcal{B}', \tag{2.16}$$

$$\overset{6}{R} = \overset{0}{R} - \mathcal{A} + \mathcal{A}' + \mathcal{B} + 3\mathcal{B}', \tag{2.17}$$

$$\overset{7}{R} = \overset{0}{R} + \mathcal{B} + \mathcal{B}', \tag{2.18}$$

$$\overset{8}{R} = \overset{0}{R} - \mathcal{B} + \mathcal{B}', \tag{2.19}$$

$$\overset{9}{R} = \overset{0}{R} - \mathcal{B} - \mathcal{B}', \tag{2.20}$$

$$\overset{10}{R} = \overset{0}{R} + \frac{1}{3}(-\mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' - 2\mathcal{C}), \tag{2.21}$$

$$\overset{11}{R} = \overset{0}{R} - \mathcal{A} + \mathcal{A}' - 3\mathcal{B} - \mathcal{B}', \tag{2.22}$$

$$\overset{12}{R} = \overset{0}{R} + \mathcal{A} - \mathcal{A}' - 3\mathcal{B} - \mathcal{B}'. \tag{2.23}$$

Consider the equations (2.12) – (2.23) as a system of linear algebraic equations in the unknowns $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{A}', \mathcal{B}'$. This system is compatible and has the rank 5. We can take the first 5 equations (independent), from which we obtain

$$\mathcal{A} = \frac{1}{4}(2\overset{1}{R} + \overset{3}{R} + \overset{4}{R}) - \overset{0}{R}, \quad \mathcal{B} = \frac{1}{4}(\overset{1}{R} + 2\overset{2}{R} + \overset{5}{R}) - \overset{0}{R}, \quad \mathcal{C} = \frac{1}{4}(\overset{4}{R} - \overset{3}{R}),$$
$$\mathcal{A}' = \frac{1}{4}(2\overset{2}{R} + \overset{3}{R} + \overset{4}{R}) - \overset{0}{R}, \quad \mathcal{B}' = \frac{1}{4}(\overset{5}{R} - \overset{1}{R}).$$

By putting these values into (2.17) – (2.23), we obtain

$$\overset{6}{R} = -\overset{1}{R} + \overset{2}{R} + \overset{5}{R}, \quad \overset{7}{R} = \frac{1}{2}(\overset{2}{R} + \overset{5}{R}), \quad \overset{8}{R} = 2\overset{0}{R} - \frac{1}{2}(\overset{1}{R} + \overset{2}{R}),$$
$$\overset{9}{R} = 2\overset{0}{R} - \frac{1}{2}(\overset{2}{R} + \overset{5}{R}), \quad \overset{10}{R} = \frac{1}{6}(8\overset{0}{R} - 2\overset{1}{R} + \overset{3}{R} - \overset{4}{R}), \quad \overset{11}{R} = 4\overset{0}{R} - \overset{1}{R} - \overset{2}{R} - \overset{5}{R},$$
$$\overset{12}{R} = 4\overset{0}{R} - 2\overset{2}{R} - \overset{5}{R}.$$

□

Remark. In the case of a symmetric connection ($\overset{1}{\nabla} \equiv \overset{2}{\nabla} \equiv \overset{0}{\nabla}$) all tensors $\overset{\theta}{R}$ $(\theta = 1, ..., 12)$ reduce to $\overset{0}{R}$. This can be concluded from (0.3), (1.1) – (1.8) or (2.10) – (2.23).

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