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Some elementary zero-free regions for Dirichlet series and power series

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Abstract. Adapting some elementary methods used by a number of authors to investigate the location of roots of polynomials with complex coefficients, we present some results which provide zero-free regions for Dirichlet series and power series.

1. Introduction

A classical result of Cauchy states that all the zeros of a complex polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie in the disc $|z| \leq r$, where r is the unique positive root of the algebraic equation $|a_n| z^n = \sum_{i=0}^{n-1} |a_i| z^i$. An important generalization of this result is Pellet's theorem [19]:

If the equation $|a_k| z^k = \sum_{i=0, i\neq k}^{n} |a_i| z^i$ $(0 < k < n,\ a_0 a_n \neq 0)$ has two positive roots r_k and R_k $(0 < r_k < R_k)$, then the complex polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ has no zeros in the annulus $r_k < |z| < R_k$ and precisely k zeros in the disc $|z| \leq r_k$.

Walsh [23] established another more direct proof of this result and remarked that his proof remains valid in the case of a power series and its zeros inside its disc of convergence. He also devised a sort of converse of Pellet's theorem (see also Ostrowski [18]). Fujiwara [7] employed a very simple method to obtain the following sharp result:

Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients, of degree $n \geq 2$, and $\mu_1, \ldots, \mu_n \in (0, \infty)$ such that $\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n} \leq 1$. Then

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all the roots of P are contained in the disk $|z| \le R$, where

$$R = \max_{1\le j\le n}\left(\mu_j \frac{|a_{n-j}|}{|a_n|}\right)^{\frac{1}{j}}.$$

Another old result is the following theorem of Kojima (see [10], [11]):

All the zeros of the polynomial $f(z) = a_0 + a_1 z^{n_1} + \cdots + a_k z^{n_k}$ with all $a_j \neq 0$, $0 = n_0 < n_1 < \cdots < n_k$, lie on the disc $|z| \le r$, where $r = \max[|a_0/a_1|^{m_1}, |2a_j/a_{j-1}|^{m_j}]$, $m_j = (n_j - n_{j-1})^{-1}$, $j = 1, 2, \ldots, k$.

Further estimates for the absolute values of the roots of a given polynomial with complex coefficients have been provided by Cowling and Thron (see [4], [5]):

Let $f(z) = a_0 + a_1 z^{n_1} + \cdots + a_k z^{n_k}$ with all $a_j \neq 0$, $0 = n_0 < n_1 < \cdots < n_k$, and $m_j = (n_j - n_{j-1})^{-1}$, $j = 1, 2, \ldots, k$. Let $r_0 = 0$, $r_k = 1$ and $r_1, \ldots, r_{k-1}$ be arbitrary positive constants. Then all the zeros of f lie in the disc

$$|z| \le M = \max_{j=1,\ldots,k}\left\{\frac{(1 + r_{j-1})}{r_j} \cdot \frac{|a_{j-1}|}{|a_j|}\right\}^{m_j}.$$

Fujiwara [7] also provided information on the possible location of zeros of a given power series with complex coefficients. Some refinements of Pellet's theorem, and of Fujiwara's bounds, have been obtained by Marden in [14] (see also [15]). Results on the location of zeros in the case of multivariate polynomials have been established by Cargo and Shisha [2], and Mond and Shisha [17].

Riddell [20] considered the case when a complex polynomial $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ has one $|a_k|$ large in comparison with the other $|a_i|$ and proved using Pellet's theorem that then $P(z)$ has $n - k$ zeros near 0 and one zero near each of the k values of $(-a_k)^{\frac{1}{k}}$.

Some bounds for the zeros of a complex polynomial $P(z)$ can be obtained by using results on the numerical range and the numerical radius of the Frobenius companion matrix of $P(z)$. Some sharp results in this direction have been obtained by Chien [3], Fujii and Kubo [6], and Kittaneh [9]. Other companion matrices can be obtained by a similarity transformation of the Frobenius companion matrix of $P(z)$. Linden [12], [13] used such generalized companion matrices based on some special multiplicative decompositions of the coefficients of a polynomial $P(z)$, in order to find estimates for the zeros of $P(z)$. This was settled by mainly applying Gershgorin's theorem to the companion matrices, or by computing their singular values and using majorization relations of Weyl between the singular values and the eigenvalues of a matrix.

Another method to investigate the location of zeros of a complex polynomial is to consider its expansion with respect to a system of orthogonal polynomials. A famous result in this direction was obtained by Turán [22]:

If a complex polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree n has the Hermite expansion $P(z) = \sum_{k=0}^{n} \alpha_k H_k(z)$, then all the zeros of $P(z)$ lie in the strip

$$|Im(z)| \leq \frac{1}{2}(1 + \max_{0\leq k\leq n-1} \frac{|\alpha_k|}{|\alpha_n|}).$$

The same holds for the strip

$$|Im(z)| \leq \frac{1}{2}\sum_{k=0}^{n-1} | \frac{\alpha_k}{\alpha_n} |^{1/(n-k)} .$$

Further results in this direction using other systems of orthogonal polynomials were obtained by Schmeisser [21], Gautschi [8], de Bruin [1], Milovanović and Rajković [16].

In the present paper we adapt the methods from [7], [10], [11], [4] and [5] to provide some elementary zero-free regions for Dirichlet series and power series. The results in Section 2 below, which have very simple proofs, are quite flexible, and might prove to be useful in various applications.

2. Zero-free regions for Dirichlet series and power series

In this section we present some results which provide zero-free regions for Dirichlet series and power series with complex coefficients.

Proposition 1. *Let $\varphi(s)$ be a Dirichlet series convergent for $\Re(s) = \sigma > A$, and let f_i, $i = 1, 2, \ldots$, be a sequence of functions defined for $\Re(s) > A$ and bounded on vertical lines, such that*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{f_n(s)}{n^s}$$

for any s with $\Re(s) > A$. We assume that there exist an integer $j \geq 1$, a real number $\sigma > A$ and a sequence of positive real numbers $\mu_1, \mu_2, \mu_3, \ldots$ such that

$$\sum_{k=1,\ k\neq j}^{\infty} \mu_k \leq 1 \tag{1}$$

and

$$\sup_{k>j} \log_{\frac{k}{j}} \left(\frac{1}{\mu_k} \cdot \frac{M_k}{m_j}\right) < \sigma < \inf_{k<j} \log_{\frac{j}{k}} \left(\mu_k \cdot \frac{m_j}{M_k}\right), \tag{2}$$

where $m_i = \inf_{\Re(s)=\sigma} |f_i(s)|$ and $M_i = \sup_{\Re(s)=\sigma} |f_i(s)|$ for any i.

Then $\varphi(s) \neq 0$ for any s with $\Re(s) = \sigma$.

Proof. Let us fix a complex number s, with $\Re(s) = \sigma$. Using the triangle inequality, we write

$$|\varphi(s)| \geq \frac{|f_j(s)|}{|j^s|} - \sum_{k=1,\ k\neq j}^{\infty} \frac{|f_k(s)|}{|k^s|} \geq \frac{m_j}{j^\sigma} - \sum_{k=1,\ k\neq j}^{\infty} \frac{M_k}{k^\sigma}. \tag{3}$$

The first inequality from (2) implies that $\left(\frac{k}{j}\right)^\sigma > \frac{1}{\mu_k} \cdot \frac{M_k}{m_j}$ for any $k > j$, or $\mu_k \frac{m_j}{j^\sigma} > \frac{M_k}{k^\sigma}$ for any $k > j$. The second inequality from (2) implies that $\left(\frac{j}{k}\right)^\sigma > \mu_k \cdot \frac{m_j}{M_k}$ for any $k < j$, or $\mu_k \frac{m_j}{j^\sigma} > \frac{M_k}{k^\sigma}$ for any $k < j$. We add the above inequalities for all $k \neq j$, and, using also (1), we obtain

$$\frac{m_j}{j^\sigma} \geq \sum_{k=1,\ k\neq j}^{\infty} \mu_k \frac{m_j}{j^\sigma} > \sum_{k=1,\ k\neq j}^{\infty} \frac{M_k}{k^\sigma}.$$

Therefore $|\varphi(s)| > 0$, by (3). Hence $\varphi(s) \neq 0$ for any s with $\Re(s) = \sigma$. □

Next, we use similar arguments to establish further results on the possible location of zeros of the corresponding functions.

Proposition 2. *Let $\varphi(s)$ be a Dirichlet series convergent for $\Re(s) > A$, and let f_i, $i = 1, 2, \ldots$, be a sequence of functions defined for $\Re(s) > A$ and bounded on vertical lines, such that*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{f_n(s)}{n^s}$$

for any s with $\Re(s) > A$. Let $\mu_1 = 0$, let $\mu_2, \mu_3, \ldots$ be a bounded sequence of positive real numbers, and let $j \geq 2$ be an integer for which $\mu_j > 1$. Then any zero s of φ with $\Re(s) = \sigma$ for which $\sum_{n=1}^{\infty} M_n/n^\sigma$ is convergent belongs to the halfplane

$$\begin{aligned} \Re(s) \quad &\geq \quad \min\left\{ \log_{\frac{j}{j-1}} \frac{\mu_j m_j}{(1+\mu_{j-1})M_{j-1}},\ \log_{\frac{j+1}{j}} \frac{\mu_{j+1} M_{j+1}}{(-1+\mu_j)m_j}, \right. \\ &\qquad \left. \inf_{k\neq j-1,\ j} \log_{\frac{k+1}{k}} \frac{\mu_{k+1} M_{k+1}}{(1+\mu_k)M_k} \right\}, \end{aligned}$$

where $M_n = \sup_{\Re(s)=\sigma} |f_n(s)|$ and $m_n = \inf_{\Re(s)=\sigma} |f_n(s)|$ for any n.

Proof. Let s be a zero of φ with $\Re(s) = \sigma$ and the sum $\sum_{n=1}^{\infty} M_n/n^\sigma$ convergent. By the triangle inequality,

$$|\varphi(s)| \geq \frac{|f_j(s)|}{|j^s|} - \sum_{k=1,\ k\neq j}^{\infty} \frac{|f_k(s)|}{|k^s|} \geq \frac{m_j}{j^\sigma} - \sum_{k=1,\ k\neq j}^{\infty} \frac{M_k}{k^\sigma}. \tag{4}$$

Let us assume that s does not belong to the disc from the statement of the proposition. Then we have successively:

$$\begin{aligned}
\frac{\mu_2 M_2}{2^\sigma} &> \frac{(1+\mu_1)M_1}{1^\sigma}\\
&\cdots\cdots\\
\frac{\mu_{j-1} M_{j-1}}{(j-1)^\sigma} &> \frac{(1+\mu_{j-2})M_{j-2}}{(j-2)^\sigma}\\
\frac{\mu_j m_j}{j^\sigma} &> \frac{(1+\mu_{j-1})M_{j-1}}{(j-1)^\sigma}\\
\frac{\mu_{j+1} M_{j+1}}{(j+1)^\sigma} &> \frac{(-1+\mu_j)m_j}{j^\sigma}\\
\frac{\mu_{j+2} M_{j+2}}{(j+2)^\sigma} &> \frac{(1+\mu_{j+1})M_{j+1}}{(j+1)^\sigma}\\
&\cdots\cdots
\end{aligned}$$

Adding these inequalities and using the fact that the sequence $\{\mu_i\}$ is bounded and the sum $\sum_{n=1}^{\infty} M_n/n^\sigma$ is convergent, so that the sums on both sides are finite, after appropriate cancellation we obtain

$$m_j/j^\sigma > \sum_{k=1,\ k\neq j}^{\infty} M_k/k^\sigma.$$

Therefore by (4) it follows that $|\varphi(s)| > 0$, hence s is not a zero of φ, and we obtain a contradiction. $\square$

Proposition 3. *Let $\varphi(s)$ be a Dirichlet series convergent on the halfplane $\Re(s) > A$, and let f_i, $i = 1, 2, \ldots$, be a sequence of functions defined for $\Re(s) > A$ and bounded on vertical lines, such that*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{f_n(s)}{n^s}$$

for any s with $\Re(s) > A$. Let $\mu_1, \mu_2, \mu_3, \ldots$ be a bounded sequence of positive real numbers, with $\mu_1 = 1$. Then any zero s of φ with $\Re(s) = \sigma$ for which $\sum_{n=1}^{\infty} M_n/n^\sigma$ is convergent lies in the halfplane

$$\Re(s) \leq \max\left\{ \log_2 \frac{(1+\mu_2)M_2}{\mu_1 m_1},\ \sup_{k\geq 2} \log_{\frac{k+1}{k}} \frac{(1+\mu_{k+1})M_{k+1}}{\mu_k M_k} \right\},$$

where $m_1 = \inf_{\Re(s)=\sigma} |f_1(s)|$ and $M_n = \sup_{\Re(s)=\sigma} |f_n(s)|$ for any n.

Proof. Let s be a zero of φ with $\Re(s) = \sigma$ such that the sum $\sum_{n=1}^{\infty} M_n/n^\sigma$ is convergent. Using the triangle inequality, we have

$$|\varphi(s)| \geq |f_1(s)| - \sum_{k=2}^{\infty} \frac{|f_k(s)|}{|k^s|} \geq m_1 - \sum_{k=2}^{\infty} \frac{M_k}{k^\sigma}. \tag{5}$$

Let us assume that

$$\Re(s) > \max\left\{\log_2 \frac{(1+\mu_2)M_2}{\mu_1 m_1},\ \sup_{k\geq 2}\log_{\frac{k+1}{k}} \frac{(1+\mu_{k+1})M_{k+1}}{\mu_k M_k}\right\}.$$

Then we obtain successively

$$\begin{aligned}
\frac{\mu_1 m_1}{1^\sigma} &> \frac{(1+\mu_2)M_2}{2^\sigma}\\
\frac{\mu_2 M_2}{2^\sigma} &> \frac{(1+\mu_3)M_3}{3^\sigma}\\
\frac{\mu_3 M_3}{3^\sigma} &> \frac{(1+\mu_4)M_4}{4^\sigma}\\
\frac{\mu_4 M_4}{4^\sigma} &> \frac{(1+\mu_5)M_5}{5^\sigma}\\
&\cdots\cdots
\end{aligned}$$

Adding these inequalities, the sums on both sides are finite, and after appropriate cancellation of terms we obtain $m_1 > \sum_{k=2}^{\infty} M_k/k^\sigma$. Then, by (5) it follows that $|\varphi(s)| > 0$, so s is not a zero for φ, and we obtain a contradiction. □

Denote as usual $\mathbb{N} = \{0,1,2,\dots\}$, $\mathbb{N}^* = \{1,2,\dots\}$. We consider now a suitable class of permutations on $\mathbb{N}$.

Proposition 4. *Let $\varphi(s)$ be a Dirichlet series convergent in the halfplane $\Re(s) > M$, and let f_i, $i = 1,2,\dots$, be a sequence of functions defined for $\Re(s) > A$ and bounded on vertical lines, such that*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{f_n(s)}{n^s}$$

for any s with $\Re(s) > M$. Let $\theta \in \Sigma_{\mathbb{N}^*}$ be a permutation of $\mathbb{N}^*$ without fixed points and let $A = \{i \in \mathbb{N}^* : i > \theta(i)\}$, $B = \{i \in \mathbb{N}^* : i < \theta(i)\}$. We assume that there exist a real number $\sigma > M$, a bounded sequence of positive real numbers $\mu_1, \mu_2, \dots$ and an integer $j \geq 1$ with $\mu_{\theta(j)} > 1$ such that

$$\sup_{i\in B, i\neq j,\theta(j)} \log_{\frac{\theta(i)}{i}} \frac{(1+\mu_{\theta(i)})M_{\theta(i)}}{\mu_i M_i} < \sigma < \inf_{i\in A, i\neq j,\theta(j)} \log_{\frac{i}{\theta(i)}} \frac{\mu_i M_i}{(1+\mu_{\theta(i)})M_{\theta(i)}}$$

and

$$\begin{aligned}
\frac{\mu_j M_j}{j^\sigma} &> \frac{(-1+\mu_{\theta(j)})m_{\theta(j)}}{\theta(j)^\sigma},\\
\frac{\mu_{\theta(j)} m_{\theta(j)}}{\theta(j)^\sigma} &> \frac{(1+\mu_{\theta^2(j)})M_{\theta^2(j)}}{\theta^2(j)^\sigma},
\end{aligned}$$

where $M_n = \sup_{\Re(s)=\sigma} |f_n(s)|$ and $m_n = \inf_{\Re(s)=\sigma} |f_n(s)|$ for any n.

Then $\varphi(s) \neq 0$ *for any* s *with* $\Re(s) = \sigma$ *for which the sum* $\sum_{n=1}^{\infty} M_n/n^{\sigma}$ *is convergent.*

Proof. Let us assume that s is a zero of φ, with $\Re(s) = \sigma$, and such that the sum $\sum_{n=1}^{\infty} M_n/n^{\sigma}$ is convergent. Using as before the triangle inequality, we have

$$0 = |\varphi(s)| \geq \frac{|f_{\theta(j)}(s)|}{|\theta(j)^s|} - \sum_{k=1,\ k\neq\theta(j)}^{\infty} \frac{|f_k(s)|}{|k^s|} \geq \frac{m_{\theta(j)}}{\theta(j)^{\sigma}} - \sum_{k=1,\ k\neq\theta(j)}^{\infty} \frac{M_k}{k^{\sigma}}. \tag{6}$$

On the other hand, by the conditions from the statement of the proposition we derive successively that

$$\begin{aligned}
\frac{\mu_1 M_1}{1^{\theta}} &> \frac{(1+\mu_{\theta(1)})M_{\theta(1)}}{\theta(1)^{\sigma}} \\
\frac{\mu_2 M_2}{2^{\sigma}} &> \frac{(1+\mu_{\theta(2)})M_{\theta(2)}}{\theta(2)^{\sigma}} \\
&\;\;\cdots\cdots \\
\frac{\mu_{j-1} M_{j-1}}{(j-1)^{\sigma}} &> \frac{(1+\mu_{\theta(j-1)})M_{\theta(j-1)}}{\theta(j-1)^{\sigma}} \\
\frac{\mu_j M_j}{j^{\sigma}} &> \frac{(-1+\mu_{\theta(j)})m_{\theta(j)}}{\theta(j)^{\sigma}} \\
\frac{\mu_{j+1} M_{j+1}}{(j+1)^{\sigma}} &> \frac{(1+\mu_{\theta(j+1)})M_{\theta(j+1)}}{\theta(j+1)^{\sigma}} \\
&\;\;\cdots\cdots \\
\frac{\mu_{\theta(j)} m_{\theta(j)}}{\theta(j)^{\sigma}} &> \frac{(1+\mu_{\theta^2(j)})M_{\theta^2(j)}}{\theta^2(j)^{\sigma}} \\
&\;\;\cdots\cdots
\end{aligned}$$

Since the sequence (μ_i) is bounded and the sum $\sum_{n=1}^{\infty} M_n/n^{\sigma}$ is convergent, we may add these inequalities to obtain $m_{\theta(j)}/\theta(j)^{\sigma} > \sum_{k=1,\ k\neq\theta(j)}^{\infty} M_k/k^{\sigma}$. Combining with (6), this leads to a contradiction. □

Using a standard change of variable, one obtains the following analogs of the above results for power series with complex coefficients.

Corollary 1. *Let* $\varphi(z)$ *be a power series with complex coefficients, with radius of convergence* $R > 0$, *and let* f_i, $i = 0, 1, 2, \ldots$, *be a sequence of locally bounded functions defined on the disc* $|z| < R$, *such that*

$$\varphi(z) = \sum_{i=0}^{\infty} f_i(z) \cdot z^i$$

for any z *with* $|z| < R$. *We assume that there exist an integer* $j \geq 0$, *a real number* $0 < \rho < R$ *and a sequence of positive real numbers* $\mu_0, \mu_1, \mu_2, \ldots$

such that

$$\sum_{k=0,\ k\neq j}^{\infty} \mu_k \leq 1 \tag{7}$$

and

$$\sup_{k<j}\left(\frac{1}{\mu_k}\cdot\frac{M_k}{m_j}\right)^{\frac{1}{j-k}} < \rho < \inf_{k>j}\left(\mu_k\cdot\frac{m_j}{M_k}\right)^{\frac{1}{k-j}}, \tag{8}$$

where $m_i = \inf_{|z|=\rho}|f_i(z)|$ *and* $M_i = \sup_{|z|=\rho}|f_i(z)|$ *for any* i.

Then $\varphi(z) \neq 0$ *for any* z *with* $|z| = \rho$.

Corollary 2. *Let* $\varphi(z)$ *be a power series with complex coefficients, with radius of convergence* $R > 0$, *and let* f_i, $i = 0, 1, 2, \ldots$, *be a sequence of locally bounded functions defined on the disc* $|z| < R$, *such that*

$$\varphi(z) = \sum_{i=0}^{\infty} f_i(z)\cdot z^i$$

for any z *with* $|z| < R$. *Let* $\mu_0 = 0$, *let* $\mu_1, \mu_2, \ldots$ *be a bounded sequence of positive real numbers, and let* $j \geq 1$ *be an integer for which* $\mu_j > 1$. *Then any zero* z_0 *of* φ *for which* $\sum_{i=0}^{\infty} M_i \cdot |z_0|^i$ *is convergent belongs to the disc*

$$|z| \leq \max\left\{\frac{(1+\mu_{j-1})M_{j-1}}{\mu_j m_j},\ \frac{(-1+\mu_j)m_j}{\mu_{j+1}M_{j+1}},\ \sup_{k\neq j-1,\ j}\frac{(1+\mu_k)M_k}{\mu_{k+1}M_{k+1}}\right\},$$

where $M_i = \sup_{|z|=|z_0|}|f_i(z)|$ *and* $m_i = \inf_{|z|=|z_0|}|f_i(z)|$ *for any* i.

Corollary 3. *Let* $\varphi(z)$ *be a power series with complex coefficients, with radius of convergence* $R > 0$, *and let* f_i, $i = 0, 1, 2, \ldots$, *be a sequence of locally bounded functions defined on the disc* $|z| < R$, *such that*

$$\varphi(z) = \sum_{i=0}^{\infty} f_i(z)\cdot z^i$$

for any z *with* $|z| < R$. *Let* $\mu_1, \mu_2, \ldots$ *be a bounded sequence of positive real numbers. Then any zero* z_0 *of* φ *for which the sum* $\sum_{i=0}^{\infty} M_i \cdot |z_0|^i$ *is convergent satisfies*

$$|z_0| \geq \min\left\{\frac{m_0}{(1+\mu_1)M_1},\ \inf_{k\geq 1}\frac{\mu_k M_k}{(1+\mu_{k+1})M_{k+1}}\right\},$$

where $m_0 = \inf_{|z|=|z_0|}|f_0(z)|$ *and* $M_i = \sup_{|z|=|z_0|}|f_i(z)|$ *for any* i.

Corollary 4. *Let* $\varphi(z)$ *be a power series with complex coefficients, with radius of convergence* $R > 0$, *and let* f_i, $i = 0, 1, 2, \ldots$, *be a sequence of locally bounded functions defined on the disc* $|z| < R$, *such that* $\varphi(z) = \sum_{i=0}^{\infty} f_i(z)\cdot z^i$ *for any* z *with* $|z| < R$. *Let* $\sigma \in \Sigma_{\mathbb{N}}$ *be a permutation of* $\mathbb{N}$ *without fixed points, and let* $A = \{i \in \mathbb{N} : i > \sigma(i)\}$, $B = \{i \in \mathbb{N} : i < \sigma(i)\}$. *We assume* *that there exist a real number* $0 < \rho < R$, *a bounded sequence of positive real numbers* $\mu_0, \mu_1, \mu_2, \ldots$ *and an integer* $j \geq 0$ *with* $\mu_{\sigma(j)} > 1$ *such that*

$$\sup_{\substack{i \in A \\ i \neq j, \sigma(j)}} \left(\frac{(1 + \mu_{\sigma(i)}) M_{\sigma(i)}}{\mu_i M_i} \right)^{\frac{1}{i - \sigma(i)}} < \rho < \inf_{\substack{i \in B \\ i \neq j, \sigma(j)}} \left(\frac{\mu_i M_i}{(1 + \mu_{\sigma(i)}) M_{\sigma(i)}} \right)^{\frac{1}{\sigma(i) - i}}$$

and

$$\begin{aligned} \mu_j M_j \rho^j &> (-1 + \mu_{\sigma(j)}) m_{\sigma(j)} \rho^{\sigma(j)}, \\ \mu_{\sigma(j)} m_{\sigma(j)} \rho^{\sigma(j)} &> (1 + \mu_{\sigma^2(j)}) M_{\sigma^2(j)} \rho^{\sigma^2(j)}, \end{aligned}$$

where $m_i = \inf_{|z|=\rho} |f_i(z)|$ *and* $M_i = \sup_{|z|=\rho} |f_i(z)|$ *for any* i.

Then $\varphi(z) \neq 0$ *for any* z *with* $|z| = \rho$ *and for which the sum* $\sum_{i=0}^{\infty} M_i \cdot \rho^i$ *is convergent.*

We remark that the results from this section could be restated in the context of lacunary series, and the corresponding sums would run over those indices i_j for which $f_{i_j} \neq 0$.

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