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Boundedness of superposition operators on the sequence spaces $(w_0)_p(\Phi)$

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ABSTRACT. For a solid sequence space λ and a sequence of modulus function $\Phi = (\varphi_k)$, let $\lambda(\Phi) = \{x = (x_k) \colon \Phi(x) = (\varphi_k(|x_k|)) \in \lambda\}$. Provided another sequence of moduli $\Psi = (\psi_k)$, we give neccesary and sufficent conditions for the local boundedness and boundedness of superposition operators from $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ in the case $1 \leq p, q < \infty$.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers, respectively. Let ω be the vector space of all real sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$. By the term *sequence space* we shall mean any linear subspace of ω.

A well-known classical Banach sequence space is

$$\ell_p = \left\{ x = (x_k) \in \omega \colon \|x\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \right\} \quad (1 \leq p < \infty).$$

By $(w_0)_p$ $(1 \leq p < \infty)$ we denote the space of all sequences $x = (x_k) \in \omega$ such that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0.$$

For $p = 1$ we write w_0 instead of $(w_0)_1$.

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It is known that $(w_0)_p$ can be equipped with the norm

$$\|x\|_{(w_0)_p} = \sup_{i\geq 0} \left(\frac{1}{2^i} \sum_{k=2^i}^{2^{i+1}-1} |x_k|^p \right)^{1/p}.$$

We remark that on the space $(w_0)_p$ also the norm

$$\|x\| = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}$$

is determined which is equivalent to $\|\cdot\|_{(w_0)_p}$ (see, for example, [7], p. 39). Moreover (see [9], p. 523),

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \Longleftrightarrow \lim_{i\to\infty} \frac{1}{2^i} \sum_{k=2^i}^{2^{i+1}-1} |x_k|^p = 0. \tag{1.1}$$

Let λ and μ be two sequence spaces and let $f\colon \mathbb{N}\times\mathbb{R}\to\mathbb{R}$ be a function with $f(k,0)=0$ $(k\in\mathbb{N})$. A *superposition operator* $P_f\colon \lambda\to\mu$ is defined by

$$P_f(x) = (f(k,x_k)) \in \mu \quad (x=(x_k)\in\lambda).$$

In some results we need the following conditions:

(B) the functions $f(k,\cdot)$ $(k\in\mathbb{N})$ are bounded on every bounded subset of real numbers;
(C) the functions $f(k,\cdot)$ $(k\in\mathbb{N})$ are continuous.

The local boundedness and boundedness of superposition operators on w_0 have been studied by Pluciennik [12]. The ideas presented by him are very useful for our considerations. In this paper we give necessary and sufficent conditions for local boundedness and boundedness of superposition operators on certain generalized sequence spaces defined by w_0 and a sequence of modulus functions.

Following Maddox [8] and Ruckle [13] we give

Definition. A function $\varphi\colon [0,\infty)\to[0,\infty)$ is called a *modulus function* (or, simply, a *modulus*), if

(i) $\varphi(t)=0 \quad \Leftrightarrow \quad t=0$,
(ii) $\varphi(t+u)\leq\varphi(t)+\varphi(u) \qquad (t,\ u\geq 0)$,
(iii) φ is nondecreasing,
(iv) φ is continuous from the right at 0,
(v) φ is unbounded.

From (i)–(iv) it follows that a modulus φ is continuous everywhere on $[0,\infty)$.

The sequence space λ is called *solid* if $(y_k) \in \lambda$ whenever $(x_k) \in \lambda$ and $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$.

Let $\Phi = (\varphi_k)$ be a sequence of moduli. For a solid sequence space λ we consider the solid sequence space (cf. [4]–[6])

$$\lambda(\Phi) = \{x = (x_k) \colon \Phi(x) = (\varphi_k(|x_k|)) \in \lambda\} .$$

It is known (see [5] or Proposition 2.3) that on $\lambda(\Phi)$ we may define certain natural F-seminorm (F-norm) under some restrictions on F-seminormed (normed) sequence space λ.

Let $\Psi = (\psi_k)$ be another sequence of moduli. We investigate the local boundedness and boundedness of superposition operators $P_f \colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ $(1 \leq p, q < \infty)$. Our results extend corresponding theorems about the boundedness of superposition operators from [12].

2. Auxiliary results

In this section we formulate some definitions, known propositions and lemmas which are needed in the proofs of main results.

Recall that an F-*seminorm* on a vector space V is a functional $g \colon V \to \mathbb{R}$ satisfying, for all $x, y \in V$, the axioms

(N1) $g(0) = 0$,
(N2) $g(x + y) \leq g(x) + g(y)$,
(N3) $g(\alpha x) \leq g(x)$ for all scalars α with $|\alpha| \leq 1$,
(N4) $\lim\limits_n g(\alpha_n x) = 0$ for every scalar sequence (α_n) with $\lim\limits_n \alpha_n = 0$.

A *Fréchet norm* (or an F-*norm*) is an F-seminorm with the condition

(N5) $g(x) = 0 \Rightarrow x = 0$.

A topological sequence space in which all coordinate functionals π_k, $\pi_k(x) = x_k$, are continuous is called a K-*space*. A BK-*space* is defined as a K-space which is also a Banach space.

An F-seminorm g on a solid sequence space λ is said to be *absolutely monotone* if $g(y) \leq g(x)$ for all $x = (x_k)$, $y = (y_k)$ from λ with $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$.

An F-seminormed solid sequence space (λ, g) is called an AK-*space* if for any $x = (x_k) \in \lambda$,

$$\lim_{m \to \infty} x^{[m]} = x,$$

where $x^{[m]} = (x_k^{[m]})$ with $x_k^{[m]} = x_k$ if $k \leq m$ and $x_k^{[m]} = 0$ otherwise.

Let (λ, g) and (μ, h) be two F-seminormed sequence spaces. Recall that the superposition operator $P_f \colon \lambda \to \mu$ is said to be *locally bounded* if for any $z \in \lambda$ there exist numbers $\alpha > 0$ and $\beta > 0$ such that for all $x \in \lambda$ with

$g(x-z) \le \alpha$ we have $h(P_f(x) - P_f(z)) \le \beta$. The superposition operator P_f is called *bounded* if $\sup\{h(P_f(x)) \colon g(x) \le \varrho\} < \infty$ for every $\varrho > 0$.

For a sequence space λ we use the notation

$$\lambda^+ = \{(x_k) \in \lambda : x_k \ge 0 \quad (k \in \mathbb{N})\}.$$

Kolk [6] characterized the superposition operators $P_f \colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ in the case when $0 < p, q < \infty$ and (C) holds. Using, in addition, the remarks of Płuciennik ([11], Remark 1; [12], Remark 1) we may formulate

Proposition 2.1. *Let $1 \le p, q < \infty$, $\Phi = (\varphi_k)$ be a sequence of strictly increasing moduli and $\Psi = (\psi_k)$ a sequence of moduli. If there exist a number $\delta > 0$ and sequences $(a_k) \in \ell^+$ and $(c_i)_{i=0}^{\infty} \in \ell^+$ such that*

$$(\psi_k(|f(k,t)|))^q \le a_k + c_i 2^{-i} (\varphi_k(|t|))^p, \tag{2.1}$$

whenever $(\varphi_k(|t|))^p \le 2^i \delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0 = \{0, 1, \dots\}$, then P_f acts $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$. Condition (2.1) *is necessary for $P_f \colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ whenever* (B) *is satisfied.*

Proposition 2.1 may be modified as follows.

Proposition 2.2. *Let $1 \le p, q < \infty$, $\Phi = (\varphi_k)$ be a sequence of strictly increasing moduli and $\Psi = (\psi_k)$ a sequence of moduli. If there exist a number $\delta > 0$ and sequences $(b_k) \in \ell_q^+$ and $(d_i)_{i=0}^{\infty} \in \ell_q^+$ such that*

$$\psi_k(|f(k,t)|) \le b_k + d_i 2^{-i/q} (\varphi_k(|t|))^{p/q}, \tag{2.2}$$

whenever $\varphi_k(|t|) \le 2^{i/p}\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$, then P_f acts $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$. Condition (2.2) *is necessary for $P_f \colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ whenever* (B) *is satisfied.*

Proof. Let $a_k = b_k^q$ and $c_i = d_i^q$. If $1 \le q < \infty$, then (2.1) gives

$$\psi_k(|f(k,t)|) \le a_k^{1/q} + c_i^{1/q} 2^{-i/q} (\varphi_k(|t|))^{p/q},$$

whenever $\varphi_k(|t|) \le 2^{i/p}\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$. So, we get (2.2).

Conversely, by (1.1) it is not difficult to see that (2.2) yields $P_f \colon (w_0)_p(\Phi) \to \ell_q(\Psi)$. □

If (λ, g) is an F-seminormed space, then for the topologization of $\lambda(\Phi)$ it is natural to use the functional g_Φ, where

$$g_\Phi(x) = g(\Phi(x)).$$

Kolk ([5], Theorem 2) proved the following statement about the topologization of $\lambda(\Phi)$.

Proposition 2.3. *Let (λ, g) be a solid F-seminormed (F-normed or normed) AK-space. If g is absolutely monotone, then g_Φ is an absolutely monotone F-seminorm (F-norm) on $\lambda(\Phi)$ for an arbitrary sequence of moduli $\Phi = (\varphi_k)$. Moreover, $(\lambda(\Phi), g_\Phi)$ is an AK-space.*

It is known that for $1 \le p < \infty$ the spaces $(w_0)_p$ and ℓ_p are solid BK-AK-spaces with absolutely monotone norms $g = \|\cdot\|_{(w_0)_p}$ and $h = \|\cdot\|_{\ell_p}$, respectively. So, by Proposition 2.3, the topologies on the sequence spaces $(w_0)_p(\Phi)$ and $\ell_p(\Psi)$ are given by F-norms

$$g_\Phi(x) = \|\Phi(x)\|_{(w_0)_p}, \quad h_\Psi(x) = \|\Psi(x)\|_{\ell_p}.$$

By a *finite sequence* we mean a sequence $x = (x_k)$ for which there exists $k_0 \in \mathbb{N}$ such that $x_k = 0$ if $k \ge k_0$. Let ϕ be the set of all finite sequences. For any $k \in \mathbb{N}$ let $e^k = (\delta_{ki})_{i \in \mathbb{N}}$ with $\delta_{ki} = 1$ if $k = i$ and $\delta_{ki} = 0$ otherwise.

The following three lemmas are proved by the author in [10], Lemmas 2.3–2.5.

Lemma 2.4. *Let λ, μ be two solid BK-AK-spaces with absolutely monotone norms and let $\Phi = (\varphi_k)$, $\Psi = (\psi_k)$ be two sequences of moduli. Assume that $\phi \subset \lambda$ and P_f maps $\lambda(\Phi)$ into $\mu(\Psi)$. If P_f is locally bounded, then f satisfies* (B).

Lemma 2.5. *Let $\Psi = (\psi_k)$ be the sequence of moduli, $r \in \mathbb{N}$ and $1 \le q < \infty$. If the functions $f(k,\cdot)$ $(k = 1, \dots, r)$ are bounded on a bounded subset of real numbers $T \subset \mathbb{R}$, then there exists a number $M > 0$ such that*

$$\sup_{t_1,\dots,t_r \in T} \left\| \sum_{k=1}^{r} \psi_k(|f(k,t_k)|) e^k \right\|_{\ell_q} \le M.$$

Lemma 2.6. *Let $1 \le p, q < \infty$ and assume that f satisfies* (B). *If for every $\beta > 0$ there is a number $\vartheta(\beta) > 0$ such that for every finite sequence $x = (x_k)$ we have*

$$\|\Psi(P_f(x))\|_{\ell_q} \le \vartheta(\beta),$$

provided

$$\sum_{k=1}^{\infty} (\varphi_k(|x_k|))^p \le \beta^p,$$

then there exists a sequence $a(\beta) = (a_k(\beta)) \in \ell_q^+$ with $\|a(\beta)\|_{\ell_q} \le \vartheta(\beta)$ such that for each $k \in \mathbb{N}$,

$$\psi_k(|f(k,t)|) \le a_k(\beta) + 2^{1/q} \beta^{-p/q} \vartheta(\beta) (\varphi_k(|t|))^{p/q}$$

whenever $\varphi_k(|t|) \le \beta$.

3. Boundedness of P_f on $(w_0)_p(\Phi)$

Let $\Phi = (\varphi_k)$ and $\Psi = (\psi_k)$ be two sequences of moduli. By the definition of a modulus it is not difficult to see that, for a fixed sequence $z = (z_k)$, the sets

$$T_m(\varkappa) = \{t \in \mathbb{R}\colon \max_{1\le k\le m} \varphi_k(|t - z_k|) \le \varkappa\}$$

are bounded for every $m \in \mathbb{N}$ and $\varkappa > 0$.

First we characterize the local boundedness of superposition operators on $(w_0)_p(\Phi)$.

Theorem 3.1. *Let $1 \le p, q < \infty$. If the moduli φ_k $(k \in \mathbb{N})$ are strictly increasing, then a superposition operator $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ is locally bounded if and only if f satisfies* (B).

Proof. Necessity of condition (B) follows from Lemma 2.4.

Conversely, suppose that f satisfies (B), P_f maps $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ and $z = (z_k) \in (w_0)_p(\Phi)$. By Proposition 2.1, there exist a number $\delta > 0$ and sequences $(a_k) \in \ell^+$ and $(c_i)_{i=0}^{\infty} \in \ell^+$ such that

$$(\psi_k(|f(k,t)|))^q \le a_k + c_i 2^{-i}(\varphi_k(|t|))^p, \tag{3.1}$$

whenever $(\varphi_k(|t|))^p \le 2^i\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$. Since by (1.1),

$$\lim_{i\to\infty} 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p = 0,$$

there exists $\tilde{r} \in \mathbb{N}$ with

$$2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p < 2^{-p}\delta \quad (i \ge \tilde{r}). \tag{3.2}$$

Let $x = (x_k) \in (w_0)_p(\Phi)$ be such that

$$\|\Phi(x - z)\|_{(w_0)_p} \le 2^{-1}\delta^{1/p}. \tag{3.3}$$

Then by (ii) from the definition of modulus function, Minkowski's inequality, (3.2) and (3.3), for $i \geq \tilde{r}$, we have

$$\begin{aligned}
&\left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p\right)^{1/p} \\
&\leq \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k - z_k|))^p\right)^{1/p} + \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p\right)^{1/p} \\
&\leq \|\Phi(x-z)\|_{(w_0)_p} + \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p\right)^{1/p} \\
&\leq 2^{-1}\delta^{1/p} + 2^{-1}\delta^{1/p} = \delta^{1/p}.
\end{aligned}$$

Consequently, if $i \geq \tilde{r}$, then

$$\sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p \leq 2^i\delta \tag{3.4}$$

and so $(\varphi_k(|x_k|))^p \leq 2^i\delta \quad (2^i \leq k < 2^{i+1})$. Thus, in this case, by (3.1) we get

$$(\psi_k(|f(k, x_k)|))^q \leq a_k + c_i 2^{-i}(\varphi_k(|x_k|))^p. \tag{3.5}$$

Further, since the functions $f(k, \cdot)$ $(i < \tilde{r},\ 2^i \leq k < 2^{i+1})$ are bounded on bounded subset of real numbers $T_m(\varkappa)$ with $m = 2^{\tilde{r}} - 1$ and $\varkappa = 2^{-1}\delta^{1/p}$, by Lemma 2.5 there exists $M > 0$ such that

$$\sum_{k=1}^{2^{\tilde{r}}-1} (\psi_k(|f(k, x_k)|))^q \leq M^q. \tag{3.6}$$

Finally, by (ii), Minkowski's inequality and (3.4)–(3.6), we conclude

$$\begin{aligned}
&\|\Psi(P_f(x) - P_f(z))\|_{\ell_q} \\
&\leq \left(\sum_{k=1}^{2^{\tilde{r}}-1} (\psi_k(|f(k, x_k)|))^q\right)^{1/q} + \left(\sum_{i=\tilde{r}}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, x_k)|))^q\right)^{1/q} \\
&\quad + \left(\sum_{k=1}^{\infty} (\psi_k(|f(k, z_k)|))^q\right)^{1/q} \\
&\leq M + \left(\sum_{i=\tilde{r}}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} a_k\right)^{1/q} + \left(\sum_{i=\tilde{r}}^{\infty} c_i 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p\right)^{1/q} \\
&\quad + \|\Psi(P_f(z))\|_{\ell_q}
\end{aligned}$$

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$$\leq M + \|(a_k)\|_\ell^{1/q} + \left(\delta \sum_{i=\bar{r}}^{\infty} c_i\right)^{1/q} + \|\Psi(P_f(z))\|_{\ell_q}$$
$$\leq M + \|(a_k)\|_\ell^{1/q} + (\delta \ \|(c_i)_{i=0}^{\infty}\|_\ell)^{1/q} + \|\Psi(P_f(z))\|_{\ell_q}.$$

So, putting $\alpha = 2^{-1}\delta^{1/p}$ and $\beta = M + \|(a_k)\|_\ell^{1/q} + (\delta \ \|(c_i)_{i=0}^{\infty}\|_\ell)^{1/q} + \|\Psi(P_f(z))\|_{\ell_q}$, we have $\|\Phi(P_f(x)-P_f(z))\|_{\ell_p} \leq \beta$ whenever $g_\Phi(x-z) \leq \alpha$. □

Now we give necessary and sufficent conditions for boundedness of superposition operators from $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$.

Theorem 3.2. *Let $1 \leq p, q < \infty$. A superposition operator P_f: $(w_0)_p(\Phi) \to \ell_q(\Psi)$ is bounded if and only if for every $\varrho > 0$ there are sequences $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and $c(\varrho) = (c_i(\varrho))_{i=0}^{\infty} \in \ell_q^+$ such that*

$$\psi_k(|f(k,t)|) \leq a_k(\varrho) + c_i(\varrho) 2^{-i/q} (\varphi_k(|t|))^{p/q} \tag{3.7}$$

whenever $\varphi_k(|t|) \leq 2^{i/p}\varrho$, $2^i \leq k < 2^{i+1}$, $i \in \mathbb{N}_0$. Furthermore, for every $\varrho > 0$,

$$\eta_f(\varrho) \leq \nu_f(\varrho) \leq (1 + 2^{1/q})\eta_f(\varrho),$$

where

$$\eta_f(\varrho) = \sup\left\{\|\Psi(P_f(x))\|_{\ell_q} \ : \ \|\Phi(x)\|_{(w_0)_p} \leq \varrho\right\}$$

and

$$\nu_f(\varrho) = \inf\big\{\|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q} \ : \text{(3.7) holds for } \varphi_k(|t|) \leq 2^{i/p}\varrho, 2^i \leq k < 2^{i+1}, \ i \in \mathbb{N}_0\big\}.$$

Proof. Sufficiency. Suppose that for every $\varrho > 0$ there are sequences $a(\varrho)$ and $c(\varrho)$ from ℓ_q^+ such that the inequality (3.7) holds if $\varphi_k(|t|) \leq 2^{i/p}\varrho$, $2^i \leq k < 2^{i+1}$, $i \in \mathbb{N}_0$. Let $\varrho > 0$ and $x = (x_k) \in (w_0)_p(\Phi)$ be such that

$$\|\Phi(x)\|_{(w_0)_p} \leq \varrho.$$

Then $\varphi_k(|x_k|) \leq 2^{i/p}\varrho \quad (2^i \leq k < 2^{i+1}, \ i \in \mathbb{N}_0)$ and (3.7) yields

$$\psi_k(|f(k,x_k)|) \leq a_k(\varrho) + c_i(\varrho) 2^{-i/q} (\varphi_k(|x_k|))^{p/q}.$$

So we have

$$\begin{aligned}
\|\Psi(P_f(x))\|_{\ell_q} &= \left(\sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}(\psi_k(|f(k,x_k)|))^q\right)^{1/q} \\
&\le \left(\sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}(a_k(\varrho))^q\right)^{1/q} + \left(\sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}\left(c_i(\varrho)2^{-i/q}(\varphi_k(|x_k|))^{p/q}\right)^q\right)^{1/q} \\
&\le \|a(\varrho)\|_{\ell_q} + \left(\sum_{i=0}^{\infty}(c_i(\varrho))^q 2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p\right)^{1/q} \\
&\le \|a(\varrho)\|_{\ell_q} + \left(\sum_{i=0}^{\infty}(c_i(\varrho))^q\varrho^p\right)^{1/q} \\
&\le \|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q} < \infty
\end{aligned}$$

whenever $\|\Phi(x)\|_{(w_0)_p} \le \varrho$.

The inequality $\eta_f(\varrho) \le \nu_f(\varrho)$ is obvious because

$$\|\Psi(P_f(x))\|_{\ell_q} \le \|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q}$$

and $\|\Phi(x)\|_{(w_0)_p} \le \varrho$.

Necessity. Let P_f be a bounded superposition operator acting from $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ and $x = (x_k) \in (w_0)_p(\Phi)$. For fixed $\varrho > 0$ we have

$$\|\Psi(P_f(x))\|_{\ell_q} = \left(\sum_{k=1}^{\infty}(\psi_k(|f(k,x_k)|))^q\right)^{1/q} \le \eta_f(\varrho)$$

whenever

$$\|\Phi(x)\|_{(w_0)_p} = \sup_{i\ge 0}\left(2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p\right)^{1/p} \le \varrho.$$

We define, for every $i \in \mathbb{N}_0$,

$$\tilde{c}_i(\varrho) = \sup\left\{\left(\sum_{k=2^i}^{2^{i+1}-1}(\psi_k(|f(k,x_k)|))^q\right)^{1/q} : 2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p \le \varrho^p\right\}.$$

Since f satisfies (B) by Lemma 2.4, we see that $\tilde{c}_i(\varrho) < \infty$ $(i \in \mathbb{N}_0)$. Therefore, by definition of $\tilde{c}_i(\varrho)$, for every $\varepsilon > 0$ there exists a sequence

$y(\varrho,\varepsilon) = (y_k(\varrho,\varepsilon))$ such that

$$\sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|y_k(\varrho,\varepsilon)|))^p \leq 2^i \varrho^p \tag{3.8}$$

and

$$\tilde{c}_i(\varrho) \leq \left(\sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, y_k(\varrho,\varepsilon))|))^q \right)^{1/q} + \frac{\varepsilon}{2^i} \tag{3.9}$$

for any $i \in \mathbb{N}_0$. If $\tilde{r} \in \{0,1,2,\dots\}$ and $\tilde{y}(\varrho,\varepsilon) = (\tilde{y}_k(\varrho,\varepsilon))$ is a sequence with

$$\tilde{y}_k(\varrho,\varepsilon) = \begin{cases} y_k(\varrho,\varepsilon) & \text{if } 1 \leq k \leq 2^{\tilde{r}}, \\ 0 & \text{if } k > 2^{\tilde{r}}, \end{cases}$$

then by (3.8) we have

$$\|\Phi(\tilde{y}(\varrho,\varepsilon))\|_{(w_0)_p} \leq \varrho.$$

Next we show that $\tilde{c}(\varrho) = (\tilde{c}_i(\varrho))_{i=0}^{\infty} \in \ell_q^+$ and $\|\tilde{c}(\varrho)\|_{\ell_q} \leq \eta_f(\varrho)$. Indeed, using (3.9), we get

$$\begin{aligned}
\left(\sum_{i=0}^{\tilde{r}} (\tilde{c}_i(\varrho))^q \right)^{1/q} &\leq \left(\sum_{i=0}^{\tilde{r}} \left(\left(\sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, y_k(\varrho,\varepsilon))|))^q \right)^{1/q} + \frac{\varepsilon}{2^i} \right)^q \right)^{1/q} \\
&\leq \left(\sum_{i=0}^{\tilde{r}} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, y_k(\varrho,\varepsilon))|))^q \right)^{1/q} + \left(\sum_{i=0}^{\tilde{r}} \left(\frac{\varepsilon}{2^i} \right)^q \right)^{1/q} \\
&\leq \|\Psi(P_f(\tilde{y}(\varrho,\varepsilon)))\|_{\ell_q} + \varepsilon \leq \eta_f(\varrho) + \varepsilon < \infty.
\end{aligned}$$

Thus

$$\|\tilde{c}(\varrho)\|_{\ell_q} = \left(\sum_{i=0}^{\infty} (\tilde{c}_i(\varrho))^q \right)^{1/q} = \lim_{\tilde{r}\to\infty} \left(\sum_{i=0}^{\tilde{r}} (\tilde{c}_i(\varrho))^q \right)^{1/q} \leq \eta_f(\varrho) + \varepsilon.$$

While $\varepsilon > 0$ is arbitrary, then $\tilde{c}(\varrho) \in \ell_q^+$ with $\|\tilde{c}(\varrho)\|_{\ell_q} \leq \eta_f(\varrho)$.

On the other hand, for every $i \in \mathbb{N}_0$,

$$\left(\sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, x_k)|))^q \right)^{1/q} \leq \tilde{c}_i(\varrho)$$

whenever

$$2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p \leq \varrho^p.$$

Applying Lemma 2.6 to the previous inequality with $\beta^p = 2^i \varrho^p$, $\vartheta(\beta) = \tilde{c}_i(\varrho)$ and $f(k,t) = 0$ for $k \neq 2^i, 2^i+1, \ldots, 2^{i+1}-1$, we can find a sequence $a(\varrho) \in \ell_q^+$ such that

$$\sum_{k=2^i}^{2^{i+1}-1} (a_k(\varrho))^q \leq \|a(\varrho)\|_{\ell_q}^q \leq (\tilde{c}_i(\varrho))^q,$$

$$\psi_k(|f(k,t)|) \leq a_k(\varrho) + 2^{1/q}\, 2^{-i/q} \varrho^{-p/q} \tilde{c}_i(\varrho)(\varphi_k(|t|))^{p/q} \tag{3.10}$$

provided $\varphi_k(|x_k|) \leq 2^{i/p}\varrho$, $2^i \leq k < 2^{i+1}$. Putting $c_i(\varrho) = 2^{1/q}\ \varrho^{-p/q}\tilde{c}_i(\varrho)$ we have (3.7).

So we get

$$\|a(\varrho)\|_{\ell_q}^q = \sum_{k=1}^{\infty}(a_k(\varrho))^q = \sum_{i=0}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (a_k(\varrho))^q \leq \sum_{i=0}^{\infty}(\tilde{c}_i(\varrho))^q = \|\tilde{c}(\varrho)\|_{\ell_q}^q$$

which yields

$$\|a(\varrho)\|_{\ell_q} \leq \|\tilde{c}(\varrho)\|_{\ell_q} \leq \eta_f(\varrho).$$

By (3.10) it follows

$$\begin{aligned}
a_k(\varrho) + 2^{-i/q} c_i(\varrho)(\varphi_k(|t|))^{p/q} &\leq a_k(\varrho) + 2^{-i/q} 2^{1/q} \varrho^{-p/q} \tilde{c}_i(\varrho)(\varphi_k(|t|))^{p/q} \\
&\leq \|a(\varrho)\|_{\ell_q} + 2^{-i/q} 2^{1/q} \varrho^{-p/q} \|c(\varrho)\|_{\ell_q} (2^{i/p}\varrho)^{p/q} \\
&\leq \eta_f(\varrho) + 2^{1/q}\|\tilde{c}(\varrho)\|_{\ell_q} \leq \eta_f(\varrho) + 2^{1/q}\eta_f(\varrho) \\
&= (1 + 2^{1/q})\eta_f(\varrho)
\end{aligned}$$

whenever $\varphi_k(|x_k|) \leq 2^{i/p}\varrho$ and $i \in \mathbb{N}_0$. Hence

$$\nu_f(\varrho) \leq (1 + 2^{1/q})\eta_f(\varrho).$$

□

4. Applications

The sequence spaces ℓ_p and $(w_0)_p$ $(1 \leq p < \infty)$ can be considered as the spaces $\ell_p(\Phi)$ and $(w_0)_p(\Phi)$, where $\Phi = (\varphi_k)$ with $\varphi_k(t) = t$ $(k \in \mathbb{N})$. So, Theorems 3.1 and 3.2 allow us to formulate extensions of the results of Płuciennik ([12], Theorems 2 and 3) about the boundedness of superposition operators on w_0.

Proposition 4.1. *Let $1 \leq p, q < \infty$. A superposition operator P_f: $(w_0)_p \to \ell_q$ is locally bounded if and only if f satisfies* (B).

Proposition 4.2. *Let $1 \leq p, q < \infty$. A superposition operator P_f: $(w_0)_p \to \ell_q$ is bounded if and only if for every $\varrho > 0$ there are sequences $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and $c(\varrho) = (c_i(\varrho))_{i=0}^{\infty} \in \ell_q^+$ such that*

$$|f(k,t)| \leq a_k(\varrho) + c_i(\varrho) 2^{-i/q} |t|^{p/q} \tag{4.1}$$

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whenever $|t| \le 2^{i/p}\, \varrho$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$. *Furthermore,*

$$\overline{\eta}_f(\varrho) \le \overline{\nu}_f(\varrho) \le (1 + 2^{1/q})\overline{\eta}_f(\varrho)$$

for every $\varrho > 0$ *with*

$$\overline{\eta}_f(\varrho) = \sup\left\{\|P_f(x)\|_{\ell_q} \ : \ \|x\|_{(w_0)_p} \le \varrho^{1/p}\right\}$$

and

$$\overline{\nu}_f(\varrho) = \inf\{\|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q} : (4.1) \text{ holds for } |t| \le 2^{i/p}\varrho,\ 2^i \le k < 2^{i+1},\ i \in \mathbb{N}_0\}.$$

As generalizations of the spaces ℓ_p and w_0 we consider the multiplier sequence spaces of Maddox type

$$\ell(p,u) = \left\{x \in \omega : \sum_{k=1}^{\infty} |u_k x_k|^{p_k} < \infty\right\},$$

$$w_0(p,u) = \left\{x \in \omega : \lim_n \frac{1}{n}\sum_{k=1}^{n} |u_k x_k|^{p_k} = 0\right\},$$

where $u = (u_k)$ is a sequence with $u_k \ne 0$ ($k \in \mathbb{N}$) and $p = (p_k)$ is a bounded sequence of positive numbers (cf. [3]).

In the case $u_k = 1$ ($k \in \mathbb{N}$) the spaces $\ell(p,u)$ and $w_0(p,u)$ are known as the sequence spaces of Maddox type $\ell(p)$ and $w_0(p)$, respectively (see, for example, [2] and [7]). Some authors ([1], [14]) consider the space $\ell(p,v)$ for special multipliers

$$u_k = k^{-\alpha/p_k}, \quad v_k = k^{\alpha/p_k} \quad (\alpha > 0).$$

To apply our theorems for the multiplier spaces of Maddox type, we put $r = \max\{1, \sup_k p_k\}$ and define the sequence of moduli $\Phi = (\varphi_k)$ by

$$\varphi_k(t) = (|u_k|t)^{p_k/r} \qquad (k \in \mathbb{N}).$$

Then the spaces $\ell(p,u)$ and $w_0(p,u)$ we may consider as the spaces $\ell_r(\Phi)$ and $(w_0)_r(\Phi)$, respectively. The corresponding F-norms on $\ell(p,u)$ and $w_0(p,u)$ are determined, respectively, by

$$g_\Phi(x) = \left(\sum_{k=1}^{\infty} |u_k x_k|^{p_k}\right)^{1/r}$$

and

$$g_\Phi(x) = \sup_{i \ge 0}\left(\frac{1}{2^i}\sum_{k=2^i}^{2^{i+1}-1} |u_k x_k|^{p_k}\right)^{1/r}.$$

Let $q = (q_k)$ be another bounded sequence of strictly positive numbers and $v = (v_k)$ be a sequence such that $v_k \neq 0$ $(k \in \mathbb{N})$. Now, putting $s = \max\{1, \sup_k q_k\}$ and defining the sequence of moduli $\Psi = (\psi_k)$ by

$$\psi_k(t) = (|v_k|t)^{q_k/s} \quad (k \in \mathbb{N}),$$

from Theorems 3.1 and 3.2 we get the following statements about the boundedness of superposition operators on multiplier sequence spaces of Maddox type.

Proposition 4.3. *A superposition operator $P_f\colon w_0(p,u) \to \ell(q,v)$ is locally bounded if and only if f satisfies* (B).

Proposition 4.4. *A superposition operator $P_f\colon w_0(p,u) \to \ell(q,v)$ is bounded if and only if for every $\varrho > 0$ there are sequences $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and $c(\varrho) = (c_i(\varrho))_{i=0}^{\infty} \in \ell_q^+$ such that*

$$|v_k f(k,t)| \leq a_k(\varrho) + c_i(\varrho) 2^{-i/q_k} |u_k t|^{p_k/q_k} \tag{4.2}$$

whenever $|u_k t| \leq 2^i\ \varrho$, $2^i \leq k < 2^{i+1}$, $i \in \mathbb{N}_0$. Furthermore,

$$\tilde{\eta}_f(\varrho) \leq \tilde{\nu}_f(\varrho) \leq (1 + 2^{1/s})\tilde{\eta}_f(\varrho),$$

for every $\varrho > 0$ with

$$\tilde{\eta}_f(\varrho) = \sup\left\{\|P_f(x)\|_{\ell(q,v)}^s \ :\ \|x\|_{w_0(p,u)} \leq \varrho^{1/r}\right\}$$

and

$$\tilde{\nu}_f(\varrho) = \inf\big\{\|a(\varrho)\|_\ell + \varrho\|c(\varrho)\|_\ell \ :\ (4.2) \text{ holds for } |u_k t|^{p_k} \leq 2^i \varrho,\ 2^i \leq k < 2^{i+1},\ i \in \mathbb{N}_0\big\}.$$

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