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On the degree of approximation of functions of two variables by some operators

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ABSTRACT. We introduce operators of Szász-Mirakyan and Baskakov type in polynomial weighted spaces of functions of two variables. We prove a theorem on the degree of approximation and a Voronovskaya type theorem for these operators. Similar results for Bernstein, Szász-Mirakyan and Baskakov operators of functions of one variable were given in [3–6].

1. Introduction

1.1. Approximation properties of Szász-Mirakyan and Baskakov operators, i.e.,

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

$$V_n(f;x) := \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right),$$

$x \in \mathbb{R}_0 = [0,\infty)$, $n \in \mathbb{N} = \{1,2,...\}$, in polynomial weighted spaces C_p, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, were examined in [1].

In [3–6] approximation theorems for modified Bernstein, Szász-Mirakyan and Baskakov operators in spaces of differentiable functions of one variable were given.

In this paper we shall establish analogues of results given in [3–5] for certain positive linear operators in polynomial weighted spaces of differentiable functions of two variables. The operators introduced in this note contain classical Szász-Mirakyan and Baskakov operators of functions of two variables.

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Theorems given in Section 3 show that the order of approximation of r times differentiable functions by considered operators improves as $r \in \mathbb{N}$ increases.

1.2. Let $p, q \in \mathbb{N}_0$ and let

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1}, \qquad \text{if} \quad p \geq 1, \; x \in \mathbb{R}_0, \tag{1}$$

$$w_{p,q}(x,y) := w_p(x) w_q(y), \qquad (x,y) \in \mathbb{R}_0^2 = \mathbb{R}_0 \times \mathbb{R}_0. \tag{2}$$

Denote by $C_{p,q} \equiv C_{p,q}(\mathbb{R}_0^2)$ the polynomial weighted space, i.e., the set of all real-valued functions f continuous on $\mathbb{R}_0^2$ for which $w_{p,q}f$ is bounded on $\mathbb{R}_0^2$ and the norm is defined by the formula

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup\{w_{p,q}(x,y)|f(x,y)| : (x,y) \in \mathbb{R}_0^2\}. \tag{3}$$

Moreover, let $C^r \equiv C^r(\mathbb{R}_0^2)$, with a fixed $r \in \mathbb{N}_0$, be the class of all $f \in C_{r,r}$ having the r-th partial derivatives on $\mathbb{R}_0^2$ and $f^{(m)}_{x^{m-i}y^i} \in C_{r-m,r-m}$ for all $0 \leq i \leq m \leq r$.

For example, $f(x,y) = (ax+by+c)^r$, a, b, $c =$ const. $\in \mathbb{R}$ and $r \in \mathbb{N}_0$, is function belonging to C^r.

It is obvious that $C^0 \equiv C_{0,0}$ and $C_{p_1,q_1} \subseteq C_{p_2,q_2}$ if $p_1, p_2, q_1, q_2 \in \mathbb{N}_0$ and $p_1 \leq p_2$, $q_1 \leq q_2$. Moreover for $f \in C_{p_1,q_1} \subseteq C_{p_2,q_2}$ we have $\|f\|_{p_2,q_2} \leq \|f\|_{p_1,q_1}$. In particular we have $C_{0,0} \subseteq C_{p,q}$ for $p, q \in \mathbb{N}_0$ and $\|f\|_{p,q} \leq \|f\|_{0,0}$ for $f \in C_{0,0}$.

1.3. Let $f \in C_{0,0}$ and let $\omega(f)$ be its modulus of continuity ([7]), i.e.,

$$\begin{aligned} \omega(f;s,t) : = \sup\{&|f(x,y) - f(u,v)| \,:\, (x,y),(u,v) \in \mathbb{R}_0^2, \\ &|x-u| \leq s, \; |y-v| \leq t\}, \quad s,t \in \mathbb{R}_0. \end{aligned} \tag{4}$$

It is known ([7]) that if $f \in C_{0,0}$ and $\lambda_1, \lambda_2 =$ const. > 0, then

$$\begin{aligned} \omega(f;\lambda_1 s, \lambda_2 t) &\leq \omega(f;\lambda_1 s, 0) + \omega(f;0,\lambda_2 t) \\ &\leq (\lambda_1+1)\,\omega(f;s,0) + (\lambda_2+1)\,\omega(f;0,t) \\ &\leq (\lambda_1+\lambda_2+2)\,\omega(f;s,t) \quad \text{for} \quad s,t \geq 0. \end{aligned} \tag{5}$$

If, moreover, f is uniformly continuous on $\mathbb{R}_0^2$, then $\lim\limits_{s,t\to 0+} \omega(f;s,t) = 0$.

1.4. Denote by Ω the set of all infinite matrices $A = [a_{nk}(\cdot)]$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, of continuous functions on $\mathbb{R}_0$ satisfying the following conditions:

(i) $a_{nk}(x) \geq 0$ for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$,

(ii) $\sum\limits_{k=0}^{\infty} a_{nk}(x) = 1$ for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$,

(iii) the series $\sum\limits_{k=0}^{\infty} k^s a_{nk}(x)$ is uniformly convergent on $\mathbb{R}_0$ for every $n, s \in \mathbb{N}$ and its sum $F_{n,s}(\cdot; A)$ is function such that $w_s F_{n,s}$ is bounded on $\mathbb{R}_0$,

(iv) for every $s \in \mathbb{N}$ there exists a positive constant $M_1(s, A)$ independent on $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ such that for the function

$$T_{n;s}(x; A) := \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x\right)^s, \quad x \in \mathbb{R}_0, \ n \in \mathbb{N}, \tag{6}$$

there holds the inequality

$$w_{2s}(x) \left|T_{n;2s}(x; A)\right| \leq M_1(s, A)\, n^{-s}, \quad x \in \mathbb{R}_0, \ n \in \mathbb{N}. \tag{7}$$

Definition. Let $A, B \in \Omega$, $A = [a_{nk}(\cdot)]$, $B = [b_{nk}(\cdot)]$, and let $r \in \mathbb{N}_0$. For $f \in C^r$ we define the operators

$$\begin{aligned} L_{n;r}(f; x, y) &\equiv L_{n;r}(f; A, B; x, y) \\ &:= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{nj}(x)\, b_{nk}(y) \sum_{s=0}^{r} \frac{1}{s!}\, d^s f\left(\frac{j}{n}, \frac{k}{n}\right), \end{aligned} \tag{8}$$

$(x, y) \in \mathbb{R}_0^2$, $n \in \mathbb{N}$, where $d^s f\left(\frac{j}{n}, \frac{k}{n}\right)$ is the s-th differential of f at point $\left(\frac{j}{n}, \frac{k}{n}\right)$ and $\Delta x = x - \frac{j}{n}$, $\Delta y = y - \frac{k}{n}$, i.e. $d^0 f\left(\frac{j}{n}, \frac{k}{n}\right) = f\left(\frac{j}{n}, \frac{k}{n}\right)$ and

$$d^s f\left(\frac{j}{n}, \frac{k}{n}\right) = \sum_{i=0}^{s} \binom{s}{i} f^{(s)}_{x^{s-i} y^i}\left(\frac{j}{n}, \frac{k}{n}\right) \left(x - \frac{j}{n}\right)^{s-i} \left(y - \frac{k}{n}\right)^i. \tag{9}$$

If $r = 0$, then

$$L_{n;0}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{nj}(x)\, b_{nk}(y)\, f\left(\frac{j}{n}, \frac{k}{n}\right), \quad (x, y) \in \mathbb{R}_0^2, \ n \in \mathbb{N}. \tag{10}$$

We mention that formulas (8) and (10) contain the case of identical matrices A, B.

In Section 2 we shall prove that $L_{n;r}(f)$, $n \in \mathbb{N}$, is an operator from the space C^r into $C_{r,r}$, and we shall give some auxiliary inequalities.

In Section 3 we shall give a theorem on the degree of approximation of $f \in C^r$ by $L_{n;r}(f)$ and a Voronovskaya type theorem.

We shall denote by $M_k(\alpha, \beta)$, $k \in \mathbb{N}$, suitable positive constants depending only on indicated parameters α, β.

2. Lemmas

The properties (i)–(iv) of $A, B \in \Omega$ and (8)–(10) imply that

$$L_{n;r}(1; x, y) = 1, \qquad (x, y) \in \mathbb{R}_0^2, \ n \in \mathbb{N}, \ r \in \mathbb{N}_0, \tag{11}$$

and, moreover, $L_{n;r}(f)$ is well-defined for every $f(x, y) = x^p y^q$, $p, q \in \mathbb{N}_0$.

Applying Hölder's inequality and (1), (2), (6), (7) and (10), we immediately obtain the following

Lemma 1. *For fixed $A, B \in \Omega$ and $p, q \in \mathbb{N}$ there exists $M_2 = M_2(p, q, A, B) > 0$ such that*

$$w_{p,q}(x,y)L_{n;0}\left(|t-x|^p|z-y|^q; A, B; x, y\right) \\ \leq \left(w_{2p}(x)\, T_{n;2p}(x;A)\right)^{\frac{1}{2}} \left(w_{2q}(y)\, T_{n;2q}(y;B)\right)^{\frac{1}{2}} \leq M_2\, n^{-\frac{p+q}{2}}$$

for all $(x, y) \in \mathbb{R}_0^2$ and $n \in \mathbb{N}$.

Now we shall prove the main lemma.

Lemma 2. *Let $A, B \in \Omega$ and $r \in \mathbb{N}_0$. If $r = 0$, then for every $f \in C^0$ we have*

$$\|L_{n;0}(f)\|_{0,0} \leq \|f\|_{0,0}, \qquad n \in \mathbb{N}. \tag{12}$$

If $r \in \mathbb{N}$, then there exists $M_3 = M_3(r, A, B) > 0$ such that for every $f \in C^r$ and $n \in \mathbb{N}$ we have

$$\|L_{n;r}(f)\|_{r,r} \leq M_3 \sum_{s=0}^{r} \sum_{i=0}^{s} \|f^{(s)}_{x^{s-i}y^i}\|_{r-s,r-s}. \tag{13}$$

The formulas (8)–(10) and (i) and inequalities (12) and (13) show that $L_{n;r}(f)$ is a positive linear operator from the space C^r into $C_{r,r}$.

Proof. If $r = 0$, then by (1)–(3), (10) and (11) we get

$$\|L_{n;0}(f)\|_{0,0} \leq \|f\|_{0,0}\, \|L_{n;0}(1;\cdot,\cdot)\|_{0,0} = \|f\|_{0,0},$$

for every $f \in C^0$ and $n \in \mathbb{N}$.

If $r \in \mathbb{N}$, then by (8)–(10) it results that

$$|L_{n;r}(f;x,y)| \leq \sum_{s=0}^{r} \frac{1}{s!} \sum_{i=0}^{s} \binom{s}{i} \|f^{(s)}_{x^{s-i}y^i}\|_{r-s,r-s} \\ \times L_{n;0}\left(\frac{|t-x|^{s-i}|z-y|^i}{w_{r-s,r-s}(t,z)}; x, y\right).$$

From (1) and (2) we get

$$\left(w_{r-s,r-s}(t,z)\right)^{-1} = \left(1+t^{r-s}\right)\left(1+z^{r-s}\right) \\ \leq 4^{r-s}\left(1+x^{r-s}+|t-x|^{r-s}\right)\left(1+y^{r-s}+|z-y|^{r-s}\right)$$

for $(x, y), (t, z) \in \mathbb{R}_0^2$ and $0 \leq s \leq r$, which implies that

$$w_{r,r}(x,y)\, \|L_{n;r}(f;x,y)\|_{r,r} \\ \leq \sum_{s=0}^{r} \frac{4^{r-s}}{s!} \sum_{i=0}^{s} \binom{s}{i} \|f^{(s)}_{x^{s-i}y^i}\|_{r-s,r-s} \sum_{j=1}^{4} Q_j(x,y), \tag{14}$$

where

$$Q_1(x,y) := \frac{w_{r,r}(x,y)}{w_{r-s,r-s}(x,y)} L_{n;0}\left(|t-x|^{s-i}\,|z-y|^{i};x,y\right),$$

$$Q_2(x,y) := \frac{w_{r,r}(x,y)}{w_{r-s}(x)} L_{n;0}\left(|t-x|^{s-i}\,|z-y|^{r-s+i};x,y\right),$$

$$Q_3(x,y) := \frac{w_{r,r}(x,y)}{w_{r-s}(y)} L_{n;0}\left(|t-x|^{r-i}\,|z-y|^{i};x,y\right),$$

$$Q_4(x,y) := w_{r,r}(x,y)\, L_{n;0}\left(|t-x|^{r-i}\,|z-y|^{r-s+i};x,y\right).$$

Applying Lemma 1 and (1) and (2), we get

$$\begin{aligned} Q_1(x,y) &\le M_2(i,s,A,B)\, n^{-s/2} \frac{w_r(x)}{w_{r-s}(x)\, w_{s-i}(x)} \frac{w_r(y)}{w_{r-s}(y)\, w_i(y)} \\ &\le M_2(i,s,A,B)\, \frac{(1+x^{r-s})(1+x^{s-i})}{1+x^r} \frac{(1+y^{r-s})(1+y^i)}{1+y^r} \\ &\le M_4(i,s,r,A,B), \end{aligned} \tag{15}$$

for $n \in \mathbb{N}$, $(x,y) \in \mathbb{R}_0^2$, and $0 \le i \le s \le r$. Analogously we obtain

$$Q_j(x,y) \le M_5(i,s,r,A,B), \qquad j = 2,3,4, \tag{16}$$

for $(x,y) \in \mathbb{R}_0^2$, $n \in \mathbb{N}$ and $0 \le i \le s \le r$, where $M_5(i,s,r,A,B)$ is a positive constant.

The inequality (13) follows from (14)–(16). □

3. Theorems

3.1. First we shall give a theorem on the degree of approximation of function $f \in C^r$ by $L_{n;r}(f)$. The formulas (8) and (11) and Lemma 2 imply that

$$\begin{aligned} &L_{n;r}(f;x,y) - f(x,y) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{nj}(x)\, b_{nk}(y) \left(\sum_{s=0}^{r} \frac{1}{s!} d^s f\left(\frac{j}{n},\frac{k}{n}\right) - f(x,y)\right) \end{aligned} \tag{17}$$

for all $f \in C^r$, $r \in \mathbb{N}_0$, $(x,y) \in \mathbb{R}_0^2$ and $n \in \mathbb{N}$.

Theorem 1. *Let $A, B \in \Omega$ and $r \in \mathbb{N}_0$. Then there exists $M_6 = M_6(r,A,B) > 0$ such that for every $f \in C^r$ and $n \in \mathbb{N}$ we have*

$$\|L_{n;r}(f) - f\|_{r+1,r+1} \le M_6\, n^{-r/2} \sum_{i=0}^{r} \omega\left(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right), \tag{18}$$

where $\omega(f)$ is the modulus of continuity of f defined by (4).

Proof. Let $r=0$. The property (i) of A, B and (17) imply that

$$|L_{n;0}(f;x,y)-f(x,y)| \leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{nj}(x)\, b_{nk}(y) \left| f\left(\frac{j}{n},\frac{k}{n}\right) - f(x,y)\right|,$$

and by (4) and (5) we have

$$\begin{aligned}\left| f\left(\frac{j}{n},\frac{k}{n}\right) - f(x,y)\right| &\leq \omega\left(f; \left|\frac{j}{n}-x\right|, \left|\frac{k}{n}-y\right|\right)\\ &\leq \left(\sqrt{n}\left|\frac{j}{n}-x\right| + \sqrt{n}\left|\frac{k}{n}-y\right| + 2\right)\omega\left(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right).\end{aligned}$$

From the above and by (1), (2), (11) and Lemma 1 we get

$$\begin{aligned} &w_{1,1}(x,y)|L_{n;0}(f;x,y)-f(x,y)|\\ &\quad\leq \omega\left(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)\left(\sqrt{n}\, w_{1,1}(x,y)\, L_{n;0}\left(|t-x|;x,y\right)\right.\\ &\qquad\qquad \left. +\sqrt{n}\, w_{1,1}(x,y)\, L_{n;0}\left(|z-y|;x,y\right)+2\right)\\ &\quad\leq (2M_2+2)\,\omega\left(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right),\end{aligned}$$

for all $(x,y)\in\mathbb{R}_0^2$ and $n\in\mathbb{N}$, which yields (18) for $r=0$.

Considering $r\in\mathbb{N}$, we apply the following Taylor formula of $f\in C^r$ at a fixed point $(x_0,y_0)\in\mathbb{R}_0^2$:

$$\begin{aligned} f(x,y) &= \sum_{s=0}^{r}\frac{1}{s!}\, d^s f\left(x_0,y_0\right)\\ &\qquad + \frac{1}{(r-1)!}\int_0^1 (1-t)^{r-1}\left(d^r f\left(\tilde{x},\tilde{y}\right) - d^r f\left(x_0,y_0\right)\right) dt \end{aligned} \tag{19}$$

for $(x,y)\in\mathbb{R}_0^2$, where $(\tilde{x},\tilde{y}) := (x_0+t(x-x_0), y_0+t(y-y_0))$ and $d^r f(x_0,y_0)$ and $d^r f(\tilde{x},\tilde{y})$ are differentials of f with $\Delta x = x-x_0$ and $\Delta y = y-y_0$ (see, e.g., [2]). Taking $(x_j,y_k) := \left(\frac{j}{n},\frac{k}{n}\right)$ in (19) instead of (x_0,y_0), we obtain from (17) and (19) that

$$\begin{aligned} L_{n;r}(f;x,y)-f(x,y) &= \frac{1}{(r-1)!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{nj}(x)\, a_{nk}(y)\\ &\qquad \times \int_0^1 (1-t)^{r-1}\left(d^r f\left(\tilde{x}_j,\tilde{y}_k\right) - d^r f\left(x_j,y_k\right)\right) dt. \end{aligned} \tag{20}$$

By assumptions on $f \in C^r$ and by (9), (4) and (5) we can write

$$
\begin{aligned}
&|d^r f(\tilde{x}_j, \tilde{y}_k) - d^r f(x_j, y_k)| \\
&\quad \le \sum_{i=0}^{r} \binom{r}{i} \left| f^{(r)}_{x^{r-i}y^i}(\tilde{x}_j, \tilde{y}_k) - f^{(r)}_{x^{r-i}y^i}(x_j, y_k) \right| \left| x - \frac{j}{n} \right|^{r-i} \left| y - \frac{k}{n} \right|^{i} \\
&\quad \le \sum_{i=0}^{r} \binom{r}{i} \omega\left(f^{(r)}_{x^{r-i}y^i}; t\left| x - \frac{j}{n} \right|, t\left| y - \frac{k}{n} \right| \right) \left| x - \frac{j}{n} \right|^{r-i} \left| y - \frac{k}{n} \right|^{i} \qquad (21) \\
&\quad \le \sum_{i=0}^{r} \binom{r}{i} \omega\left(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \left(\sqrt{n}\left| x - \frac{j}{n} \right| + \sqrt{n}\left| y - \frac{k}{n} \right| + 2 \right) \\
&\qquad \times \left| x - \frac{j}{n} \right|^{r-i} \left| y - \frac{k}{n} \right|^{i}
\end{aligned}
$$

for $0 \le t \le 1$, $n \in \mathbb{N}$ and $j, k \in \mathbb{N}_0$. Using (21) to (20), we get

$$
\begin{aligned}
&|L_{n;r}(f; x, y) - f(x, y)| \\
&\quad \le \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} \omega\left(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \sum_{p=1}^{3} Y_p(x, y) \qquad (22)
\end{aligned}
$$

for $(x, y) \in \mathbb{R}_0^2$ and $n \in \mathbb{N}$, where

$$Y_1(x, y) := \sqrt{n}\, L_{n;0}\left(|t - x|^{r-i+1} |z - y|^{i}; x, y \right),$$

$$Y_2(x, y) := \sqrt{n}\, L_{n;0}\left(|t - x|^{r-i} |z - y|^{i+1}; x, y \right),$$

$$Y_3(x, y) := 2\, L_{n;0}\left(|t - x|^{r-i} |z - y|^{i}; x, y \right).$$

Next, by (1), (2) and Lemma 1 we get

$$Y_1(x, y) \le M_7(i, r)\, n^{-\frac{r}{2}} \left(1 + x^{r-i+1} \right) \left(1 + y^{i} \right),$$

$$Y_2(x, y) \le M_8(i, r)\, n^{-\frac{r}{2}} \left(1 + x^{r-i} \right) \left(1 + y^{i+1} \right),$$

$$Y_3(x, y) \le M_9(i, r)\, n^{-\frac{r}{2}} \left(1 + x^{r-i} \right) \left(1 + y^{i} \right),$$

which imply that

$$w_{r+1,r+1}(x, y)\, Y_p(x, y) \le M_{10}(i, r)\, n^{-\frac{r}{2}}, \qquad (23)$$

for all $(x, y) \in \mathbb{R}_0^2$, $n \in \mathbb{N}$, $0 \le i \le r$ and $p = 1, 2, 3$, where $M_{7-10}(i, r)$ are positive constants. From (22) and (23) the estimate (18) immediately follows.

The proof is complete. □

From Theorem 1 we derive the following

Corollary 1. *If $f \in C^r$, $r \in \mathbb{N}_0$, and the partial derivatives $f^{(r)}_{x^{r-i}y^i}$, $0 \le i \le r$, are uniformly continuous on $\mathbb{R}_0^2$, then*

$$\lim_{n\to\infty} n^{\frac{r}{2}} \|L_{n;r}(f) - f\|_{r+1,r+1} = 0,$$

for every $A, B \in \Omega$. This shows that

$$\lim_{n\to\infty} n^{\frac{r}{2}} \left(L_{n;r}(f;x,y) - f(x,y)\right) = 0$$

at every $(x,y) \in \mathbb{R}_0^2$.

3.2. Now we shall prove the following Voronovskaya type theorem.

Theorem 2. *Let $A, B \in \Omega$, $r \in \mathbb{N}_0$ and let $f \in C^{r+2}$. Then*

$$\begin{aligned} &L_{n;r}(f;A,B;x,y) - f(x,y) \\ &\quad = \frac{(-1)^r}{(r+1)!} \sum_{i=0}^{r+1} f^{(r+1)}_{x^{r+1-i}y^i}(x,y)\, T_{n;r+1-i}(x;A)\, T_{n;i}(y;B) \\ &\quad + \frac{(-1)^r (r+1)}{(r+2)!} \sum_{i=0}^{r+2} f^{(r+2)}_{x^{r+2-i}y^i}(x,y)\, T_{n;r+2-i}(x;A)\, T_{n;i}(y;B) \\ &\quad + o_{x,y}\left(n^{-\frac{r+2}{2}}\right) \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{24}$$

at every $(x,y) \in \mathbb{R}_0^2$, where $T_{n;s}$ is defined by (6). In particular, if $r = 0$ and $f \in C^2$, then

$$\begin{aligned} &L_{n;0}(f;A,B;x,y) - f(x,y) \\ &\quad = f'_x(x,y)\, T_{n;1}(x;A) + f'_y(x,y) T_{n;1}(y;B) \\ &\quad + \frac{1}{2}\Big(f''_{x^2}(x,y)\, T_{n;2}(x;A) + 2 f''_{xy}(x,y)\, T_{n;1}(x;A)\, T_{n;1}(y;B) \\ &\quad + f''_{y^2}(x,y)\, T_{n;2}(y;B)\Big) + o_{x,y}\left(n^{-1}\right) \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{25}$$

at every $(x,y) \in \mathbb{R}_0^2$.

Proof. Let $(x,y) \in \mathbb{R}_0^2$ be a fixed point. First we shall prove (25).

By the Taylor formula of $f \in C^2$ at the point (x, y) we have

$$\begin{aligned} f\left(\frac{j}{n}, \frac{k}{n}\right) &= f(x,y) + f'_x(x,y)\left(\frac{j}{n} - x\right) \\ &+ f'_y(x,y)\left(\frac{k}{n} - y\right) + \frac{1}{2}\left(f''_{x^2}(x,y)\left(\frac{j}{n} - x\right)^2\right. \\ &\left. + 2f''_{xy}(x,y)\left(\frac{j}{n} - x\right)\left(\frac{k}{n} - y\right) + f''_{y^2}(x,y)\left(\frac{k}{n} - y\right)^2\right) \\ &+ \sum_{i=0}^{2} \varphi_i\left(\frac{j}{n}, \frac{k}{n}; x, y\right)\left(\frac{j}{n} - x\right)^{2-i}\left(\frac{k}{n} - y\right)^i, \end{aligned} \tag{26}$$

for all $j, k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, where $\varphi_i(t,z) \equiv \varphi_i(t,z;x,y)$, $i = 0,1,2$, are functions belonging to $C_{0,0}$ and $\lim\limits_{(t,z)\to(x,y)} \varphi_i(t,z) = \varphi_i(x,y) = 0$. From (10), (26) and (6), we get that

$$\begin{aligned} &L_{n;0}(f; x, y) \\ &\quad = f(x,y) + f'_x(x,y)\, T_{n;1}(x; A) \\ &\quad + f'_y(x,y)\, T_{n;1}(y; B) + \frac{1}{2}\left\{ f''_{x^2}(x,y)\, T_{n;2}(x; A)\right. \\ &\quad \left. + 2f''_{xy}(x,y)\, T_{n;1}(x; A)\, T_{n;1}(y; B) + f''_{y^2}(x,y)\, T_{n;2}(y; B)\right\} \\ &\quad + \sum_{i=0}^{2} L_{n;0}\left(\varphi_i(t,z)(t-x)^{2-i}(z-y)^i; x, y\right) \end{aligned} \tag{27}$$

By Hölder's inequality we have

$$\begin{aligned} &\left|L_{n;0}\left(\varphi_i(t,z)(t-x)^{2-i}(z-y)^i; x, y\right)\right| \\ &\quad \le \left(L_{n;0}\left(\varphi_i^2(t,z); x, y\right)\right)^{\frac{1}{2}} \left(L_{n;0}\left((t-x)^{4-2i}(z-y)^{2i}; x, y\right)\right)^{\frac{1}{2}} \end{aligned} \tag{28}$$

for $n \in \mathbb{N}$ and $i = 0, 1, 2$. The properties of φ_i and Corollary 1 imply that

$$\lim_{n\to\infty} L_{n;0}\left(\varphi_i^2(t,z); x, y\right) = \varphi_i^2(, y) = 0 \quad \text{for} \quad i = 0, 1, 2. \tag{29}$$

From (28), (29) and Lemma 1, we deduce that

$$L_{n;0}\left(\varphi_i(t,z)(t-x)^{2-i}(z-y)^i; x, y\right) = o_{xy}\left(n^{-1}\right) \quad \text{as} \quad n \to \infty, \tag{30}$$

for $i = 0, 1, 2$. Now (25) immediately follows from (30) and (27).

Let $r \in \mathbb{N}$. Denoting $\Delta x_j = \frac{j}{n} - x$ and $\Delta y_k = \frac{k}{n} - y$, we get from (8) and (9)

$$L_{n;r}(f; x, y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{nj}(x)\, b_{nk}(y) \sum_{s=0}^{r} \frac{(-1)^s}{s!}\, \widetilde{d^s f}\left(\frac{j}{n}, \frac{k}{n}\right), \tag{31}$$

where $\widetilde{d^s f}\left(\frac{j}{n}, \frac{k}{n}\right)$, $0 \le s \le r$, are defined for Δx_j and Δy_k, i.e.,

$$\widetilde{d^s f}\left(\frac{j}{n}, \frac{k}{n}\right) = \sum_{i=0}^{s} \binom{s}{i} f^{(s)}_{x^{s-i}y^i}\left(\frac{j}{n}, \frac{k}{n}\right) (\Delta x_j)^{s-i} (\Delta y_k)^i . \tag{32}$$

Since $f^{(s)}_{x^{s-i}y^i} \in C^{r+2-s}$, we can write the Taylor formula at the point (x, y):

$$\begin{aligned} f^{(s)}_{x^{s-i}y^i}\left(\frac{j}{n}, \frac{k}{n}\right) &= \sum_{p=0}^{r+2-s} \frac{1}{p!} \widetilde{d^p f^{(s)}_{x^{s-i}y^i}}(x, y) \\ &+ \sum_{q=0}^{r+2-s} \varphi_{i,q,s}\left(\frac{j}{n}, \frac{k}{n}; x, y\right) (\Delta x_j)^{r+2-s-q} (\Delta y_k)^q , \end{aligned} \tag{33}$$

where differentials $\widetilde{d^p f^{(s)}_{x^{s-i}y^i}}(x, y)$ are defined for Δx_j and Δy_k and $\varphi_{i,q,s}(t, z) \equiv \varphi_{i,q,s}(t, z; x, y)$ are functions belonging to the space $C_{0,0}$ and $\lim\limits_{(t,z)\to(x,y)} \varphi_{i,q,s}(t, z) = \varphi_{i,q,s}(x, y) = 0$. From (32) and (33) we deduce that

$$\begin{aligned} \widetilde{d^s f}\left(\frac{j}{n}, \frac{k}{n}\right) &= \sum_{p=0}^{r+2-s} \frac{1}{p!} \widetilde{d^{s+p} f}(x, y) \\ &\quad + \sum_{p=0}^{r+2} \psi_{p,s}\left(\frac{j}{n}, \frac{k}{n}\right) (\Delta x_j)^{r+2-p} (\Delta y_k)^p , \end{aligned}$$

where $\psi_{p,s} \in C_{0,0}$ and $\lim\limits_{(t,z)\to(x,y)} \psi_{p,s}(t, z) = \psi_{p,s}(x, y) = 0$. Consequently we have

$$\begin{aligned} \sum_{s=0}^{r} \frac{(-1)^s}{s!} \widetilde{d^s f}\left(\frac{j}{n}, \frac{k}{n}\right) &= \sum_{s=0}^{r} \frac{(-1)^s}{s!} \sum_{p=0}^{r+2-s} \frac{1}{p!} \widetilde{d^{s+p} f}(x, y) \\ &\quad + \sum_{p=0}^{r+2} h_{p,r}\left(\frac{j}{n}, \frac{k}{n}\right) (\Delta x_j)^{r+2-p} (\Delta y_k)^p , \end{aligned} \tag{34}$$

where $h_{p,r} \in C_{0,0}$ and $\lim\limits_{(t,z)\to(x,y)} h_{p,r}(t,z) = h_{p,r}(x,y) = 0$. Next, by elementary calculations, we get that

$$\begin{aligned}
\sum_{s=0}^{r} \frac{(-1)^s}{s!} \sum_{p=0}^{r+2-s} \frac{1}{p!} \widetilde{d^{s+p} f}(x,y) &= \sum_{s=0}^{r} \frac{(-1)^s}{s!} \sum_{q=s}^{r+2} \frac{\widetilde{d^q f}(x,y)}{(q-s)!} \\
&= \sum_{s=0}^{r} \frac{(-1)^s}{s!} \left(\sum_{q=s}^{r} \frac{\widetilde{d^q f}(x,y)}{(q-s)!} + \frac{\widetilde{d^{r+1} f}(x,y)}{(r+1-s)!} + \frac{\widetilde{d^{r+2} f}(x,y)}{(r+2-s)!} \right) \\
&= \sum_{q=0}^{r} \frac{\widetilde{d^q f}(x,y)}{q!} \sum_{s=0}^{r} \binom{q}{s} (-1)^s + \frac{\widetilde{d^{r+1} f}(x,y)}{(r+1)!} \sum_{s=0}^{r} \binom{r+1}{s} (-1)^s \\
&\quad + \frac{\widetilde{d^{r+2} f}(x,y)}{(r+2)!} \sum_{s=0}^{r} \binom{r+2}{s} (-1)^s.
\end{aligned}$$

It is well known that

$$\sum_{s=0}^{r} \binom{r}{s} (-1)^s = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } r \in \mathbb{N}, \end{cases}$$

$$\sum_{s=0}^{r} \binom{r+1}{s} (-1)^s = (-1)^r, \qquad \sum_{s=0}^{r} \binom{r+2}{s} (-1)^s = (r+1)(-1)^r,$$

for $r \in \mathbb{N}_0$. Consequently we have

$$\begin{aligned}
\sum_{s=0}^{r} \frac{(1-)^s}{s!} \sum_{p=0}^{r+2-s} \frac{\widetilde{d^{p+s} f}(x,y)}{p!} &= f(x,y) + \frac{(-1)^r}{(r+1)!} \widetilde{d^{r+1} f}(x,y) \\
&\quad + \frac{(-1)^r (r+1)}{(r+2)!} \widetilde{d^{r+2} f}(x,y).
\end{aligned} \tag{35}$$

From (31), (34), (35) and (10) we deduce that

$$\begin{aligned}
L_{n;r}(f;x,y) &= f(x,y) L_{n;0}(1;x,y) + \frac{(-1)^r}{(r+1)!} L_{n;0} \left(\widetilde{d^{r+1} f}(x,y); x, y \right) \\
&\quad + \frac{(-1)^r (r+1)}{(r+2)!} L_{n;0} \left(\widetilde{d^{r+2} f}(x,y); x, y \right) \\
&\quad + \sum_{p=0}^{r+2} L_{n;0} \left(h_{p,r}(t,z)(t-x)^{r+2-p} (z-y)^p; x, y \right)
\end{aligned} \tag{36}$$

for $n \in \mathbb{N}$. Further by (10), (32) and (6) we get that

$$\begin{aligned} L_{n;0}&\left(\widetilde{d^{r+1}f}(x,y);A,B;x,y\right)\\ &=\sum_{i=0}^{r+1}\binom{r+1}{i}f^{(r+1)}_{x^{r+1-i}y^i}(x,y)\times L_{n;0}\left((t-x)^{r+1-i}(z-y)^i;x,y\right)\\ &=\sum_{i=0}^{r+1}\binom{r+1}{i}f^{(r+1)}_{x^{r+1-i}y^i}(x,y)T_{n;r+1-i}(x;A)\,T_{n;i}(y;B)\end{aligned}\tag{37}$$

and, analogously,

$$\begin{aligned} L_{n;0}&\left(\widetilde{d^{r+2}f}(x,y);A,B;x,y\right)\\ &=\sum_{i=0}^{r+2}\binom{r+2}{i}f^{(r+2)}_{x^{r+2-i}y^i}(x,y)\,T_{n;r+2-i}(x;A)\,T_{n;i}(y;B).\end{aligned}\tag{38}$$

By Hölder's inequality we have

$$\begin{aligned} &\left|L_{n;0}\left(h_{p,r}(t,z)(t-x)^{r+2-p}(z-p)^p;x,y\right)\right|\\ &\qquad\le\left(L_{n;0}\left(h^2_{p,r}(t,z);x,y\right)\right)^{\frac12}\\ &\qquad\times\left(L_{n;0}\left((t-x)^{2(r+2-p)}(z-y)^{2p};x,y\right)\right)^{\frac12}\end{aligned}\tag{39}$$

and by properties of functions $h_{p,r}$, $0\le p\le r+2$, and Corollary 1 we have

$$\lim_{n\to\infty}L_{n;0}\left(h^2_{p,r}(t,z);x,y\right)=h^2_{p,r}(x,y)=0.\tag{40}$$

From (39), (40) and Lemma 1 it results that

$$L_{n;0}\left(h_{p,r}(t,z)(t-x)^{r+2-p}(z-y)^p;x,y\right)=o_{x,y}\left(n^{-\frac{r+2}{2}}\right)\quad\text{as } n\to\infty,\tag{41}$$

at every fixed point $(x,y)\in\mathbb{R}^2_0$ and for $0\le p\le r+2$.

Using (36), (37), (38) and (41) we immediately obtain the desired assertion (24) and we complete the proof. □

4. Examples

4.1. Consider matrices $\bar{A}=[\bar{a}_{nk}(\cdot)]$ and $\bar{B}=[\bar{b}_{nk}(\cdot)]$ with

$$\begin{aligned}\bar{a}_{nk}(x)&=e^{-nx}\,\frac{(nx)^k}{k!},\\ \bar{b}_{nk}(x)&=\binom{n-1+k}{k}x^k(1+x)^{-n-k},\end{aligned}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. By results given in [1] we see that $\bar{A}, \bar{B} \in \Omega$ and analogously as in (8)–(10) we can define the following operators for $f \in C^r(R_0^2)$, $r \in \mathbb{N}_0$, and $n \in \mathbb{N}$:

1. Szász-Mirakyan operator $L_{n;r}^{\{1\}}(f;\cdot,\cdot) \equiv L_{n;r}^{\{1\}}(f;\bar{A},\bar{A},\cdot,\cdot)$,
2. Baskakov operator $L_{n;r}^{\{2\}}(f;\cdot,\cdot) \equiv L_{n;r}^{\{2\}}(f;\bar{B},\bar{B},\cdot,\cdot)$,
3. Szasz-Mirakyan and Baskakov operator $L_{n;r}^{\{3\}}(f;\cdot,\cdot) \equiv L_{n;r}^{\{3\}}(f;\bar{A},\bar{B},\cdot,\cdot)$.

For the classical Szaśz-Mirakyan and Baskakov operators the next formulas hold [1]

$$S_n(t-x;x) = \mathrm{O} = V_n(t-x;x), \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N},$$

which by the above formulas and (10) yield

$$L_{n;0}^{\{i\}}(t-x;x,y) = \mathrm{O} = L_{n;0}^{\{i\}}(z-y;x,y)$$

for $(x,y) \in \mathbb{R}_0^2$, $n \in \mathbb{N}$, $i = 1,2,3$. Hence Theorem 1 and Theorem 2 imply the following corollaries.

Corollary 2. *Let $f \in C^r$, $r \in \mathbb{N}_0$. Then*

$$\lim_{n\to\infty} n^{\frac{r}{2}} \left(L_{n;r}^{\{i\}}(f;x,y) - f(x,y) \right) = 0, \qquad i = 1,2,3,$$

at every $(x,y) \in \mathbb{R}_0^2$.

Corollary 3. *Let $f \in C^2$. Then*

$$\lim_{n\to\infty} n \left(L_{n;r}^{\{i\}}(f;x,y) - f(x,y) \right) = \begin{cases} \varphi_i(x) f''_{x^2}(x,y) + \varphi_i(y) f''_{y^2}(x,y) & \text{if } i = 1,2, \\ \varphi_1(x) f''_{x^2}(x,y) + \varphi_2(y) f''_{y^2}(x,y) & \text{if } i = 3, \end{cases}$$

at every $(x,y) \in \mathbb{R}_0^2$, where $\varphi_1(x) = \frac{x}{2}$ and $\varphi_2(x) = \frac{x(1+x)}{2}$.

4.2. Now we shall consider continuous functions $f(\cdot,\cdot)$ with partial derivatives of order m, $0 \le m \le r$, on $I^2 = I \times I$, where $I = [0,1]$ and $r \in \mathbb{N}_0$ is a fixed number. We define the Bernstein polynomial

$$B_{n;r}(f;x,y) := \sum_{j=0}^{n} \sum_{k=0}^{n} p_{nj}(x)\, p_{nk}(y) \sum_{s=0}^{r} \frac{d^s f\left(\frac{j}{n}, \frac{k}{n}\right)}{s!},$$

$(x,y) \in I^2$, $n \in \mathbb{N}$, where

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k \in \mathbb{N}_0, \ x \in I,$$

and $d^s f\left(\frac{j}{n}, \frac{k}{n}\right)$ is the s-th differential of f defined by (9).

Arguing as in the proof of Theorem 1, we obtain for $B_{n;r}(f)$ the following inequality:

$$\max_{(x,y)\in I^2} |B_{n;r}(f;x,y) - f(x,y)| \le M_{11}(r)\, n^{-\frac{r}{2}} \sum_{i=0}^{r} \omega\left(f^{(r)}_{x^{r-i}y^i}; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)$$

for $n \in \mathbb{N}$, where $M_{11}(r)$ is a positive constant and $\omega(g;\cdot,\cdot)$ is the modulus of continuity of g continuous on I^2 (see [7]).

It is obvious that for $B_{n;r}(f;\cdot,\cdot)$ we also can derive an analogue of Theorem 2.

Similar theorems for functions of one variable were given in [3].

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