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Uniform factorization for compact sets of operators acting from a Banach space to its dual space

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ABSTRACT. Let X be a Banach space. We prove a uniform factorization result that describes the factorization of compact sets of compact and weakly compact operators acting from X to X^* via Hölder continuous homeomorphisms having Lipschitz continuous inverses. This yields a similar factorization result for compact sets of 2-homogeneous polynomials.

1. Introduction and the main result

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators from X to Y and by $\mathcal{K}(X,Y)$ and $\mathcal{W}(X,Y)$ its subspaces of compact and weakly compact operators.

Our main result relies on the following isometric version of the famous Davis–Figiel–Johnson–Pełczyński factorization construction [DFJP] due to Lima, Nygaard, and Oja [LNO].

Let a be the unique solution of the equation

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1, \ a > 1.$$

Let X be a Banach space and let K be a closed absolutely convex subset of B_X, the closed unit ball of X. For each $n \in \mathbb{N}$, put $B_n = a^{n/2}K + a^{-n/2}B_X$.

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The gauge of B_n gives an equivalent norm $\|\cdot\|_n$ on X. Set

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{1/2},$$

define $X_K = \{x \in X : \|x\|_K < \infty\}$, and let $J_K : X_K \to X$ denote the identity embedding. Then $X_K = (X_K, \|\cdot\|_K)$ is a Banach space and $\|J_K\| \leq 1$. Moreover X_K is reflexive if and only if K is weakly compact, and J_K is compact if and only if K is compact; in this case X_K is separable.

For a Banach space X, let us consider the following infinite direct sums in the sense of ℓ_2:

$$W_X = \Big(\sum_K (X^*)_K^*\Big)_2 \text{ and } Z_X = \Big(\sum_L (X^*)_L^*\Big)_2,$$

where K and L run, respectively, through the weakly compact and compact absolutely convex subsets of B_{X^*}. The spaces W_X and Z_X are reflexive. In Theorem 1 below, which is our main result, they will, respectively, serve as universal factorization spaces for all compact sets of the spaces $\mathcal{W}(X, X^*)$ and $\mathcal{K}(X, X^*)$.

Theorem 1. *Let X be a Banach space. Let $W = W_X$ and $Z = Z_X$. Then, for every compact subset $\mathcal{C}$ of $\mathcal{W}(X, X^*)$, there exist norm one operators $u, v \in \mathcal{W}(X, W)$, and a linear mapping Φ : span $\mathcal{C} \to \mathcal{W}(W, W^*)$ which preserves finite rank and compact operators such that $S = v^* \circ \Phi(S) \circ u$, for all $S \in$ span $\mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying*

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\|$$

$$\leq \min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S - T\|^{1/4}\Big\}, \quad S, T \in \mathcal{C} \cup \{0\},$$

where $\mathrm{d} =$ diam $\mathcal{C} \cup \{0\}$. *In particular, if $-S \in \mathcal{C}$ for some $S \in \mathcal{C}$, then*

$$\|\Phi(S)\| \leq \min\Big\{\frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S\|^{1/4}\Big\}.$$

Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X, X^*)$, then W is everywhere replaced by Z, and u and v are compact operators.

Remark 1. By [LNO] a "good" estimate of a is $\exp(4/9)$. Hence

$$\frac{1}{4} + \frac{1}{\ln a} \approx \frac{5}{2}.$$

Remark 2. Observe that diam $\Phi(\mathcal{C} \cup \{0\}) =$ diam $\mathcal{C} \cup \{0\}$ in Theorem 1.

The proof of Theorem 1 is contained in Section 2. It uses techniques from the recent paper [MO]. In Section 3, Theorem 1 is applied to get a similar factorization result for compact sets of 2-homogeneous polynomials.

Our notation is rather standard. A Banach space X will always be regarded as a subspace of its bidual X^{**} under the canonical embedding. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The linear span of A is denoted by span A.

2. Proof of Theorem 1

The proof of Theorem 1 uses Lemmas 2 and 3 below. These lemmas are, respectively, immediate consequences of Lemmas 4 and 5 of [MO] (because $\mathcal{W}(X,Y)$ and $\mathcal{K}(X,Y)$ are canonically isometrically isomorphic (under the mapping $S \to S^{**}$) with the subspaces of the weak*-weak continuous operators of $\mathcal{W}(X^{**},Y)$ and $\mathcal{K}(X^{**},Y)$, respectively).

Lemma 2. *Let X and Y be Banach spaces. Let C be a compact subset of $\mathcal{W}(Y,X^*)$. Then there exist a weakly compact absolutely convex subset K of B_{X^*}, which is compact whenever C is contained in $\mathcal{K}(Y,X^*)$, and a linear mapping $\Phi : \operatorname{span} C \to \mathcal{W}(Y,(X^*)_K)$ such that $S = J_K \circ \Phi(S)$, for all $S \in \operatorname{span} C$, and $\|J_K\| = 1$. Moreover, if $S \in \operatorname{span} C$, then*

(i) *S has finite rank if and only if $\Phi(S)$ has finite rank,*

(ii) *S is compact if and only if $\Phi(S)$ is compact.*

The mapping Φ restricted to $C \cup \{0\}$ is a homeomorphism satisfying

$$\|S-T\| \le \|\Phi(S)-\Phi(T)\|$$

$$\le \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{1/2}\|S-T\|^{1/2}\Big\},\ S,T \in C\cup\{0\},$$

where $\mathrm{d} = \operatorname{diam} C \cup \{0\}$; in particular, if $-S \in C$ for some $S \in C$, then

$$\|\Phi(S)\| \le \min\Big\{\frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{1/2}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{1/2}\|S\|^{1/2}\Big\}.$$

Lemma 3. *Let X be a Banach space. Let C be a compact subset of $\mathcal{W}(X,X^*)$. Then there exist a weakly compact absolutely convex subset K of B_{X^*}, a norm one operator $J \in \mathcal{W}(X,(X^*)^*_K)$, and a linear mapping $\Phi : \operatorname{span} C \to \mathcal{W}((X^*)^*_K, X^*)$ satisfying conditions* (i) *and* (ii) *of Lemma 2 such that $S = \Phi(S) \circ J$, for all $S \in \operatorname{span} C$. Moreover, if C is contained in $\mathcal{K}(X,X^*)$, then K is compact and $J \in \mathcal{K}(X,(X^*)^*_K)$. The mapping Φ restricted to $C \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma* 2.

Proof of Theorem 1. Let $K \subset B_{X^*}$, $J \in \mathcal{W}(X,(X^*)^*_K)$, and $\varphi : \operatorname{span} C \to \mathcal{W}((X^*)^*_K, X^*)$, respectively, be the weakly compact absolutely convex subset, the norm one operator, and the linear mapping given by Lemma 3.

Since $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{W}((X^*)^*_K, X^*)$, we can apply Lemma 2. Let $L \subset B_{X^*}$ and ψ : span $\varphi(\mathcal{C}) \to \mathcal{W}((X^*)^*_K, (X^*)_L)$, respectively, be the weakly compact subset and the linear mapping given by Lemma 2.

Let $I_K : (X^*)^*_K \to W$ and $I_L : (X^*)^*_L \to W$ denote the natural norm one embeddings, and let $P_K : W \to (X^*)^*_K$ and $P_L : W \to (X^*)^*_L$ denote the natural norm one projections. It is straightforward to verify (observing that diam $\varphi(\mathcal{C} \cup \{0\}) = \mathrm{d}$) that the mappings $u = I_K \circ J$, Φ, $v = I_L \circ J^*_L|_X$, and Φ defined by $\Phi(S) = P^*_L \circ \psi(\varphi(S)) \circ P_K$, $S \in$ span $\mathcal{C}$, have desired properties. In particular, for all $S \in$ span $\mathcal{C}$,

$$\begin{aligned} S &= \varphi(S) \circ J = J_L \circ \psi(\varphi(S)) \circ J \\ &= J_L \circ (P_L \circ I_L)^* \circ \psi(\varphi(S)) \circ P_K \circ I_K \circ J \\ &= J_L \circ I^*_L \circ P^*_L \circ \psi(\varphi(S)) \circ P_K \circ u \\ &= v^* \circ \Phi(S) \circ u, \end{aligned}$$

and therefore

$$\|S - T\| \le \|\Phi(S) - \Phi(T)\|, \quad S, T \in \mathcal{C} \cup \{0\}.$$

The "moreover" part uses that φ and ψ preserve compact operators. It also uses that K is a compact set and $J \in \mathcal{K}(X, (X^*)^*_K)$ whenever $\mathcal{C} \subset \mathcal{K}(X, X^*)$ (see Lemma 3) and that, in this case, $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{K}((X^*)^*_K, X^*)$, implying (see Lemma 2) the compacity of the set L and operator J_L.

□

3. Uniform factorization for compact sets of 2-homogeneous polynomials

Let us point out the following immediate consequence of Theorem 1.

Corollary 4. *Let X be a Banach space and let $Z = Z_X$. Then, for every compact subset $\mathcal{C}$ of $\mathcal{K}(X, X^*)$, there exist norm one operators $u, v \in \mathcal{K}(X, Z)$ and a linear mapping Φ :* span $\mathcal{C} \to \mathcal{K}(Z, Z^*)$ *such that $S = v^* \circ \Phi(S) \circ u$, for all $S \in$* span $\mathcal{C}$. *The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Theorem 1.*

We will apply Corollary 4 to 2-homogeneous polynomials on a Banach space X.

Let $\mathcal{L}(^nX)$ denote the Banach space of all continuous n-linear forms on X and let $\mathcal{L}^s(^nX)$ denote its subspace of symmetric n-linear forms. Denote by $s : \mathcal{L}(^nX) \to \mathcal{L}^s(^nX)$ the symmetrization operator and recall that s is a linear norm one projection onto $\mathcal{L}^s(^nX)$.

Let $\mathcal{P}(^nX)$ denote the Banach space of all continuous n-homogeneous polynomials on X. Then for each $P \in \mathcal{P}(^nX)$ there is a unique $A_P \in$

$\mathcal{L}^s({}^nX)$ satisfying $P(x) = A_P(x, \dots, x)$ for each $x \in X$. The correspondence $P \to A_P$ is an isomorphism between $\mathcal{P}({}^nX)$ and $\mathcal{L}^s({}^nX)$ satisfying

$$\|P\| \le \|A_P\| \le \frac{n^n}{n!}\|P\|, \ P \in \mathcal{P}({}^nX),$$

(see, e.g., [D, p. 5, Corollary 1.6 and Theorem 1.7]).

Recall that $P \in \mathcal{P}({}^nX)$ is *weakly uniformly continuous* on the closed unit ball B_X of X if for each $\epsilon > 0$ there are $x_1^*, \dots, x_n^* \in X^*$ and $\delta > 0$ such that if $x, y \in B_X$, $|x_i^*(x-y)| < \delta$ for $i = 1, \dots, n$, then $|P(x) - P(y)| < \epsilon$. Let $\mathcal{P}_{wu}({}^nX)$ denote the subspace of $\mathcal{P}({}^nX)$ consisting of the polynomials that are weakly uniformly continuous on B_X. The corresponding subspace of $\mathcal{L}^s({}^nX)$ is denoted by $\mathcal{L}^s_{wu}({}^nX)$. Notice that $\mathcal{P}_{wu}({}^nX)$, with the norm induced from $\mathcal{P}({}^nX)$, is a Banach space (see [AP]).

For each $P \in \mathcal{P}({}^nX)$ there is a linear operator $T_P : X \to \mathcal{L}^s({}^{n-1}X)$ defined by $(T_Px_1)(x_2, ..., x_n) = A_P(x_1, x_2, ..., x_n)$. Clearly, the correspondence $A_P \to T_P$ is a linear isometry. According to [AP], $P \in \mathcal{P}_{wu}({}^nX)$ if and only if $T_P \in \mathcal{K}(X, \mathcal{L}^s({}^{n-1}X))$. Moreover, if $P \in \mathcal{P}_{wu}({}^nX)$, then $T_P \in \mathcal{K}(X, \mathcal{L}^s_{wu}({}^{n-1}X))$.

In this paper we shall be interested in the case $n = 2$. In this case $\mathcal{L}^s_{wu}({}^1X) = \mathcal{L}^s({}^1X) = X^*$ and therefore $\mathcal{K}(X, \mathcal{L}^s_{wu}({}^1X)) = \mathcal{K}(X, \mathcal{L}^s({}^1X)) = \mathcal{K}(X, X^*)$. This enables us to apply Corollary 4 to get the following uniform factorization result for compact sets of 2-homogeneous polynomials. Recall that

$$s(A)(x_1, x_2) = \frac{1}{2}\Big(A(x_1, x_2) + A(x_2, x_1)\Big), \ x_1, x_2 \in X, \ A \in \mathcal{L}({}^2X).$$

Theorem 5. *Let X be a Banach space and let $Z = Z_X$. Then, for every compact subset C of $\mathcal{P}_{wu}({}^2X)$, there exist norm one operators $u, v \in \mathcal{K}(X, Z)$, and linear mappings Ψ : span $C \to \mathcal{P}_{wu}({}^2Z)$ and ψ : span $C \to \mathcal{L}({}^2Z)$ such that, for all $P \in$ span C,*

$$P(x) = \psi(P)(ux, vx), \ x \in X,$$

and

$$s(\psi(P)) = A_{\Psi(P)}.$$

The mappings Ψ and ψ restricted to $C \cup \{0\}$ satisfy

$$\max\Big\{\|P - Q\|, \|\Psi(P) - \Psi(Q)\|\Big\} \le \|\psi(P) - \psi(Q)\|$$

$$\le 2\min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|P - Q\|^{1/4}\Big\}, \ P, Q \in C \cup \{0\},$$

where $\mathrm{d} = \operatorname{diam} C \cup \{0\}$. *In particular, if $-P \in C$ for some $P \in C$, then*

$$\|\Psi(P)\| \le \|\psi(P)\| \le \min\Big\{\mathrm{d}, 2^{1/4}\mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|P\|^{1/4}\Big\}.$$

Proof. Let $\mathcal{C}$ be a compact subset of $\mathcal{P}_{wu}(^2X)$. Then

$$\mathcal{K} := \{T_P : \ P \in \mathcal{C}\} \subset \mathcal{K}(X, X^*).$$

The set $\mathcal{K}$ is compact because the correspondence $P \to A_P \to T_P$ is continuous. Notice that

$$\operatorname{diam} \mathcal{K} \cup \{0\} \leq 2\mathrm{d}$$

because $\|T_P - T_Q\| = \|A_P - A_Q\| \leq 2\|P - Q\|$ for all $P, Q \in \mathcal{P}(^2X)$.

Applying Corollary 4 to the compact subset $\mathcal{K} \subset \mathcal{K}(X, X^*)$, there are norm one operators $u, v \in \mathcal{K}(X, Z)$ and a linear mapping Φ : span $\mathcal{K} \to \mathcal{K}(Z, Z^*)$ such that $T_P = v^* \circ \Phi(T_P) \circ u$, for all $T_P \in$ span $\mathcal{K}$. Now, $\Phi(T_P) \in \mathcal{K}(Z, Z^*)$, but $\Phi(T_P)$ need not be of the form T_Q for some $Q \in \mathcal{P}(^2Z)$. Let us therefore consider the mapping $\sigma \in \mathcal{L}(\mathcal{K}(Z, Z^*), \mathcal{L}^s(^2Z))$ defined by

$$\sigma(S)(z_1, z_2) = \frac{1}{2}\Big((Sz_1)(z_2) + (Sz_2)(z_1)\Big), \ S \in \mathcal{K}(Z, Z^*), \ z_1, z_2 \in Z.$$

Observe that, in fact, $\sigma(S) \in \mathcal{L}^s_{wu}(^2Z)$ for all $S \in \mathcal{K}(Z, Z^*)$. Indeed, let $S \in \mathcal{K}(Z, Z^*)$. Then $\sigma(S) = A_Q$ for some $Q \in \mathcal{P}(^2Z)$. Since

$$(\sigma(S))(z_1, z_2) = \frac{1}{2}\Big((Sz_1)(z_2) + (S^*z_1)(z_2)\Big), \ z_1, z_2 \in Z,$$

we have $T_Q = (S + S^*)/2$. Hence $T_Q \in \mathcal{K}(Z, Z^*)$ and therefore $Q \in \mathcal{P}_{wu}(^2Z)$ meaning that $\sigma(S) \in \mathcal{L}^s_{wu}(^2Z)$.

This permits us to define a linear mapping Ψ : span $\mathcal{C} \to \mathcal{P}_{wu}(^2Z)$ by

$$\Psi(P)(z) = \sigma(\Phi(T_P))(z, z), \ z \in Z, \ P \in \text{span } \mathcal{C},$$

meaning that

$$A_{\Psi(P)} = \sigma(\Phi(T_P)), \ P \in \text{span } \mathcal{C}.$$

We also define a linear mapping ψ : span $\mathcal{C} \to \mathcal{L}(^2Z)$ by

$$\psi(P)(z_1, z_2) = (\Phi(T_P)z_1)(z_2), \quad z_1, z_2 \in Z, \ P \in \text{span } \mathcal{C}.$$

Let now $P \in$ span $\mathcal{C}$. We have for all $x \in X$

$$P(x) = (T_Px)(x) = (v^*\Phi(T_P)ux)(x) = \psi(P)(ux, vx)$$

and we have for all $z_1, z_2 \in Z$

$$\begin{aligned} s(\psi(P))(z_1, z_2) &= \frac{1}{2}\Big(\psi(P)(z_1, z_2) + \psi(P)(z_2, z_1)\Big) \\ &= \sigma(\Phi(T_P))(z_1, z_2) = A_{\Psi(P)}(z_1, z_2). \end{aligned}$$

Let us finally consider the mappings Ψ and ψ restricted to $\mathcal{C} \cup \{0\}$. For all $P, Q \in$ span $\mathcal{C}$, we have, since $\|u\| = \|v\| = 1$,

$$\begin{aligned} \|P - Q\| &= \sup_{\|x\| \leq 1} \|(P - Q)(x)\| = \sup_{\|x\| \leq 1} |(\psi(P) - \psi(Q))(ux, vx)| \\ &\leq \|\psi(P) - \psi(Q)\|. \end{aligned}$$

We also have

$$\begin{aligned}\|\Psi(P)-\Psi(Q)\| &\le \|A_{\Psi(P)}-A_{\Psi(Q)}\| = \|s(\psi(P)-\psi(Q))\| \\ &\le \|\psi(P)-\psi(Q)\|.\end{aligned}$$

For all $P, Q \in \mathcal{C} \cup \{0\}$, using the definition of ψ and Corollary 4, we have

$$\begin{aligned}\|\psi(P)-\psi(Q)\| &= \|\Phi(T_P)-\Phi(T_Q)\| \\ &\le \min\Big\{2\mathrm{d}, 2^{3/4}\mathrm{d}^{3/4}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{3/4}\|T_P-T_Q\|^{1/4}\Big\}.\end{aligned}$$

Since

$$\|T_P - T_Q\| = \|A_P - A_Q\| \le 2\|P-Q\|,$$

we have

$$\begin{aligned}\|\psi(P)-\psi(Q)\| &\le \min\Big\{2\mathrm{d}, 2^{3/4}\mathrm{d}^{3/4}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{3/4}2^{1/4}\|P-Q\|^{1/4}\Big\} \\ &= 2\min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{3/4}\|P-Q\|^{1/4}\Big\}\end{aligned}$$

as needed.

If, in particular, $P, -P \in \mathcal{C}$, then the desired estimate for the norm of $\psi(P) = (\psi(P)-\psi(-P))/2$ immediately follows from the above.

□

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