Uniform factorization for compact sets of operators acting from a Banach space to its dual space

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ABSTRACT. Let $X$ be a Banach space. We prove a uniform factorization result that describes the factorization of compact sets of compact and weakly compact operators acting from $X$ to $X^*$ via Hölder continuous homeomorphisms having Lipschitz continuous inverses. This yields a similar factorization result for compact sets of 2-homogeneous polynomials.

1. Introduction and the main result

Let $X$ and $Y$ be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from $X$ to $Y$ and by $\mathcal{K}(X, Y)$ and $\mathcal{W}(X, Y)$ its subspaces of compact and weakly compact operators.

Our main result relies on the following isometric version of the famous Davis–Figiel–Johnson–Pełczyński factorization construction [DFJP] due to Lima, Nygaard, and Oja [LNO].

Let $a$ be the unique solution of the equation

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2} = 1, \; a > 1.$$ 

Let $X$ be a Banach space and let $K$ be a closed absolutely convex subset of $B_X$, the closed unit ball of $X$. For each $n \in \mathbb{N}$, put $B_n = a^{n/2} K + a^{-n/2} B_X$.

Received December 6, 2005.
2000 Mathematics Subject Classification. Primary 46B04, 46B20, 46B28, 46B50, 47A68, 47B07; Secondary 46B25.

Key words and phrases. Banach spaces, compact subsets of weakly compact operators, uniform factorization, uniform compact factorization, 2-homogeneous polynomials.
This research was partially supported by Estonian Science Foundation Grant 5704.
The gauge of \( B_N \) gives an equivalent norm \( \| \cdot \|_n \) on \( X \). Set
\[
\|x\|_K = \left( \sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2},
\]
define \( X_K = \{ x \in X : \|x\|_K < \infty \} \), and let \( J_K : X_K \to X \) denote the identity embedding. Then \( X_K = (X_K, \| \cdot \|_K) \) is a Banach space and \( \|J_K\| \leq 1 \). Moreover \( X_K \) is reflexive if and only if \( K \) is weakly compact, and \( J_K \) is compact if and only if \( K \) is compact; in this case \( X_K \) is separable.

For a Banach space \( X \), let us consider the following infinite direct sums in the sense of \( \ell_2 \):\[
W_X = \left( \sum_K (X^*)_K^* \right)_2 \quad \text{and} \quad Z_X = \left( \sum_L (X^*)_L \right)_2,
\]
where \( K \) and \( L \) run, respectively, through the weakly compact and compact absolutely convex subsets of \( B_X \). The spaces \( W_X \) and \( Z_X \) are reflexive. In Theorem 1 below, which is our main result, they will, respectively, serve as universal factorization spaces for all compact sets of the spaces \( \mathcal{W}(X, X^*) \) and \( \mathcal{K}(X, X^*) \).

**Theorem 1.** Let \( X \) be a Banach space. Let \( W = W_X \) and \( Z = Z_X \). Then, for every compact subset \( C \) of \( \mathcal{W}(X, X^*) \), there exist norm one operators \( u, v \in \mathcal{W}(W, W) \), and a linear mapping \( \Phi : \text{span } C \to \mathcal{W}(W, W^*) \) which preserves finite rank and compact operators such that \( S = v^* \circ \Phi(S) \circ u \), for all \( S \in \text{span } C \). The mapping \( \Phi \) restricted to \( C \cup \{ 0 \} \) is a homeomorphism satisfying
\[
\|S - T\| \leq \|\Phi(S) - \Phi(T)\| \leq \min \left\{ d, d^{3/4} \left( \frac{1}{4} + \frac{1}{\ln a} \right)^{3/4} \|S - T\|^{1/4} \right\}, \quad S, T \in C \cup \{0\},
\]
where \( d = \text{diam } C \cup \{0\} \). In particular, if \( -S \in C \) for some \( S \in C \), then
\[
\|\Phi(S)\| \leq \min \left\{ \frac{d}{2}, \left( \frac{d}{2} \right)^{3/4} \left( \frac{1}{4} + \frac{1}{\ln a} \right)^{3/4} \|S\|^{1/4} \right\}.
\]
Moreover, if \( C \) is contained in \( \mathcal{K}(X, X^*) \), then \( W \) is everywhere replaced by \( Z \), and \( u \) and \( v \) are compact operators.

**Remark 1.** By [LNO] a “good” estimate of \( a \) is \( \exp(4/9) \). Hence
\[
\frac{1}{4} + \frac{1}{\ln a} \approx \frac{5}{2}.
\]

**Remark 2.** Observe that \( \text{diam } \Phi(C \cup \{0\}) = \text{diam } C \cup \{0\} \) in Theorem 1.

The proof of Theorem 1 is contained in Section 2. It uses techniques from the recent paper [MO]. In Section 3, Theorem 1 is applied to get a similar factorization result for compact sets of 2-homogeneous polynomials.
Our notation is rather standard. A Banach space $X$ will always be regarded as a subspace of its bidual $X^{**}$ under the canonical embedding. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The linear span of $A$ is denoted by $\text{span} \ A$.

2. Proof of Theorem 1

The proof of Theorem 1 uses Lemmas 2 and 3 below. These lemmas are, respectively, immediate consequences of Lemmas 4 and 5 of [MO] (because $W(X, Y)$ and $K(X, Y)$ are canonically isometrically isomorphic (under the mapping $S \to S^{**}$) with the subspaces of the weak*-weak continuous operators of $W(X^{**}, Y)$ and $K(X^{**}, Y)$, respectively).

**Lemma 2.** Let $X$ and $Y$ be Banach spaces. Let $C$ be a compact subset of $W(Y, X^*)$. Then there exist a weakly compact absolutely convex subset $K$ of $B_{X^*}$, which is compact whenever $C$ is contained in $K(Y, X^*)$, and a linear mapping $\Phi : \text{span} \ C \to W(Y, (X^*)_K)$ such that $S = J_K \circ \Phi(S)$, for all $S \in \text{span} \ C$, and $\|J_K\| = 1$. Moreover, if $S \in \text{span} \ C$, then

(i) $S$ has finite rank if and only if $\Phi(S)$ has finite rank,

(ii) $S$ is compact if and only if $\Phi(S)$ is compact.

The mapping $\Phi$ restricted to $C \cup \{0\}$ is a homeomorphism satisfying

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\| \leq \min \left\{d, q^{1/2} \left(\frac{1}{4} + \frac{1}{\ln a}\right)^{1/2} \|S - T\|^{1/2}\right\}, \ S, T \in C \cup \{0\},$$

where $d = \text{diam} \ C \cup \{0\}$; in particular, if $-S \in C$ for some $S \in C$, then

$$\|\Phi(S)\| \leq \min \left\{\frac{d}{2}, \left(\frac{d}{2}\right)^{1/2} \left(\frac{1}{4} + \frac{1}{\ln a}\right)^{1/2} \|S\|^{1/2}\right\}.$$

**Lemma 3.** Let $X$ be a Banach space. Let $C$ be a compact subset of $W(X, X^*)$. Then there exist a weakly compact absolutely convex subset $K$ of $B_{X^*}$, a norm one operator $J \in W(X, (X^*)_K)$, and a linear mapping $\Phi : \text{span} \ C \to W((X^*)_K, X^*)$ satisfying conditions (i) and (ii) of Lemma 2 such that $S = \Phi(S) \circ J$, for all $S \in \text{span} \ C$. Moreover, if $C$ is contained in $K(X, X^*)$, then $K$ is compact and $J \in K(X, (X^*)_K)$. The mapping $\Phi$ restricted to $C \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 2.

**Proof of Theorem 1.** Let $K \subset B_{X^*}$, $J \in W(X, (X^*)_K)$, and $\varphi : \text{span} \ C \to W((X^*)_K, X^*)$, respectively, be the weakly compact absolutely convex subset, the norm one operator, and the linear mapping given by Lemma 3.
Since $\varphi(C)$ is a compact subset of $W((X^*)_K^*, X^*)$, we can apply Lemma 2. Let $L \subset B_{X^*}$ and $\psi: \text{span } \varphi(C) \to W((X^*)_K^*, (X^*)_L^*)$, respectively, be the weakly compact subset and the linear mapping given by Lemma 2.

Let $I_K : (X^*)_K^* \to W$ and $I_L : (X^*)_L^* \to W$ denote the natural norm one embeddings, and let $P_K : W \to (X^*)_K^*$ and $P_L : W \to (X^*)_L^*$ denote the natural norm one projections. It is straightforward to verify (observing that $\text{diam } \varphi(C \cup \{0\}) = d$) that the mappings $u = I_K \circ J, \Phi, v = I_L \circ J^*_L|_X$, and $\Phi$ defined by $\Phi(S) = P_L^* \circ (\varphi(S)) \circ P_K, \ S \in \text{span } C,$ have desired properties. In particular, for all $S \in \text{span } C,$

\[
S = \varphi(S) \circ J = J_L \circ \psi(\varphi(S)) \circ J = J_L \circ (P_L \circ I_L)^* \circ \psi(\varphi(S)) \circ P_K \circ I_K \circ J = J_L \circ I_L^* \circ P_L^* \circ \psi(\varphi(S)) \circ P_K \circ u = u^* \circ \Phi(S) \circ u,
\]

and therefore

$$||S - T|| \leq ||\Phi(S) - \Phi(T)||, \ S, T \in C \cup \{0\}.$$ 

The “moreover” part uses that $\varphi$ and $\psi$ preserve compact operators. It also uses that $K$ is a compact set and $J \in K(X, (X^*)_K^*)$ whenever $C \in K(X, X^*)$ (see Lemma 3) and that, in this case, $\varphi(C)$ is a compact subset of $K((X^*)_K^*, X^*)$, implying (see Lemma 2) the compacity of the set $L$ and operator $J_L$.

\[\square\]

3. Uniform factorization for compact sets of 2-homogeneous polynomials

Let us point out the following immediate consequence of Theorem 1.

Corollary 4. Let $X$ be a Banach space and let $Z = Z_X$. Then, for every compact subset $C$ of $K(X, X^*)$, there exist norm one operators $u, v \in K(X, Z)$ and a linear mapping $\Phi: \text{span } C \to K(Z, Z^*)$ such that $S = u^* \circ \Phi(S) \circ u$, for all $S \in \text{span } C$. The mapping $\Phi$ restricted to $C \cup \{0\}$ is a homeomorphism satisfying the conclusions of Theorem 1.

We will apply Corollary 4 to 2-homogeneous polynomials on a Banach space $X$.

Let $L^r(X)$ denote the Banach space of all continuous $r$-linear forms on $X$ and let $L^s(X)$ denote its subspace of symmetric $r$-linear forms. Denote by $s: L^r(X) \to L^s(X)$ the symmetrization operator and recall that $s$ is a linear norm one projection onto $L^s(X)$.

Let $P^r(X)$ denote the Banach space of all continuous $r$-homogeneous polynomials on $X$. Then for each $P \in P^r(X)$ there is a unique $A_P \in$
$\mathcal{L}^s(nX)$ satisfying $P(x) = A_P(x, \ldots, x)$ for each $x \in X$. The correspondence $P \rightarrow A_P$ is an isomorphism between $\mathcal{P}(nX)$ and $\mathcal{L}^s(nX)$ satisfying

$$\|P\| \leq \|A_P\| \leq \frac{n^3}{n!} \|P\|, \ P \in \mathcal{P}(nX),$$

(see, e.g., [D, p. 5, Corollary 1.6 and Theorem 1.7]).

Recall that $P \in \mathcal{P}(nX)$ is weakly uniformly continuous on the closed unit ball $B_X$ of $X$ if for each $\epsilon > 0$ there are $x_1^*, \ldots, x_n^* \in X^*$ and $\delta > 0$ such that if $x, y \in B_X$, $|x_i^*(x - y)| < \delta$ for $i = 1, \ldots, n$, then $|P(x) - P(y)| < \epsilon$.

Let $\mathcal{P}_{wu}(nX)$ denote the subspace of $\mathcal{P}(nX)$ consisting of the polynomials that are weakly uniformly continuous on $B_X$. The corresponding subspace of $\mathcal{L}^s(nX)$ is denoted by $\mathcal{L}_{wu}^s(nX)$. Notice that $\mathcal{P}_{wu}(nX)$, with the norm induced from $\mathcal{P}(nX)$, is a Banach space (see [AP]).

For each $P \in \mathcal{P}(nX)$ there is a linear operator $T_P : X \rightarrow \mathcal{L}^s(n-1X)$ defined by $(T_Px_1)(x_2, \ldots, x_n) = A_P(x_1, x_2, \ldots, x_n)$. Clearly, the correspondence $A_P \rightarrow T_P$ is a linear isometry. According to [AP], $P \in \mathcal{P}_{wu}(nX)$ if and only if $T_P \in \mathcal{K}(X, \mathcal{L}^s(n-1X))$. Moreover, if $P \in \mathcal{P}_{wu}(nX)$, then $T_P \in \mathcal{K}(X, \mathcal{L}_{wu}^s(n-1X))$.

In this paper we shall be interested in the case $n = 2$. In this case $\mathcal{L}_{wu}^s(2X) = \mathcal{L}^s(2X) = X^*$ and therefore $\mathcal{K}(X, \mathcal{L}_{wu}^s(2X)) = \mathcal{K}(X, \mathcal{L}^s(2X)) = \mathcal{K}(X, X^*)$. This enables us to apply Corollary 4 to get the following uniform factorization result for compact sets of 2-homogeneous polynomials. Recall that

$$s(A)(x_1, x_2) = \frac{1}{2} \left(A(x_1, x_2) + A(x_2, x_1)\right), \ x_1, x_2 \in X, \ A \in \mathcal{L}^2(X).$$

**Theorem 5.** Let $X$ be a Banach space and let $Z = Z_X$. Then, for every compact subset $C$ of $\mathcal{P}_{wu}(2X)$, there exist norm one operators $u, v \in \mathcal{K}(X, Z)$, and linear mappings $\Psi : \text{span } C \rightarrow \mathcal{P}_{wu}(2Z)$ and $\psi : \text{span } C \rightarrow \mathcal{L}^2(Z)$ such that, for all $P \in \text{span } C$,

$$P(x) = \psi(P)(ux, vx), \ x \in X,$$

and

$$s(\psi(P)) = A\psi(P),$$

The mappings $\Psi$ and $\psi$ restricted to $C \cup \{0\}$ satisfy

$$\max\left\{\|P - Q\|, \|\Psi(P) - \Psi(Q)\|\right\} \leq \|\psi(P) - \psi(Q)\| \leq 2 \min\left\{d, d^{3/4}\left(\frac{1}{4} + \frac{1}{\ln a}\right)^{3/4}\|P - Q\|^{1/4}\right\}, \ P, Q \in C \cup \{0\},$$

where $d = \text{diam } C \cup \{0\}$. In particular, if $-P \in C$ for some $P \in C$, then

$$\|\Psi(P)\| \leq \|\psi(P)\| \leq \min\left\{d, d^{1/4}d^{3/4}\left(\frac{1}{4} + \frac{1}{\ln a}\right)^{3/4}\|P\|^{1/4}\right\}.$$
Proof. Let $\mathcal{C}$ be a compact subset of $\mathcal{P}_{wu}(\mathcal{Z})$. Then

$$\mathcal{K} := \{T_P : P \in \mathcal{C}\} \subset \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$$

The set $\mathcal{K}$ is compact because the correspondence $P \to A_P \to T_P$ is continuous. Notice that

$$\text{diam } \mathcal{K} \cup \{0\} \leq 2d$$

because $\|T_P - T_Q\| = \|A_P - A_Q\| \leq 2\|P - Q\|$ for all $P, Q \in \mathcal{P}(\mathcal{Z})$.

Applying Corollary 4 to the compact subset $\mathcal{K} \subset \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$, there are norm one operators $u, v \in \mathcal{K}(\mathcal{Z}, \mathcal{Z})$ and a linear mapping $\Phi : \text{span } \mathcal{K} \to \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$ such that $T_P = v^* \circ \Phi(T_P) \circ u$, for all $T_P \in \mathcal{K}$. Now, $\Phi(T_P) \in \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$, but $\Phi(T_P)$ need not be of the form $T_Q$ for some $Q \in \mathcal{P}(\mathcal{Z})$. Let us therefore consider the mapping $\sigma \in \mathcal{L}(\mathcal{K}(\mathcal{Z}, \mathcal{Z}^*), \mathcal{L}^1(\mathcal{Z}))$ defined by

$$\sigma(S)(z_1, z_2) = \frac{1}{2} \left( (Sz_1)(z_2) + (Sz_2)(z_1) \right), \quad S \in \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*), \quad z_1, z_2 \in \mathcal{Z}.$$

Observe that, in fact, $\sigma(S) \in \mathcal{L}_{wu}^1(\mathcal{Z})$ for all $S \in \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$. Indeed, let $S \in \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$. Then $\sigma(S) = A_Q$ for some $Q \in \mathcal{P}(\mathcal{Z})$. Since

$$(\sigma(S))(z_1, z_2) = \frac{1}{2} \left( (Sz_1)(z_2) + (Sz_2)(z_1) \right), \quad z_1, z_2 \in \mathcal{Z},$$

we have $T_Q = (S + S^*)/2$. Hence $T_Q \in \mathcal{K}(\mathcal{Z}, \mathcal{Z}^*)$ and therefore $Q \in \mathcal{P}_{wu}(\mathcal{Z})$ meaning that $\sigma(S) \in \mathcal{L}_{wu}^1(\mathcal{Z})$.

This permits us to define a linear mapping $\Psi : \text{span } \mathcal{C} \to \mathcal{P}_{wu}(\mathcal{Z})$ by

$$\Psi(P)(z) = \sigma(\Phi(T_P))(z, z), \quad z \in \mathcal{Z}, \quad P \in \text{span } \mathcal{C},$$

meaning that

$$A_{\Psi(P)} = \sigma(\Phi(T_P)), \quad P \in \text{span } \mathcal{C}.$$

We also define a linear mapping $\psi : \text{span } \mathcal{C} \to \mathcal{L}(\mathcal{Z})$ by

$$\psi(P)(z_1, z_2) = (\Phi(T_P)z_1)(z_2), \quad z_1, z_2 \in \mathcal{Z}, \quad P \in \text{span } \mathcal{C}.$$

Let now $P \in \text{span } \mathcal{C}$. We have for all $x \in \mathcal{X}$

$$P(x) = (T_Px)(x) = (v^*\Phi(T_P)ux)(x) = \psi(P)(ux, vx)$$

and we have for all $z_1, z_2 \in \mathcal{Z}$

$$s(\psi(P))(z_1, z_2) = \frac{1}{2} \left( \psi(P)(z_1, z_2) + \psi(P)(z_2, z_1) \right) = \sigma(\Phi(T_P))(z_1, z_2) = A_{\Psi(P)}(z_1, z_2).$$

Let us finally consider the mappings $\Psi$ and $\psi$ restricted to $\mathcal{K} \cup \{0\}$. For all $P, Q \in \text{span } \mathcal{C}$, we have, since $\|u\| = \|v\| = 1$,

$$\|P - Q\| = \sup_{\|\xi\| \leq 1} \|(P - Q)(\xi)\| = \sup_{\|\xi\| \leq 1} \|(\psi(P) - \psi(Q))(ux, vx)\|$$

$$\leq \|\psi(P) - \psi(Q)\|.$$
We also have
\[
\|\Psi(P) - \Psi(Q)\| \leq \|A\psi_p(P) - A\psi_q(Q)\| = \|s(\psi(P) - \psi(Q))\| \\
\leq \|\psi(P) - \psi(Q)\|.
\]

For all \(P, Q \in \mathcal{C} \cup \{0\}\), using the definition of \(\psi\) and Corollary 4, we have
\[
\|\psi(P) - \psi(Q)\| = \|\Phi(T_P) - \Phi(T_Q)\| \\
\leq \min \left\{ 2d, 2^{3/4}d^{3/4}\left(\frac{1}{4} + \frac{1}{\ln a}\right)^{3/4}\|T_P - T_Q\|^{1/4}\right\}.
\]
Since
\[
\|T_P - T_Q\| = \|A_P - A_Q\| \leq 2\|P - Q\|,
\]
we have
\[
\|\psi(P) - \psi(Q)\| \leq \min \left\{ 2d, 2^{3/4}d^{3/4}\left(\frac{1}{4} + \frac{1}{\ln a}\right)^{3/4}\|P - Q\|^{1/4}\right\} \\
= 2 \min \left\{ d, d^{3/4}\left(\frac{1}{4} + \frac{1}{\ln a}\right)^{3/4}\|P - Q\|^{1/4}\right\}
\]
as needed.

If, in particular, \(P, -P \in \mathcal{C}\), then the desired estimate for the norm of \(\psi(P) = (\psi(P) - \psi(-P))/2\) immediately follows from the above.

\[\square\]

References


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