ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 9, 2005

Banach-Stone theorems for Banach bundles

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ABSTRACT. We prove two Banach-Stone type theorems for section spaces of real Banach bundles. The first theorem assumes that the duals of all fibers are strictly convex, and the second considers disjointness-preserving operators. In each case, the result generalizes the corresponding Banach-Stone theorem for spaces of continuous vector-valued functions.

The classic Banach-Stone theorem has the form: let X and Y be compact Hausdorff spaces, and let $T : C(X) \to C(Y)$ be an isometric isomorphism. Then there is a homeomorphism $\phi : Y \to X$ and a map $\psi \in C(Y)$ with $|\psi(y)| = 1$ for all $y \in Y$, such that for each $a \in C(X)$ and $y \in Y$, we have $(Ta)(y) = \psi(y)a(\phi(y))$. (Note that if we are speaking of real Banach spaces, then the scalar-valued map ψ has values ± 1.) Conversely, every such function ψ and homeomorphism ϕ together determine an isometric isomorphism of $C(X)$ and $C(Y)$.

Since the time of the first formulation of this theorem (for compact metric spaces) by Banach [3, page 172], and its extension by Stone [17], there has been a large literature on pairs of spaces (E, F) which have what in the survey article [7] is termed the *strong Banach-Stone property*: let X and Y be compact Hausdorff spaces, and E and F be Banach spaces such that $C(X, E)$ (the space of continuous E-valued functions on X) is isometrically isomorphic to $C(Y, F)$ via the map $T : C(X, E) \to C(Y, F)$; then there exists a homeomorphism $\phi : Y \to X$ and a map $\psi : Y \to I(E, F)$, the space of isometric isomorphisms from E to F, such that ψ is continuous in the strong operator topology, and $(T\sigma)(y) = \psi(y)[\sigma(\phi(y))]$.

Noting that, say, elements of $C(X, E)$ take their values in a fixed space E, it is reasonable to ask what might happen in a situation where continuous vector-valued functions take their values in spaces which vary with $x \in X$. This vague notion is made more precise in the definition of a Banach bundle.

Received May 24, 2005.
2000 *Mathematics Subject Classification.* 46E40, 46B04.
Key words and phrases. Banach bundle, Banach-Stone theorem.

In the following, X and Y will always be compact Hausdorff spaces, and all Banach spaces will be assumed real. If $\pi : \mathcal{E} \to X$ (shortly, π) is a Banach bundle (= bundle of Banach spaces) with fibers $\{E_x : x \in X\}$, we will assume that for all $x \in X$, $E_x \neq \{0\}$. The total space $\mathcal{E}$ carries a topology such that the relative topology on each fiber E_x is its original Banach space topology. We can regard $\mathcal{E}$ as the disjoint union $\dot{\bigcup}\{E_x : x \in X\}$. (Alternatively, we can think of $\mathcal{E}$ as $\bigcup\{\{x\} \times E_x : x \in X\}$; this is the approach of [8], which uses fibered vector spaces.) See [8] or [14] for details on the construction of Banach bundles; we summarize here the most important properties for our purposes. We denote by $\Gamma(\pi)$ the space of sections (= continuous choice functions) $\sigma : X \to \mathcal{E}$ of the bundle $\pi : \mathcal{E} \to X$; $\Gamma(\pi)$ is a Banach $C(X)$-module under the sup-norm, pointwise addition, and pointwise multiplication of sections σ by elements $a \in C(X)$.

As noted in, say, [15] or [12], the intuitive point to keep in mind is that when $\sigma \in \Gamma(\pi)$, then the values $\sigma(x)$ vary continuously over (possibly different) spaces E_x. For example, for a fixed Banach space E, we can regard $C(X, E)$, up to isometric isomorphism, as the space of sections of the trivial bundle $\rho : \mathcal{T} = X \times E \to X$, where $X \times E$ is given the product topology. Here, the total space $\mathcal{T} = X \times E = \bigcup\{\{x\} \times E : x \in X\}$ can be thought of as a union of copies of E, and an element $\sigma \in C(X, E)$, which we usually think of as having values which vary continuously over the fixed set E, can be interpreted as a section in $\Gamma(\rho)$ which varies in a very nice way between copies of E.

Given a Banach bundle $\pi : \mathcal{E} \to X$, the function $x \mapsto \|\sigma(x)\|$ is upper semicontinuous for each $\sigma \in \Gamma(\pi)$. If this function is continuous for each $\sigma \in \Gamma(\pi)$, we will call $\pi : \mathcal{E} \to X$ a *continuous bundle*. We say that $\pi : \mathcal{E} \to X$ is *separable* if there exists a countable collection $\{\sigma_n\} \subset \Gamma(\pi)$ such that $\{\sigma_n(x)\}$ is dense in E_x for each $x \in X$.

In any Banach bundle $\pi : \mathcal{E} \to X$, for $x \in X$, there is an isometric injection $j_x : E_x^* \to \Gamma(\pi)^*$ given by (for $\sigma \in \Gamma(\pi)$) $\langle \sigma, j_x(f) \rangle = \langle \sigma(x), f \rangle = \langle \sigma, f \circ ev_x \rangle$, where $ev_x : \Gamma(\pi) \to E_x$, $\sigma \mapsto \sigma(x)$, is the evaluation map at x. Moreover, there is a map $p_x : \Gamma(\pi)^* \to E_x^*$ such that $p_x \circ j_x$ is the identity on E_x^*; that is, E_x^* is a retract of $\Gamma(\pi)^*$. We have $\|p_x\| = 1$, and p_x is an isometry when restricted to $j_x(E_x^*)$.

Denote by $\mathcal{H} = \mathcal{H}(\pi)$ the space $Hom_X(\Gamma(\pi), C(X))$ of all $C(X)$-module homomorphisms from $\Gamma(\pi)$ to $C(X)$. As noted in [15], we can identify $\mathcal{H}$ with the space of choice functions $H : X \to \dot{\bigcup}\{E_x^* : x \in X\}$ such that the function $x \mapsto \langle \sigma(x), H(x) \rangle$ is continuous for each $\sigma \in \Gamma(\pi)$. Then $\mathcal{H}$ is a Banach space under the norm $\|H\| = \sup\{\|H(x)\| : x \in X\}$. Following [10], say that $\mathcal{H}$ *norms* $\pi : \mathcal{E} \to X$ if for each $z \in E_x \subset \mathcal{E}$ we have

$$\|z\| = \sup\{|\langle z, H(x) \rangle| : H \in \mathcal{H}, \|H\| \leq 1\}.$$

In [10], following Definition 4.4, there is a catalogue of continuous bundles π which are known to be normed by $\mathcal{H}$. These include e.g. 1) the trivial bundles $\rho : X \times E \to X$, where E is a Banach space and $\Gamma(\rho)$ is isometrically isomorphic to $C(X, E)$ (the constant maps from X to E^* will do the job); 2) bundles all of whose fibers are Hilbert spaces; 3) separable bundles; and 4) bundles whose base space X is extremally disconnected.

We say that π is *strongly normed* by $\mathcal{H}$ provided that for each $x \in X$ we have

$$\{H(x) : H \in \mathcal{H}, \|H\| = \|H(x)\|\} = E_x^*.$$

The strongly normed bundles include cases 1) and 3) above.

Note that, in either event, if $H \in \mathcal{H}$ and $\sigma \in \Gamma(\pi)$, then the map $x \mapsto j_x(H(x))$ is weak* continuous from X to $\Gamma(\pi)^*$ (because $x \mapsto \langle \sigma(x), H(x) \rangle = \langle \sigma, j_x(H(x)) \rangle$ is in $C(X)$ for each $\sigma \in \Gamma(\pi)$) and that if $a \in C(X), \sigma \in \Gamma(\pi)$, and $f_x \in E_x^*$, then $\langle a\sigma, j_x(f_x) \rangle = a(x) \langle \sigma, j_x(f_x) \rangle$.

[Historical note: Banach bundles (= "bundles of Banach spaces") and their section spaces have been around under a variety of labels for some time, to a variety of ends. The terminology includes "upper semicontinuous function modules" [5]; "Banach function modules" [4]; and "bundles of topological vector spaces" [8]. A more categorical discussion of the matter can be found in e.g. [11], and the paper [10] and some of its references are also relevant. Indeed, the basic ideas go back as far as 1949 [9] and 1951 [13], under the terminology "continuous sums" (of Banach spaces). The tie between Banach bundles and function modules can be summarized by the following: the section space of any Banach bundle is a function module, and, conversely, any Banach $C(X)$-module M has a norm-decreasing representation as a space of sections of a certain "canonical" (see [14]) Banach bundle; if M is $C(X)$-locally convex (see [11]), this representation is an isometric isomorphism. The connections between and among Banach bundles, function modules and continuous products (sums) of Banach spaces are also elucidated and exploited in [6].]

In all that follows, unless explicitly stated otherwise, let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be continuous bundles of (real) Banach spaces over the compact Hausdorff spaces X and Y. Suppose that $T : \Gamma(\pi) \to \Gamma(\xi)$ is an isometric isomorphism. We seek analogues of the Banach-Stone theorem appropriate for section spaces. The difficulty in finding them is a result of there not being a complete description of the dual spaces $\Gamma(\pi)^*$. However, if we think of $C(X)$ as being "scalars" for $\Gamma(\pi)$, then the existence of a norming $\mathcal{H}(\pi)$ provides a partially satisfactory replacement.

We note briefly that without at least some restrictions on our bundles, it is almost impossible to say anything. For example, let $X = [0, 1]$, and let $Y = [0, 1] \times [0, 1]$, and let X_d and Y_d be X and Y with their discrete topologies. If $\pi : \mathcal{E} \to X$ and $\rho : \mathcal{F} \to Y$ are the canonical bundles (as described in [14]) of the spaces $c_0(X_d)$ and $c_0(Y_d)$, viewed as modules over $C(X)$ and $C(Y)$ respectively, then $\Gamma(\pi) \simeq c_0(X_d) \simeq c_0(Y_d) \simeq \Gamma(\rho)$, but certainly X and Y are not homeomorphic.

As in the introduction, for Banach spaces E and F, denote by $I(E,F)$ the space of isometric isomorphisms $T : E \to F$. Given the bundles $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$, and a homeomorphism $\phi : Y \to X$, we can endow $\dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$, the disjoint union of the $I(E_{\phi(y)}, F_y)$, with the weak topology generated by Y, ϕ, and $\Gamma(\pi)$. In this topology, we have $S_{y_\alpha} \to S_y$ if and only if $y_\alpha \to y$ in Y and $S_{y_\alpha}[\sigma(\phi(y_\alpha))] \to S_y[\sigma(\phi(y))]$ in the bundle topology of ξ for each $\sigma \in \Gamma(\pi)$.

It is easy to write down a condition which is sufficient to generate an isometric isomorphism $T : \Gamma(\pi) \to \Gamma(\xi)$.

Proposition 1. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be Banach bundles, let $\phi : Y \to X$ be a homeomorphism, and suppose there exists a map $\psi : Y \to \dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$ which is continuous with respect to the weak topology generated by Y, ϕ, and $\Gamma(\pi)$. Then the map $T : \Gamma(\pi) \to \Gamma(\xi)$ defined by*

$$(T\sigma)(y) = \psi(y)[\sigma(\phi(y))]$$

is an isometric isomorphism.

Proof. Since each $\psi(y)$ is an isometric isomorphism, and since ϕ is a homeomorphism, it is easy to check that $\|T\sigma\| = \|\sigma\|$. Clearly T is linear. That $T\sigma \in \Gamma(\xi)$ whenever $\sigma \in \Gamma(\pi)$ follows from the continuity of ϕ, σ, and ψ. Finally, note that $T(\Gamma(\pi))$ is a (closed) $C(Y)$-submodule of $\Gamma(\xi)$: for, if $b \in C(Y)$, then for some $a \in C(X)$ we have $b = a \circ \phi$, by the scalar-valued Banach-Stone theorem. Thus, for $\sigma \in \Gamma(\pi)$, we have

$$\begin{aligned}(b \cdot T\sigma)(y) &= b(y) \cdot (T\sigma)(y) \\ &= a(\phi(y)) \cdot \psi(y)[\sigma(\phi(y))] \\ &= \psi(y)[(a\sigma)(\phi(y))],\end{aligned}$$

and since $a\sigma \in \Gamma(\pi)$, we have $b{\cdot}T\sigma \in \Gamma(\xi)$. It is easily checked that $\{(T\sigma)(y) : \sigma \in \Gamma(\pi)\} = F_y$ for each $y \in Y$, and so by the bundle version of the Stone-Weierstrass Theorem (see e.g. [8]), since $T(\Gamma(\pi))$ is dense in $\Gamma(\xi)$, it is all of $\Gamma(\xi)$. □

We prove the converse for two cases of isometric isomorphisms $T : \Gamma(\pi) \to \Gamma(\xi)$.

The first is a special case of [2, Theorem 2.8]. Denote by $extr(Z)$ the extreme points of the unit ball of the Banach space Z. Using our language of bundles and section spaces, let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be Banach bundles with all fibers non-zero such that F_y has a trivial centralizer for each $y \in Y$, and let $T : \Gamma(\pi) \to \Gamma(\xi)$ be an isomorphism such that $T^*(extr(\Gamma(\xi)^*) \subset$

$extr(\Gamma(\pi)^*)$. Then there is a function $\phi : Y \to X$ onto a dense subset of X and a map $\psi : Y \to \dot{\bigcup}\{Z_y : y \in Y\}$ (where Z_y is the family of operators from $E_{\phi(y)}$ to F_y which take extreme points of the unit ball in $E_{\phi(y)}$ to extreme points in the unit ball of F_y) such that

$$(T\sigma)(y) = \psi(y)[\sigma(\phi(y))]$$

for all $\sigma \in \Gamma(\pi)$. Since a strictly convex space has a trivial centralizer, it follows in particular that if all F_y are strictly convex, then the equality displayed above holds.

This result is also related to that of [16]: let E and F be Banach spaces such that both E^* and F^* are strictly convex; that is, such that each point of the unit spheres S_{E^*}, S_{F^*} is an extreme point of the respective unit balls B_{E^*}, B_{F^*}. Then (E, F) satisfies the strong Banach-Stone property.

The following remark is known, and the result is crucial for the succeeding development.

Lemma 2 (see [5] and [4], Theorem 4.5). *Let $\pi : \mathcal{E} \to X$ be any Banach bundle. Then $g \in extr(\Gamma(\pi)^*)$ if and only if there exists some $x \in X$ and $f_x \in extr(E_x^*)$ such that $g = f_x \circ ev_x = j_x(f_x)$.*

The next lemmas are analogous to those in [16].

Let $\pi : \mathcal{E} \to X$ be a bundle such that E_x^* is strictly convex for each $x \in X$; for convenience we will call π a *dual strictly convex bundle.* In Lemma 3 through Lemma 6, $T : \Gamma(\pi) \to \Gamma(\xi)$ will denote an isometric isomorphism between the section spaces of the dual strictly convex bundles $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$.

Lemma 3. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be dual strictly convex bundles. Given $y \in Y$ and $g_y \in F_y^*$, there is an $x \in X$ and $f_x \in E_x^*$ such that $T^*(j_y(g_y)) = j_x(f_x)$.*

Proof. Since T is an isometric isomorphism, so is $T^* : \Gamma(\xi)^* \to \Gamma(\pi)^*$; then $T^*(extr(\Gamma(\xi)^*) \subset extr(\Gamma(\pi)^*)$. If $g_y = 0$, we are done: choose any $x \in X$, and let $f_x = 0$. Otherwise, $g_y / \|g_y\|$ has norm 1, and hence is an extreme point of $B_{F_y^*}$. Thus, $T^*(j_y(g_y / \|g_y\|)) = 1/\|g_y\| \, T^*(i_y(g_y)) = j_x(f_x)$ for some $x \in X$ and $f_x \in extr(E_x^*)$. Multiply through by $\|g_y\|$. □

Lemma 4. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be dual strictly convex bundles. Let $y \in Y$, $g_y, g_y' \neq 0 \in E_y^*$. Denote $T^*(j_y(g_y)) = j_x(f_x)$ and $T^*(j_y(g_y')) = j_{x'}(f_{x'})$. Then $x = x'$.*

Proof. Suppose that $x \neq x'$. We have $T^*(j_y(g_y + g_y')) = T^*(j_y(g_y)) + T^*(j_y(g_y')) = j_x(f_x) + j_{x'}(f_{x'}') = j_{x''}(f_{x''}'')$ for some $x'' \in X$ and $f_{x''} \in E_{x''}^*$, by Lemma 3. If, say, $x'' = x$, then we can write $0 \in \Gamma(\pi)^*$ as a linear combination of elements from $j_{x''}(E_{x''}^*) = j_x(E_x^*)$ and $j_{x'}(E_{x'}^*)$; this is a contradiction, since spaces of the form $\{j_x(E_x^*) : x \in X\}$ are linearly independent in $\Gamma(\pi)^*$ (because $\sigma(x)$ and $\sigma(x')$ can be assigned arbitrarily). Similarly, $x'' \neq x'$, and so all three points are distinct. But then the same contradiction to linear independence occurs when we look at the equation $j_x(f_x) + j_{x'}(f'_{x'}) = j_{x''}(f''_{x''})$. Hence, $x = x'$. □

Note that, as a result of Lemmas 3 and 4, given $y \in Y$, there is a well-defined $x \in X$ such that $T^*(j_y(F_y^*)) \subset j_x(E_x^*)$. Write $x = \phi(y)$.

Lemma 5. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be dual strictly convex bundles, and let $y \in Y$ be fixed. Then the map $\psi(y)^* : F_y^* \to E_{\phi(y)}^*$, $\psi(y)^*(g_y) = p_{\phi(y)}(T^*(j_y(g_y))$, is an isometric isomorphism, and the map $\phi : Y \to X$, described above, is a bijection.*

Proof. Fix y. As defined above, we have $\psi(y)^* = p_{\phi(y)} \circ T^* \circ j_y$, where $p_{\phi(y)}$ is the retraction map mentioned in the discussion preceding Lemma 1. Since both j_y and T^* are isometries, and since $p_{\phi(y)}$ is a retract of $j_{\phi(y)}$, it follows easily that $\psi(y)^*$ is an isometry; evidently it is also linear. Note now that if $\psi(y)^*(g_y) = j_x(f_x)$ and $\psi(y')^*(g_{y'}) = j_x(f'_x)$, then $y' = y$; to see this, work with $(T^*)^{-1} : \Gamma(\pi)^* \to \Gamma(\xi)^*$ as in Lemma 3. (Note that this is tantamount to saying that $\phi : Y \to X$ is injective.) If $f_x \in E_x^*$, then there exists a $y \in Y$ such that $(T^*)^{-1}(j_x(f_x)) = j_y(g_y)$, and so we have $\psi(y)^*(g_y) = p_{\phi(y)}[T^*(j_y(g_y))] = f_x$. This shows both that $\psi(y)^*$ is surjective and that ϕ is surjective. □

It now follows that for each $y \in Y$ there is an isometric isomorphism $\psi(y) : E_{\phi(y)} \to F_y$, whose adjoint is the map $\psi(y)^*$ mentioned above. Let $g_y \in F_y$ and $\sigma \in \Gamma(\pi)$ be arbitrary. We have

$$\begin{aligned}\langle (T\sigma)(y), g_y \rangle &= \langle \sigma, T^*(j_y(g_y)) \rangle \\ &= \langle \sigma(\phi(y)), p_{\phi(y)}(T^*(j_y(g_y)) \rangle \\ &= \langle \sigma(\phi(y)), \psi(y)^*(g_y) \rangle \\ &= \langle \psi(y)[\sigma(\phi(y))], g_y \rangle ,\end{aligned}$$

i.e.,

$$(T\sigma)(y) = \psi(y)[\sigma(\phi(y))].$$

Lemma 6. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be dual strictly convex bundles, and suppose that the bundle ξ is normed by $\mathcal{H}(\xi)$. Then the map $\phi : Y \to X$ as given above is a homeomorphism.*

Proof. We have already noted that ϕ is bijective. Since X and Y are compact Hausdorff, we need only show that ϕ is continuous, and to do this it will suffice to show that whenever $y_\alpha \to y$ in Y, we have $a(x_\alpha) = a(\phi(y_\alpha)) \to a(x) = a(\phi(y))$ in $\mathbb{R}$ for any $a \in C(X)$. (It is here that our proof diverges from that of [16].)

Let $a \in C(X)$, let $\sigma \in \Gamma(\pi)$ be such that $(T\sigma)(y) \neq 0$, and choose $H \in \mathcal{H}(\xi)$ such that $\langle (T\sigma)(y), H(y) \rangle = \langle T\sigma, j_y(H(y)) \rangle \neq 0$. From earlier remarks, we have $j_{y_\alpha}(H(y_\alpha)) \to j_y(H(y))$ weak* in $\Gamma(\xi)^*$. Writing $T^*(j_y(H(y))) = j_x(f_x)$, etc., we have

$$\begin{aligned}
\langle T(a\sigma), j_{y_\alpha}(H(_{y_\alpha})) \rangle &= \langle a\sigma, T^*(j_{y_\alpha}(H(y_\alpha))) \rangle \\
&= a(x_\alpha) \langle \sigma, j_{x_\alpha}(f_{x_\alpha}) \rangle \\
&= a(x_\alpha) \langle T\sigma, j_{y_\alpha}(H(_{y_\alpha})) \rangle \\
&\to \langle T(a\sigma), j_y(H(y)) \rangle \\
&= a(x) \langle T\sigma, j_y(H(y)) \rangle ,
\end{aligned}$$

where the last equality follows as in the second row of the display. Since $\langle T\sigma, j_{y_\alpha}(H(y_\alpha)) \rangle \to \langle T\sigma, j_y(H(y)) \rangle \neq 0$, we must have $a(x_\alpha) \to a(x)$. □

In summary, we have the following, which as noted earlier is a special case of [2, Theorem 2.8]:

Proposition 7. : *Suppose that $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ are dual strictly convex bundles of Banach spaces. Suppose also that ξ is normed by $\mathcal{H}(\xi)$. Let $T : \Gamma(\pi) \to \Gamma(\xi)$ be an isometric isomorphism. Then there exist a homeomorphism $\phi : Y \to X$ and isometric isomorphisms $\psi(y) : E_{\phi(y)} \to F_y$ such that for each $\sigma \in \Gamma(\pi)$ we have*

$$(T\sigma)(y) = \psi(y)[\sigma(\phi(y))].$$

Moreover, the map $\psi : Y \to \dot{\bigcup} \{I(E_{\phi(y)}, F_y) : y \in Y\}$ is continuous in the weak topology generated by Y, ϕ, and $\Gamma(\pi)$.

In the second case we consider, we can eliminate restrictions on the geometry of the duals of the fibers of the bundles. To do this, we will make an assumption about the nature of the isometric isomorphism $T : \Gamma(\pi) \to \Gamma(\xi)$. Following [7], we will say that an operator $T : \Gamma(\pi) \to \Gamma(\xi)$ is *disjointness-preserving* provided that whenever $\sigma, \tau \in \Gamma(\pi)$ with $\|\sigma(x)\| \, \|\tau(x)\| = 0$ for all $x \in X$, then $\|(T\sigma)(y)\| \, \|(T\tau)(y)\| = 0$ for all $y \in Y$. (Disjointness-preserving operators in the context of vector lattices have been studied, for instance, in [1].)

Lemma 8. *Suppose that $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ are Banach bundles over the compact Hausdorff spaces X and Y, and let $T : \Gamma(\pi) \to \Gamma(\xi)$ be a disjointness-preserving isometric isomorphism. Let $y \in Y$. Then there exists a well-defined $x = \phi(y) \in X$ such that $T^*(j_y(extr(F_y^*))) \subset j_x(extr(E_x^*))$.*

Proof. As before, T^* takes $extr(\Gamma(\xi)^*)$ into $extr(\Gamma(\pi)^*)$. Thus, for $y \in Y$ and $g_y \in extr(F_y^*)$, there exist $x \in X$ and $f_x \in extr(E_x^*)$ such that $T^*(j_y(g_y)) = j_x(f_x)$. Suppose now that $T^*(j_y(g_y)) = j_x(f_x)$ and that $T^*(j_y(g_y')) = j_{x'}(f_{x'}')$, where $g_y, g_y' \in extr(F_y^*)$. We claim that $x = x'$.

If not, there are disjoint open neighborhoods V and V' of x and x', respectively, and sections $\sigma, \tau \in \Gamma(\pi)$, with arbitrarily chosen values at x and

x', respectively, such that σ is supported on V and τ is supported on V'. Since σ and τ have disjoint supports, so do $T\sigma$ and $T\tau$. Then we have

$$\begin{aligned}\left\langle T(\sigma+\tau), j_y(g_y)+j_y(g_y')\right\rangle &= \left\langle \sigma+\tau, T^*(j_y(g_y)+j_y(g_y'))\right\rangle \\ &= \left\langle \sigma+\tau,\ j_x(f_x)+j_{x'}(f_{x'}')\right\rangle \\ &= \langle \sigma(x), f_x\rangle + \left\langle \sigma(x'), f_{x'}'\right\rangle \\ &\quad + \langle \tau(x), f_x\rangle + \left\langle \tau(x'), f_{x'}'\right\rangle \\ &= \langle \sigma(x), f_x\rangle + \left\langle \tau(x'), f_{x'}'\right\rangle,\end{aligned}$$

because of our assumptions about the supports of σ and τ. On the other hand,

$$\begin{aligned}\left\langle T(\sigma+\tau), j_y(g_y)+j_y(g_y')\right\rangle &= \left\langle (T\sigma)(y)+(T\tau)(y), g_y+g_y'\right\rangle \\ &= \left\langle (T\sigma)(y), g_y+g_y'\right\rangle \text{ (say)} \\ &= \left\langle \sigma, T^*(j_y(g_y))+T^*(j_y(g_y'))\right\rangle \\ &= \langle \sigma, j_x(f_x)+j_{x'}(f_{x'}')\rangle \\ &= \langle \sigma(x), f_x\rangle + \langle \sigma(x'), f_{x'}'\rangle \\ &= \langle \sigma(x), f_x\rangle,\end{aligned}$$

again using the disjointness of the supports of σ and τ, and hence of $T\sigma$ and $T\tau$. It follows that

$$\left\langle \tau(x'), f_{x'}'\right\rangle = 0.$$

But since $\tau(x')$ can be chosen arbitrarily, this forces $f_{x'}' = 0$, a contradiction, since $f_{x'}' \in extr(E_{x'}^*)$. □

Lemma 9. *Let $T : \Gamma(\pi) \to \Gamma(\xi)$ be a disjointness-preserving isometric isomorphism. Then the map $\phi : Y \to X$ established in Lemma 8 is a bijection. Moreover, for each $y \in Y$ there is an isometric isomorphism of $E_{\phi(y)}$ and F_y.*

Proof. To show that $\phi : Y \to X$ is a bijection, we can work with $(T^*)^{-1}$ to establish that there is a map $\phi' : X \to Y$ such that $(T^*)^{-1}(j_x(extr(E_x^*))) \subset extr(F_{\phi'(x)}^*)$, and such that $\phi' \circ \phi$ is the identity on Y; this is the same argument as used in Lemma 5. It is easy to check that, say, $j_y(F_y^*)$ is weak* closed in $\Gamma(\xi)^*$ for each $y \in Y$; this, together with the fact that the weak* closed span of $extr(F_y^*)$ is all of F_y^* is enough to show that the norm-continuous map T^* takes $j_y(F_y^*)$ to $j_{\phi(y)}(E_{\phi(y)}^*)$ isometrically; $(T^*)^{-1}$ does likewise. This establishes an isometric isomorphism $\psi(y)^* : F_y^* \to E_{\phi(y)}^*$, and hence an isometric isomorphism $\psi(y) : E_{\phi(y)} \to F_y$. □

It then follows, as in the earlier discussion, that for each $\sigma \in \Gamma(\pi)$, we have

$$(T\sigma)(y) = \psi(y)[\sigma(\phi(y))].$$

If we now add the condition that ξ be normed by $\mathcal{H}(\xi)$, we obtain the analogue to Lemma 6.

Lemma 10. *Suppose that $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ are bundles of Banach spaces, and that ξ is normed by $\mathcal{H}(\xi)$. Suppose that $T : \Gamma(\pi) \to \Gamma(\xi)$ is a disjointness-preserving isometric isomorphism, and let $\phi : Y \to X$ be the bijection established in Lemma 9. Then ϕ is a homeomorphism.*

Proof. This follows in exactly the same fashion as in Lemma 6. □

Thus, we can summarize our results on disjointness-preserving operators.

Proposition 11. *Suppose that $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ are continuous bundles of real Banach spaces, and that ξ is normed by $\mathcal{H}(\xi)$. Suppose that $T : \Gamma(\pi) \to \Gamma(\xi)$ is a disjointness-preserving isometric isomorphism. Then there is a homeomorphism $\phi : Y \to X$ and a map $\psi : Y \to \dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$, the disjoint union of the spaces of isometric isomorphisms from $E_{\phi(y)}$ to F_y, such that for each $\sigma \in \Gamma(\pi)$ and $y \in Y$ we have*

$$(T\sigma)(y) = \psi(y)[\sigma(\phi(y))].$$

Moreover, the map ψ is continuous in the weak topology generated by Y, ϕ, and $\Gamma(\pi)$.

Proof. Exactly as before. □

We note that in both cases under consideration, which are summarized in Propositions 7 and 11, the relevant isometries T are represented as weighted composition operators. In particular, the case when T is a disjointness-preserving operator has precursors going back to [1].

In both Propositions 7 and 11, the weak topology on $\dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$ determined by Y, ϕ, and $\Gamma(\pi)$ has the property that, for fixed $y \in Y$, the relative topology on $I(E_{\phi(y)}, F_y)$ is the strong operator topology. In the case where both $\pi : X \times E \to X$ and $\xi : Y \times F \to Y$ are trivial bundles, with E^* and F^* both strictly convex, and section spaces isometrically isomorphic to $C(X, E)$ and $C(X, F)$, we not surprisingly obtain the result of [16].

Note that in Lemmas 6 and 10, we make no assumption on whether π is normed by $\mathcal{H}(\pi)$. It turns out that this follows automatically once we can establish the existence of the homeomorphism $\phi : Y \to X$ and the map $\psi : Y \to \dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$.

Proposition 12. *Suppose that $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ are continuous real bundles, and suppose that ξ is normed (strongly normed) by $\mathcal{H}(\xi)$. Sup-*

pose also that there exist an isometric isomorphism $T : \Gamma(\pi) \to \Gamma(\xi)$, *a homeomorphism* $\phi : Y \to X$, *and a map* $\psi : Y \to \dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$, *continuous in the weak topology generated by* Y, ϕ, *and* $\Gamma(\pi)$ *such that* $(T\sigma)(y) = \psi(y)[\sigma(\phi(y))]$ *for each* $\sigma \in \Gamma(\pi)$. *Then* $\mathcal{H}(\pi)$ *is isometrically isomorphic to* $\mathcal{H}(\xi)$, *and* π *is normed (strongly normed) by* $\mathcal{H}(\pi)$.

Proof. Let $H \in \mathcal{H}(\xi)$. Let $\sigma \in \Gamma(\pi)$, and define a choice function $\lambda(H) : X \to \dot{\bigcup}\{E_x^* : x \in X\}$ by $\langle \sigma(x), \lambda(H)(x)\rangle = \left\langle (T\sigma)\left(\phi^{-1}(x)\right), H(\phi^{-1}(x))\right\rangle = \langle \sigma(x), \psi(y)^*(j_y(H(y)))\rangle$, where $y = \phi^{-1}(x)$. Since ϕ^{-1} is continuous, and since $y \mapsto \langle (T\sigma)(y), H(y)\rangle$ is continuous on Y, it follows that $x \mapsto \langle \sigma(x), \lambda(H)(x)\rangle$ is continuous on X. Since σ was arbitrary, we have $\lambda(H) \in \mathcal{H}(\pi)$. Since E_x is isometrically isomorphic to $F_{\phi^{-1}(x)}$, it follows that $\left\|H(\phi^{-1}(x))\right\| = \|\lambda(H)(x)\|$, and hence that $\|\lambda(H)\| = \|H\|$. Evidently, $H \mapsto \lambda(H)$ is linear, and it is straightforward to check that λ is injective. If we then define, for $H' \in \mathcal{H}(\pi)$, $\mu(H') : Y \to \dot{\bigcup}\{F_y^* : y \in Y\}$ in an analogous fashion, we see that $\mu \circ \lambda$ is the identity on $\mathcal{H}(\xi)$. Finally, if $z \in E_x \subset \mathcal{E}$, and if we choose $\sigma \in \Gamma(\pi)$ such that $\sigma(x) = z$, then

$$\begin{aligned}
\|z\| = \|\sigma(x)\| &= \|\sigma(\phi(y))\| = \|\psi(y)[\sigma(\phi(y))]\| \\
&= \|(T\sigma)(y)\| \\
&= \sup\{\left|\left\langle (T\sigma)(\phi^{-1}(x)), H(\phi^{-1}(x))\right\rangle\right| : H \in \mathcal{H}(\xi), \|H\| \le 1\} \\
&= \sup\{|\langle \sigma(x), \lambda(H)(x)\rangle| : \lambda(H) \in \mathcal{H}(\pi), \|\lambda(H)\| \le 1\},
\end{aligned}$$

so that $\mathcal{H}(\pi)$ is norming. To show that $\mathcal{H}(\pi)$ is strongly norming if $\mathcal{H}(\xi)$ is strongly norming, we need only note the isometric isomorphisms of the $E_{\phi(y)}$ and F_y and apply the definition of strongly norming. □

Given the existence of the maps ϕ and ψ as in the statement of Proposition 12, we can also establish one more relationship. Recall from [10] that the space $\Gamma_w(\pi)$ of weakly continuous sections of the continuously normed bundle $\pi : \mathcal{E} \to X$ is defined by $\sigma \in \Gamma_w(\pi)$ if and only if $\sigma : X \to \dot{\bigcup}\{E_x : x \in X\}$ is a choice function such that the function $x \mapsto \langle \sigma(x), H(x)\rangle$ is in $C(X)$ for all $H \in \mathcal{H}(\pi)$. It is shown in [15] that $\Gamma_w(\pi)$ is a Banach space under the sup-norm. We then have

Proposition 13. *Let* $\pi : \mathcal{E} \to X$ *and* $\xi : \mathcal{F} \to Y$ *be bundles satisfying the conditions of Proposition 12. Then* $\Gamma_w(\pi)$ *is isometrically isomorphic to* $\Gamma_w(\xi)$.

Proof. It is enough to recall from Proposition 4 of [15] that $\Gamma(\pi)$ is dense in $\Gamma_w(\pi)$ in the topology defined by $\tau_\alpha \to \tau$ if and only if the functions $x \mapsto \langle \tau_\alpha(x), H(x)\rangle$ converge uniformly on X to the function $x \mapsto \langle \tau(x), H(x)\rangle$ for all $H \in \mathcal{H}(\pi)$. □

That is, under the conditions of Proposition 12, an isometric isomorphism of $\Gamma(\pi)$ and $\Gamma(\xi)$ induces an isometric isomorphism of the corresponding spaces of weakly continuous sections.

Propositions 12 and 13 yield two immediate corollaries in summary.

Corollary 14. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be dual strictly convex bundles such that ξ is normed by $\mathcal{H}(\xi)$, and suppose that there exists some isometric isomorphism $T : \Gamma(\pi) \to \Gamma(\xi)$. Then π is normed by $\mathcal{H}(\pi)$, $\mathcal{H}(\pi)$ and $\mathcal{H}(\xi)$ are isometrically isomorphic, and $\Gamma_w(\pi)$ and $\Gamma_w(\xi)$ are isometrically isomorphic.*

Corollary 15. *Suppose that $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ are bundles such that ξ is normed by $\mathcal{H}(\xi)$, and suppose that there exists some disjointness-preserving isometric isomorphism $T : \Gamma(\pi) \to \Gamma(\xi)$. Then π is normed by $\mathcal{H}(\pi)$, $\mathcal{H}(\pi)$ and $\mathcal{H}(\xi)$ are isometrically isomorphic, and $\Gamma_w(\pi)$ and $\Gamma_w(\xi)$ are isometrically isomorphic.*

Finally, there is a weaker topology that we can put on $\dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$ when ξ is normed by $\mathcal{H}(\xi)$, namely the topology generated by $Y, \phi, \Gamma(\pi)$, and $\mathcal{H}(\xi)$. In this topology, we have $S_{y_\alpha} \to S_y$ in $\dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$ if and only if $y_\alpha \to y$ and $\langle S_{y_\alpha}(\sigma(\phi(y_\alpha))), H(y_\alpha)\rangle \to \langle S_y(\sigma(\phi(y))), H(y)\rangle$ in $\mathbb{R}$ for each $\sigma \in \Gamma(\pi)$ and $H \in \mathcal{H}(\xi)$. As in Proposition 1, it is then straightforward to describe general sufficient conditions for $\Gamma_w(\pi)$ and $\Gamma_w(\xi)$ to be isometrically isomorphic.

Proposition 16. *Let $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$ be any continuous real bundles which are normed by $\mathcal{H}(\pi)$ and $\mathcal{H}(\xi)$. Suppose also that $T : \Gamma(\pi) \to \Gamma(\xi)$ and $S : \mathcal{H}(\pi) \to \mathcal{H}(\xi)$ are isometric isomorphisms and that $\phi : Y \to X$ is a homeomorphism. Assume further that $\psi : Y \to \dot{\bigcup}\{I(E_{\phi(y)}, F_y) : y \in Y\}$ is a choice function which is continuous with respect to the topology generated by $Y, \phi, \Gamma(\pi)$, and $\mathcal{H}(\xi)$, and that for all $\sigma \in \Gamma(\pi), H \in \mathcal{H}(\pi)$, and $y \in Y$ we have $\langle (T\sigma)(y), S(H)(y)\rangle = \langle \sigma(\phi(y)), H(\phi(y))\rangle$. Then there is an isometric isomorphism $T' : \Gamma_w(\pi) \to \Gamma_w(\xi)$ such that*

$$(T'\sigma)(y) = \psi(y)[\sigma(\phi(y))]$$

for each $\sigma \in \Gamma_w(\pi)$.

Now, if we are given normed (or strongly normed) bundles $\pi : \mathcal{E} \to X$ and $\xi : \mathcal{F} \to Y$, and an isometric isomorphism $T' : \Gamma_w(\pi) \to \Gamma_w(\xi)$, it would be pleasant to arrive at results, even in special cases, which serve as converses. We are unable to do this; the difficulties stem from not being able to completely identify the extreme points in, say, $\Gamma_w(\pi)^*$.

Acknowledgement. The authors wish to thank the referee for the careful reading, valuable comments and suggestions, which resulted in improvements to the paper.

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