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Functorial properties of Cayley constructions

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ABSTRACT. We describe the construction of the Cayley graph of a semigroup as a functor and investigate certain reflection and preservation properties of this functor. We also investigate it with respect to several product constructions including pullbacks.

We present some elementary results which describe the construction of Cayley graphs starting from semigroups with certain connection sets. The tool of description is the category theory. We will use set notation also for proper classes.

Take $\mathbf{C}= \{(S,C) \mid S$ a semigroup, $C \subseteq S\}$ and $\mathrm{Mor}_{\mathbf{C}}((S,C),(T,D)) = \{f \mid f: S \to T$ a semigroup homomorphism and $f(C) \subseteq D\}$. Then $\{\mathbf{C}, \mathrm{Mor}_{\mathbf{C}}\}$ is a category.

Let $\mathbb{D}$ be the category of digraphs, i.e., directed graphs whose morphisms are graph homomorphisms.

As usual we define the Cayley graph of a semigroup S with connection set $C \subseteq S$ as $Cay(S,C) = (S,E)$, where the pairs (s, sc) are elements of E for all $s \in S$ and $c \in C$, that is, we use the right action. The set of edges of $Cay(S,C)$ is denoted by $E(Cay(S,C))$. The notation $Cay(S,C)$ is used to denote both Cayley graph and its set of edges. As examples we also consider infinite semigroups and Cayley graphs.

In [5] we have the definitions for various graph products. Information about various graph categories can be found in [6]. The definitions of the Cayley graph concepts and of categorical concepts used can be found for example in [7], the latter also in [3].

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1. The functor *Cay*

Here we establish some basic and more or less obvious properties of Cay.

Theorem 1.1. *Let S and T be semigroups, let C and D be subsets of S and T respectively. Then $Cay : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ given by*

$$\begin{array}{ccccc} (S,C) & \longmapsto & Cay(S,C) & & s \in S \\ \Big\downarrow f & \longmapsto & \Big\downarrow Cay(f) & & \Big\downarrow \\ (T,D) & \longmapsto & Cay(T,D) & & f(s) \in T \end{array}$$

for any $f \in Mor_{\boldsymbol{C}}((S,C),(T,D))$ and $s \in S$, where $Cay(S,C)$ is the respective Cayley graph with connection set $C \subseteq S$, is a covariant functor.

Proof. Suppose (s, sc) is an arc in $Cay(S,C)$, where $s \in S$, $c \in C$. Then $(f(s), f(sc)) = (f(s), f(s)f(c))$ is an arc in $Cay(T,D)$ for each $f \in \mathrm{Mor}_{\mathbf{C}}((S,C),(T,D))$. It follows that $Cay(f)$ is a homomorphism from $Cay(S,C)$ to $Cay(T,D)$. The preservations of identities and composition of Cay are obvious. □

Let $G = (V,E)$ and $G' = (V',E')$ be two graphs in $\mathbf{D}$ with the sets of vertices V, V' and the sets of edges E, E' respectively. Recall that a graph homomorphism $f : V \to V'$ is said to be a *strong homomorphism* if $(f(x), f(y)) \in E'$ implies $(x,y) \in E$ for $x, y \in V$ (see [6],[7]). By the definition of a strong homomorphism, we have

Lemma 1.2. *Suppose that $f \in Mor_{\boldsymbol{C}}((S,C),(T,D))$. Then $Cay(f)$ is a strong homomorphism in $\boldsymbol{D}$ if and only if $(f(s), f(s')) \in Cay(T,D)$ implies that $s' = sc$ for some $c \in C$, where $s, s' \in S$.*

In the following, for $f \in Mor_{\mathbf{C}}((S,C),(T,D))$, by saying that f is injective, surjective, monomorphism, etc., we mean that both f and $f|_C$ have these properties.

Corollary 1.3. *Let $f \in Mor_{\boldsymbol{C}}((S,C),(T,D))$. If f is injective and $f(C) = D$ then $Cay(f)$ is a strong homomorphism in $\boldsymbol{D}$.*

Proof. Suppose that $(f(s), f(s')) \in Cay(T,D)$, where $s, s' \in S$. Then $f(s') = f(s)d$ for some $d \in D$. Since $f(C) = D$ there exists an element $c \in C$ such that $d = f(c)$. Consequently $f(s') = f(s)d = f(s)f(c) = f(sc)$ and hence $s' = sc$ by the assumption that f is injective. Now by Lemma 1.2 we get that $Cay(f)$ is a strong homomorphism. □

Let $f \in Mor_{\mathbf{C}}((S,C),(T,D))$. The following four examples show that the conditions

1. f is injective,
2. f is surjective,
3. $f(C) = D$,
4. $f(C) = D$ and $f^{-1}(D) = C$

are neither necessary nor sufficient for $Cay(f)$ to be a strong homomorphism in $\mathbf{D}$. Example 1.4 shows that 2, 3 and 4 are not necessary. Example 1.5 shows that 1 and 2 are not necessary and Example 1.6 shows that 4 is not sufficient.

Example 1.4. Let T be a 3-element set $\{1,2,3\}$ with the following multiplication table:

	1	2	3
1	1	2	2
2	2	1	1
3	2	1	1

Clearly this is a semigroup. Take the subsemigroup $S = \{1,2\}$ of T and $C := \{2\}$, $D := \{2,3\}$. Then $i : S \hookrightarrow T$, the natural embedding of S into T, belongs to $\mathrm{Mor}_{\mathbf{C}}((S,C),(T,D))$. Now we get the following sets of edges for the respective Cayley graphs:

$$E(Cay(S,C)) = \{(1,2),(2,1)\}, \quad E(Cay(T,D)) = \{(1,2),(2,1),(3,1)\},$$

and i is a strong homomorphism in $\mathbf{D}$. But $i(C) \neq D$ and i is not surjective.

Example 1.5. Set $S = (\mathbb{Z}_6, \cdot)$ where $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \cdots, \overline{5}\}$. Define $f : S \to S$ by

$$f(x) = \overline{x^2}$$

for all $\overline{x} \in \mathbb{Z}_6$. It is clear that f is a semigroup homomorphism. Take $C = \{\overline{1}, \overline{5}\}$, $D = \{\overline{1}\}$ which are subsets of S. Then $f \in \mathrm{Mor}_{\mathbf{C}}((S,C),(S,D))$. Since

$$\begin{aligned} E(Cay(S,C)) = &\{(\overline{0},\overline{0}),(\overline{1},\overline{1}),(\overline{2},\overline{2}),(\overline{3},\overline{3}),(\overline{4},\overline{4}),(\overline{5},\overline{5}), \\ &(\overline{1},\overline{5}),(\overline{2},\overline{4}),(\overline{4},\overline{2}),(\overline{5},\overline{1})\}, \\ E(Cay(S,D)) = &\{(\overline{0},\overline{0}),(\overline{1},\overline{1}),(\overline{2},\overline{2}),(\overline{3},\overline{3}),(\overline{4},\overline{4}),(\overline{5},\overline{5})\}, \end{aligned}$$

it is easy to check that f is a strong homomorphism. But clearly f is neither injective nor surjective.

Next we give another example to show that $f(C) = D$ or $f(C) = D$ and $f^{-1}(D) = C$ do not imply that f is a strong homomorphism.

Example 1.6. Let $C = \{a, b\}$ be considered as a 2-element left zero semigroup and $S = C^0$ which is a left zero semigroup with a zero adjoined. Let $D = \{a\}$ be a subset of S. Define a mapping $f : S \to S$ by

$$f(0) = 0, f(a) = a, f(b) = a.$$

Then $f \in \mathrm{Mor}_{\mathbf{C}}((S, C), (S, D))$, and

$$E(Cay(S, C)) = \{(0, 0), (a, a), (b, b)\}.$$

It is clear that $f(C) = D$ and $f^{-1}(D) = C$. But f is not a strong homomorphism in $\mathbf{D}$ since $(f(a), f(b)) = (a, a) \in Cay(S, D)$ does not imply $(a, b) \in Cay(S, C)$.

The following statement is straightforward.

Proposition 1.7. *The functor $Cay : \boldsymbol{C} \to \boldsymbol{D}$ is faithful.*

Note that for right zero semigroups S and T, the functor Cay is full. The reason is that in this case every mapping from S to T is a semigroup homomorphism. Moreover, every element in a connection set produces a loop in the respective Cayley graphs, and these are the only loops, cf. [1]. Since graph homomorphisms map loops onto loops, the condition $f(C) \subseteq D$ is fulfilled in $\mathbf{C}$ automatically.

In general, however, a morphism in $\mathbf{D}$ between two Cayley graphs whose vertex sets are semigroups is not a semigroup homomorphism. See the following example.

Example 1.8. Let $S = (\mathbb{Z}_3, +)$, where $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$. Let $C = \{\overline{2}\}$. Define a mapping $f : S \to S$ by

$$f(\overline{x}) = \overline{x + 2}$$

for all $\overline{x} \in S$. Then f is a morphism in $\mathbf{D}$ from $Cay(S, C)$ to $Cay(S, C)$, but obviously f is not a semigroup homomorphism. Moreover, the example also shows that the condition $f \in Mor_{\mathbf{D}}(Cay(S, C), Cay(T, D))$ does not imply that $f|_C$ is a mapping from C to D.

So we get

Proposition 1.9. *The functor $Cay : \boldsymbol{C} \to \boldsymbol{D}$ is not full.*

2. Reflection and preservation of morphisms

By the definition of the Cay functor, and the fact that Cay is covariant and faithful (see, for example, [7]), one can easily get

Proposition 2.1. *The functor Cay preserves and reflects injective mappings and surjective mappings. It preserves retractions and coretractions.*

It is known that every faithful covariant functor reflects monomorphisms and epimorphisms (see [7]). Since in the category **C**, the monomorphisms are injective and, as always, surjective mappings are epimorphisms, we have

Corollary 2.2. *The functor Cay preserves and reflects monomorphisms, and reflects epimorphisms.*

But by an example we show that the functor Cay does not preserve epimorphisms. In fact, let $i : (\mathbb{Z}, \cdot) \hookrightarrow (\mathbb{Q}, \cdot)$ denote the natural embedding, then i is an epimorphism in $\mathbf{C}$ from $(\mathbb{Z}, \mathbb{Z})$ to $(\mathbb{Q}, \mathbb{Z})$. Consider the digraph $(\mathbb{Q}, \mathbb{Z} \times \mathbb{Z})$ whose set of vertices is $\mathbb{Q}$, edges are $\mathbb{Z} \times \mathbb{Z}$, mappings $g, h : \mathbb{Q} \to \mathbb{Q}$ with $g(m) = h(m) = m$ if $m \in \mathbb{Z}$ and $g(m) = 1, h(m) = 0$ if $m \notin \mathbb{Z}$. Obviously $gi = hi$ but $g \neq h$. It is easy to see that $g, h : Cay(\mathbb{Q}, \mathbb{Z}) \to (\mathbb{Q}, \mathbb{Z} \times \mathbb{Z})$ are morphisms in $\mathbf{D}$. So $Cay(i)$ is not an epimorphism in $\mathbf{D}$.

So the preservation of epimorphisms is not granted since there exist non-surjective epimorphisms in the category of semigroups which will not be epimorphisms in the category of digraphs.

The following examples show that the functor Cay does not reflect retractions and coretractions.

Example 2.3. Let $\pi : (\mathbb{N}_0, \cdot) \to (\mathbb{Z}_6, \cdot)$ be the canonical mapping, where $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \cdots, \overline{5}\}$. Take $C = \{0\} \subseteq \mathbb{N}_0$ and $D = \{\overline{0}\} \subseteq \mathbb{Z}_6$. Then $\pi : (\mathbb{N}_0, C) \to (\mathbb{Z}_6, D)$ is a morphism in $\mathbf{C}$. If there is a morphism $g : (\mathbb{Z}_6, D) \to (\mathbb{N}_0, C)$ in $\mathbf{C}$ such that $\pi g = id_{\mathbb{Z}_6}$, where $id_{\mathbb{Z}_6}$ is the identity mapping of $\mathbb{Z}_6$, then $g(\overline{4}) = g(\overline{4})g(\overline{4})$ implies that $g(\overline{4}) = 0$ or $g(\overline{4}) = 1$, which contradicts to $\pi g(\overline{4}) = \overline{4}$. So π is not a retraction in $\mathbf{C}$. Now define $g' : \mathbb{Z}_6 \to \mathbb{N}_0$ by $g'(\overline{n}) = n$, where $n = 0, 1, \cdots, 5$. Then $g' : Cay(\mathbb{Z}_6, D) \to Cay(\mathbb{N}_0, C)$ is a morphism in $\mathbf{D}$ satisfying $\pi g' = id_{Cay(\mathbb{Z}_6, \{\overline{0}\})}$. So π is a retraction in $\mathbf{D}$.

Example 2.4. Take $S = \{\overline{2}, \overline{4}\}$ to be the subsemigroup of $(\mathbb{Z}_6, \cdot)$, where $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \cdots, \overline{5}\}$, and $i : S \to \mathbb{Z}_6$ the natural embedding. Then $i \in \mathrm{Mor}_{\mathbf{C}}((S, S), (\mathbb{Z}_6, S))$. If $g : \mathbb{Z}_6 \to S$ is a semigroup homomorphism such that $gi = id_S$, where id_S is the identity mapping of S, then $g(\overline{0}) = \overline{4}$ and $g(\overline{0})g(\overline{2}) = \overline{4}$ implies that $g(\overline{2}) = \overline{4}$, which contradicts to $g(\overline{2}) = \overline{2}$. Now define $f : \mathbb{Z}_6 \to S$ with $f(\overline{2}) = \overline{2}, f(\overline{n}) = \overline{4}$ for all $\overline{2} \neq \overline{n} \in \mathbb{Z}_6$. Then $f \in \mathrm{Mor}_{\mathbf{D}}(Cay(\mathbb{Z}_6, S), Cay(S, S))$ and $fi = id_S$ implies that i is a coretraction in $\mathbf{D}$.

3. Categorical products and equalizers

Now we turn to the categorical product, the so-called cross product, see [5] and [6], and equalizers, compare to [7].

Lemma 3.1. *Let $\{(S_i, C_i)\}_{i \in I}$ be a family of objects in category $\boldsymbol{C}$. Then $((\prod_{i \in I} S_i, \prod_{i \in I} C_i), p_i)$ is the product of $\{(S_i, C_i)\}_{i \in I}$ in $\boldsymbol{C}$, where $\prod_{i \in I} S_i$*

and $\prod_{i\in I} C_i$ *are cartesian products of* $(S_i)_{i\in I}$ *and* $(C_i)_{i\in I}$ *respectively, and* $p_i : \prod_{i\in I} S_i \to S_i, i \in I$, *are the canonical projections.*

Proof. Clearly, for each $i \in I$, p_i is a morphism in $\mathbf{C}$. For any $(T, D) \in \mathbf{C}$ and any family $(q_i \in \mathrm{Mor}_{\mathbf{C}}((T, D), (S_i, C_i)))_{i\in I}$, define $q : T \to \prod_{i\in I} S_i$ with $q(t) = (q_i(t))_{i\in I}, t \in T$. Then $q \in \mathrm{Mor}_{\mathbf{C}}((T, D), (\prod_{i\in I} S_i, \prod_{i\in I} C_i))$ is the unique morphism in $\mathbf{C}$ such that $p_i q = q_i$ for all $i \in I$. □

In what follows the notation $\{u, v\}$ is used for the edge of product graphs, where u, v are the vertices of product graphs.

Theorem 3.2. *Let* $\times$ *denote the cross product of graphs. Then for semigroups* S *and* T *with their subsets* C *and* D, *we have*

$$Cay(S \times T, C \times D) = Cay(S, C) \times Cay(T, D).$$

Proof. We have

$$\begin{aligned} &E(Cay(S, C) \times Cay(T, D)) \\ &\quad = \{\{(x, y), (x', y')\} \mid (x, x') \in Cay(S, C), (y, y') \in Cay(T, D)\} \\ &\quad = \{\{(x, y), (xc, yd)\} \mid (x, y) \in S \times T, (c, d) \in C \times D\} \\ &\quad = E(Cay(S \times T, C \times D)). \end{aligned}$$

□

It is clear that Theorem 3.2 can be generalized to the case of arbitrary (multiple) product. So we obtain the following corollary.

Corollary 3.3. *The functor* Cay *preserves products.*

Proof. Since the cross product is the product in category $\mathbf{D}$, the result follows from Theorem 3.2 and Lemma 3.1. □

However the functor Cay does not reflect products. This can be shown by the following example.

Example 3.4. Let $S = \{a, b\}$ be a 2-element left zero semigroup and $C = \{a\}$. By Theorem 3.2, $Cay(S', C')$ is the product of the 2 copies of $Cay(S, C)$ in $\mathbf{D}$, where $S' = S \times S, C' = \{(a, a), (b, b)\}$, since

$$\begin{aligned} &E(Cay(S', C')) \\ &\quad = \{\{(a, a), (a, a)\}, \{(a, b), (a, b)\}, \{(b, a), (b, a)\}, \{(b, b), (b, b)\}\} \\ &\quad = E(Cay(S \times S, C \times C)) \\ &\quad = E(Cay(S, C) \times Cay(S, C)). \end{aligned}$$

By Lemma 3.1, $(S \times S, C \times C)$ is the product of the two copies of (S, C) in $\mathbf{C}$. But obviously $(S', C') \ncong (S \times S, C \times C)$ and hence (S', C') is not the product of the two copies of (S, C) in $\mathbf{C}$.

Remark 3.5. We observe that the fact that the *Cay* functor preserves products together with the preservation of injective and surjective mappings leads to the preservation of subdirect products of semigroups. This in turn opens up many possibilities to characterize Cayley graphs of completely regular semigroups, compare to [9], and for some steps in this direction see [8] or [1].

Now we verify that the standard construction for equalizers in concrete categories applies to the category $\mathbf{C}$ and category $\mathbf{D}$.

Lemma 3.6. *Consider an equalizer situation* $f, g : (S, C) \rightrightarrows (S', C')$ *in* $\mathbf{C}$. *If* $T = \{s \in S \mid f(s) = g(s)\} \neq \emptyset$ *and* $D = \{c \in C \mid f(c) = g(c)\}$ *then* (T, D), *where* $T \subseteq S, D \subseteq C$, *with the natural embedding is the equalizer of* f *and* g *in* $\mathbf{C}$.

Proof. Suppose that $T = \{s \in S \mid f(s) = g(s)\} \neq \emptyset$. If $((E, A), \alpha)$ satisfies $f\alpha = g\alpha$, where $(E, A) \in \mathbf{C}, \alpha \in \mathrm{Mor}_{\mathbf{C}}((E, A), (S, C))$, then $\alpha(E) \subseteq T, \alpha(A) \subseteq D$ and $\overline{\alpha} : (E, A) \to (T, D)$ given by $\overline{\alpha}(e) = \alpha(e)$ for all $e \in E$ is the unique morphism in $\mathbf{C}$ such that $i\overline{\alpha} = \alpha$, where i is the natural embedding. □

Lemma 3.7. *Consider an equalizer situation* $f, g : (V_1, E_1) \rightrightarrows (V_2, E_2)$ *in* $\mathbf{D}$, *where* (V_1, E_1) *and* (V_2, E_2) *are digraphs with the sets of vertices* V_1, V_2 *and the sets of edges* E_1, E_2, *respectively. Suppose that* $V = \{v \in V_1 \mid f(v) = g(v)\} \neq \emptyset$. *Set* $E = \{(u, v) \in E_1 \mid u, v \in V\}$. *Then* (V, E) *with the natural embedding* i *is the equalizer of* f *and* g *in* $\mathbf{D}$, *i.e,* $((V, E), i) = Eq_{\mathbf{D}}(f, g)$.

Proof. Suppose that $(G, A) \in \mathbf{D}$ is a digraph with the set of vertices G and the set of edges A, and $h \in \mathrm{Mor}_{\mathbf{D}}((G, A), (V_1, E_1))$ satisfying $fh = gh$.

By the hypothesis that $fh = gh$, we have $h(x) \in V$ for all $x \in G$. Moreover, for any $(x, y) \in A$, since $(h(x), h(y)) \in E_1$, it follows that $(h(x), h(y)) \in E$ and hence $\overline{h} \in \mathrm{Mor}_{\mathbf{D}}((G, A), (V, E))$, where $\overline{h} : G \to V$ is given by $\overline{h}(x) = h(x)$ for all $x \in G$.

It is obvious that $\overline{h} \in \mathrm{Mor}_{\mathbf{D}}((G, A), (V, E))$ is the unique morphism in $\mathbf{D}$ that fulfills $i\overline{h} = h$. □

The next example shows that the functor *Cay* does not preserve equalizers.

Example 3.8. We consider again the semigroup $S = (\mathbb{Z}_6, \cdot)$ from Examples 1.5 and 2.4 where $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \cdots, \overline{5}\}$. Define $f : S \to S$ by $f(\overline{x}) = \overline{x^2}$ for all $x \in \mathbb{Z}_6$. Take $C = \{\overline{0}, \overline{5}\}, C' = \{\overline{0}, \overline{1}, \overline{5}\}$ which are subsets of S. Then $f \in \mathrm{Mor}_{\mathbf{C}}((S, C), (S, C'))$ and (T, D) with the natural embedding i is the equalizer of f and id_S in $\mathbf{C}$ by Lemma 3.6, where $T = \{\overline{0}, \overline{1}, \overline{3}, \overline{4}\}, D = \{\overline{0}\}$, id_S is the identity mapping on S.

By Lemma 3.7 we have $Eq_{\mathbf{D}}(f, id_S) = ((T, E), i)$, where

$$E = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{0}), (\overline{3}, \overline{0}), (\overline{4}, \overline{0}), (\overline{3}, \overline{3})\},$$

and i is the natural embedding. Clearly $(Cay(T,D),i) \neq Eq_{\mathbf{D}}(f, id_S)$, since

$$E(Cay(T,D)) = \{(\overline{0},\overline{0}),(\overline{1},\overline{0}),(\overline{3},\overline{0}),(\overline{4},\overline{0})\} \neq E.$$

Now we turn to find conditions for the reflection of equalizers of the functor Cay. We first give an example to show that given an equalizer $(Cay(T,D), Cay(\alpha)) = Eq_{\mathbf{D}}(Cay(f), Cay(g))$ in category **D** and $(G,A) \in$ **C**, $h \in \mathrm{Mor}_{\mathbf{C}}((G,A),(S,C))$ with $fh = gh$, if a mapping $h^* : G \to T$ with $\alpha h^* = h$ fails to satisfy $h^*(A) \subseteq D$, then we will not get that $((T,D),\alpha) = Eq_{\mathbf{C}}(f,g)$.

Example 3.9. Let $S' = \{a,b,c,d\}$ be a left zero semigroup, $C' = \{a,b,d\}$, and $S = \{a,b,c\}$. Define $g : S \to S'$ by $g(a) = a, g(b) = b, g(c) = d$. Then $g \in \mathrm{Mor}_{\mathbf{C}}((S,S),(S',C'))$. Let ι be the inclusion mapping from S to S'. Now from Lemma 3.7 and the fact that

$$E(Cay(S,S)) = \{(a,a),(b,b),(c,c)\}$$

we have $Eq_{\mathbf{D}}(g,\iota) = (Cay(T,D),i)$, where i is the natural embedding, $T = \{a,b\}$, and $D = \{a\}$.

Now take a 2-element left zero semigroup $G = \{x,y\}$. Define $h : G \to S$ with $h(x) = a, h(y) = b$. Then $h \in \mathrm{Mor}_{\mathbf{C}}((G,G),(S,S))$ and $gh = \iota h$. Moreover, $\overline{h} : G \to T$ satisfies $i\overline{h} = h$, where $\overline{h}(x) = h(x)$ for all $x \in G$. But obviously for $y \in G$, $\overline{h}(y) = b \notin D$ and so $\overline{h}(G) \not\subseteq D$. Indeed, for any $f \in \mathrm{Mor}_{\mathbf{C}}((G,G),(T,D))$, one has $if \neq h$ and thus $((T,D),i) \neq Eq_{\mathbf{C}}(g,\iota)$. We can also deduce from Lemma 3.6 that $Eq_{\mathbf{C}}(g,\iota) = ((\{a,b\},\{a,b\}),i)$ which is different from $((T,D),i)$.

Lemma 3.10. *Consider an equalizer situation*

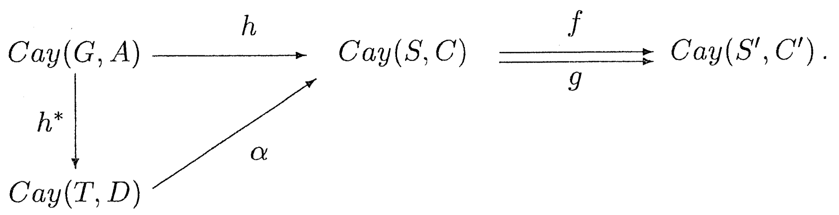
$$Cay(f), Cay(g) : Cay(S,C) \rightrightarrows Cay(S',C').$$

Suppose that $(Cay(T,D),\ Cay(\alpha)) = Eq_D(Cay(f), Cay(g))$. *Then the following statements are equivalent:*

(i) $((T,D),\alpha) = Eq_C(f,g)$;

(ii) *For every* $(G,A) \in C$ *and* $h \in Mor_C((G,A),(S,C))$ *with* $fh = gh$, *for every mapping* $h^* : G \to T$ *with* $\alpha h^* = h$, *one has* $h^*(A) \subseteq D$.

Proof. (ii) $\Rightarrow$ (i). Consider the equalizer situation $Cay(f), Cay(g) : Cay(S,C) \rightrightarrows Cay(S',C')$ in **D** such that $((G,A),h)$ fulfills $fh = gh$ in **C**. Then there exists a unique morphism $h^* : Cay(G,A) \to Cay(T,D)$ in **D** such that the following diagram commutes:



We first claim that h^* is a semigroup homomorphism. Since $(Cay(T,D),\alpha)$ is an equalizer in $\mathbf{D}$, α is a monomorphism in $\mathbf{D}$ and so α is a monomorphism in $\mathbf{C}$ by Corollary 2.2. In addition, α is injective by the fact that every monomorphism in the category of semigroups is injective. For any $x, y \in G$, we have

$$\alpha h^*(xy) = h(xy) = h(x)h(y) = \alpha(h^*(x))\alpha(h^*(y)) = \alpha(h^*(x)h^*(y)),$$

which implies that $h^*(xy) = h^*(x)h^*(y)$. Moreover, $h^*(A) \subseteq D$ by condition (ii). Thus $h^* \in \mathrm{Mor}_{\mathbf{C}}((G,A),(T,D))$ and it is the unique morphism in $\mathbf{C}$ such that $\alpha h^* = h$ by the uniqueness of h^* in $\mathbf{D}$. Therefore $((T,D),\alpha)$ is the equalizer of f and g in $\mathbf{C}$.

(i) $\Rightarrow$ (ii). Assume that $((T,D),\alpha) = Eq_{\mathbf{C}}(f,g)$ and $h \in \mathrm{Mor}_{\mathbf{C}}((G,A),(S,C))$ with $fh = gh$. Then there is a unique morphism $h' : (G,A) \to (T,D)$ in $\mathbf{C}$ such that $\alpha h' = h$. So $h^* = h'$ since α is injective, and then $h^*(A) \subseteq D$. □

Theorem 3.11. *Consider an equalizer situation*

$$Cay(f),\ Cay(g) : Cay(S,C) \rightrightarrows Cay(S',C').$$

Suppose that $(Cay(T,D), Cay(\alpha)) = Eq_{\mathbf{D}}(Cay(f), Cay(g))$. *Set* $D' = \{c \in C \mid f(c) = g(c)\}$. *Then the following statements are equivalent:*

(i) $((T,D),\alpha) = Eq_{\mathbf{C}}(f,g)$;

(ii) $D' \subseteq \alpha(D)$.

Proof. (i) $\Rightarrow$ (ii). Assume (i) holds. Then by Lemma 3.6 one has $((T',D'), i) = Eq_{\mathbf{C}}(f,g)$ with $T' = \{s \in S \mid f(s) = g(s)\}$ and i the natural embedding. Clearly, $(T',D') \in \mathbf{C}$, $i \in \mathrm{Mor}_{\mathbf{C}}((T',D'),(S,C))$ and $fi = gi$. Moreover, since $(Cay(T,D), Cay(\alpha)) = Eq_{\mathbf{D}}(Cay(f), Cay(g))$ there is a unique morphism $h^* : Cay(T',D') \to Cay(T,D)$ such that $\alpha h^* = i$. Now by Lemma 3.10 we have $h^*(D') \subseteq D$, which implies that

$$D' = i(D') = \alpha h^*(D') \subseteq \alpha(D).$$

(ii) $\Rightarrow$ (i). Assume (ii) holds. Suppose that $(G,A) \in \mathbf{C}$ and $h \in \mathrm{Mor}_{\mathbf{C}}((G,A),(S,C))$ with $fh = gh$. Then by the hypothesis since $(Cay(T,D), Cay(\alpha)) = Eq_{\mathbf{D}}(Cay(f), Cay(g))$ there exists a unique morphism $h^* : Cay(G,A) \to Cay(T,D)$ in $\mathbf{D}$ such that $\alpha h^* = h$. By the proof of (ii)$\Rightarrow$(i) in

Lemma 3.10 one has that h^* is a semigroup homomorphism. Since $h(A) \subseteq D'$ and $D' \subseteq \alpha(D)$ by (ii), we have $h(A) \subseteq \alpha(D)$ and hence $\alpha h^*(A) \subseteq \alpha(D)$. So we get that $h^*(A) \subseteq D$ since α is injective. Thus h^* belongs to $\mathbf{C}$ and is the unique morphism in $\mathbf{C}$ such that $\alpha h^* = h$. So $((T,D),\alpha) = Eq_{\mathbf{C}}(f,g)$. □

Note that in Example 3.9 we have $D' = \{a,b\}$ which is not contained in $D = \{a\}$.

Remark 3.12. We consider S and f from Example 3.8. Choose $C = \{\overline{2},\overline{3},\overline{4},\overline{5}\}$. Then $f, id_S \in \mathrm{Mor}_{\mathbf{C}}((S,C),(S,S))$, where id_S is the identity mapping of S. By the fact that

$$E(Cay(S,C)) = \{(\overline{0},\overline{0}),(\overline{1},\overline{2}),(\overline{1},\overline{3}),(\overline{1},\overline{4}),(\overline{1},\overline{5}),(\overline{2},\overline{4}),(\overline{2},\overline{0}),(\overline{2},\overline{2}),(\overline{3},\overline{0}),\\ (\overline{3},\overline{3}),(\overline{4},\overline{2}),(\overline{4},\overline{0}),(\overline{4},\overline{4}),(\overline{5},\overline{4}),(\overline{5},\overline{3})(\overline{5},\overline{2}),(\overline{5},\overline{1})\}$$

and Lemma 3.7, one has $(Cay(T,D),i) = Eq_{\mathbf{D}}(f,id_S)$, where $T = \{\overline{0},\overline{1},\overline{3},\overline{4}\}$, $D = \{\overline{3},\overline{4}\}$ and i is the natural embedding. In addition $((T,D),i) = Eq_{\mathbf{C}}(f,id_S)$ by Lemma 3.6. Here we have $D' = D$. But, nevertheless, C does not act strongly faithfully on S, since for example $\overline{3}\,\overline{3} = \overline{3}\,\overline{5}$ which means that there are "multiple edges" in $Cay(T,D)$.

Remark 3.13. Consider a pullback situation in $\mathbf{D}$

$$\begin{array}{ccc} & & (V_1,E_1) \\ & & \downarrow f_1 \\ (V_2,E_2) & \xrightarrow[f_2]{} & (V,E). \end{array}$$

One can easily check that if

$$V' = \{(v_1,v_2) \in V_1 \times V_2 \mid f_1(v_1) = f_2(v_2)\} \neq \emptyset$$

then the digraph (V',E'), where

$$E' = \{\{(v_1,v_2),(v_1',v_2')\} \in V' \times V' \mid (v_1,v_1') \in E_1, (v_2,v_2') \in E_2\},$$

with the projection $p_i : V' \to V_i, i = 1,2$, is the pullback of (f_1,f_2). By the results of preservations and reflections of products and equalizers of Cay, we conclude that the functor Cay does not preserve and reflect pullbacks.

4. Other product constructions

We consider now so-called box products (cartesian products in [5]), box cross products (strong products in [5]) and lexicographic products of graphs. Note that in the literature these products have many different names. The box product is categorically speaking the tensor product in the category $\mathbf{D}$, see for example [6]. But since we know relatively little about the tensor product in the category of semigroups, it does not make sense to talk about preservation of tensor products in this context.

Remark 4.1. We note first that because of the structure of the coproduct in the category of semigroups (see for example [7]) we cannot say anything about the preservation of coproducts by the Cay functor, but we know that $Cay(S,C) \oplus Cay(S,C') = Cay(S, C \bigcup C')$, where $\oplus$ is the edge sum.

Theorem 4.2. *Let* $\Box$, $\boxtimes$ *denote the box product and the box cross product respectively. Then for monoids* S *and* T *with their subsets* C *and* D *and identities* 1_S *and* 1_T*, we have*

(1) $Cay(S \times T, (1_S \times D) \bigcup (C \times 1_T)) = Cay(S,C) \Box Cay(T,D)$;
(2) $Cay(S \times T, (1_S \times D) \bigcup (C \times 1_T) \bigcup (C \times D)) = Cay(S,C) \boxtimes Cay(T,D)$.

Proof. (1) We have

$$\begin{aligned} E(Cay(S \times T, &(1_S \times D) \bigcup (C \times 1_T))) \\ &= \{\{(s,t),(s,td)\} \mid (s,t) \in S \times T, d \in D\} \\ &\quad \bigcup \{\{(s,t),(sc,t)\} \mid (s,t) \in S \times T, c \in C\} \\ &= E(Cay(S,C) \Box Cay(T,D)). \end{aligned}$$

(2) It is known that the box cross product of two graphs G_1 and G_2 is the edge sum of $G_1 \Box G_2$ and $G_1 \times G_2$. We denote the edge sum of graphs by $\oplus$. Now using Theorem 3.2 and Theorem 4.2 (1), we have

$$\begin{aligned} &Cay(S,C) \boxtimes Cay(T,D) \\ &\quad = (Cay(S,C) \Box Cay(T,D)) \oplus (Cay(S,C) \times Cay(T,D)) \\ &\quad = (S \times T,\ E(Cay(S,C) \Box Cay(T,D)) \bigcup E(Cay(S,C) \times Cay(T,D))) \\ &\quad = (S \times T,\ E(Cay(S \times T, (1_S \times D) \bigcup (C \times 1_T))) \\ &\quad\quad \bigcup E(Cay(S \times T, C \times D))) \\ &\quad = (S \times T,\ E(Cay(S \times T, (1_S \times D) \bigcup (C \times 1_T) \bigcup (C \times D)))) \\ &\quad = Cay(S \times T, (1_S \times D) \bigcup (C \times 1_T) \bigcup (C \times D)). \end{aligned}$$

□

For the case where S and T are groups, the statements of Theorems 3.2 and 4.2 are contained in [4]. Moreover, concerning the lexicographic product of graphs, there it is stated, also for groups G, G', that $Cay(G,C)[Cay(G', C')] \cong Cay(G \times G', (C \times G') \bigcup (1_G \times C'))$, where 1_G is the identity of G. For the situation of semigroups, we have

Theorem 4.3. *Let S be a monoid with identity 1_S, T a semigroup, C and D subsets of S and T respectively. Then $Cay(S \times T, (C \times T) \bigcup (1_S \times D)) = Cay(S, C)[Cay(T, D)]$ if and only if $tT = T$ for any $t \in T$, that is, if and only if T is a right group.*

Proof. It is known that

$$\begin{aligned} &E(Cay(S \times T, (C \times T) \bigcup (1_S \times D))) \\ &\quad = \{\{(s,t),(s,t)(c,t')\} \mid (s,t) \in S \times T, (c,t') \in C \times T\} \\ &\quad \bigcup \{\{(s,t),(s,t)(1_S,d)\} \mid (s,t) \in S \times T, (1_S,d) \in 1_S \times D\}, \\ &E((Cay(S,C)[Cay(T,D)]) \\ &\quad = \{\{(s,t),(sc,t')\} \mid (s,sc) \in Cay(S,C), t,t' \in T\} \\ &\quad \bigcup \{\{(s,t),(s,td)\} \mid s \in S, (t,td) \in Cay(T,D)\}. \end{aligned}$$

Suppose that $Cay(S \times T, (C \times T) \bigcup (1_S \times D)) = Cay(S,C)[Cay(T,D)]$. Then for any $t,t' \in T$ and $\{(s,t),(sc,t')\} \in Cay(S,C)[Cay(T,D)]$, where $(s,sc) \in Cay(S,C)$, we have $t' = tx$ for some $x \in T$. So $T \subseteq tT$ and then $T = tT$ for any $t \in T$.

On the contrary, suppose that $tT = T$ for any $t \in T$. Then for any arc $\{(s,t),(s',t')\}$ in $Cay(S,C)[Cay(T,D)]$, one has that either $s = s'$, $t' = td$ for some $d \in D$ or $s' = sc$ for some $c \in C$ and $t,t' \in T$. But for any $t,t' \in T$, there
is a $y \in T$ such that $t' = ty$ by assumption. Therefore $\{(s,t),(s',t')\}$ is an arc of $Cay(S \times T, (C \times T) \bigcup (1_S \times D))$, and so $Cay(S,C)[Cay(T,D)] \subseteq Cay(S \times T, (C \times T) \bigcup (1_S \times D))$. The inclusion $Cay(S \times T, (C \times T) \bigcup (1_S \times D)) \subseteq Cay(S,C)[Cay(T,D)]$ is obvious. □

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References

[1] Sr. Arworn, U. Knauer, and N. Na Chiangmai, *Characterization of digraphs of right (left) zero unions of groups*, Thai J. Math. **1** (2003), 131–140.

[2] P. Hell and J. Nešetřil, *Cohomomorphisms of graphs and hypergraphs*, Math. Nachr. **87** (1979), 53–61.

[3] H. Herrlich and G. E. Strecker, *Category Theory: an Introduction*, Allyn and Bacon, Boston, Mass., 1973.

[4] M.-C. Heydemann, *Cayley graphs and interconnection networks*; In: *Graph Symmetry* (Montreal, PQ, 1996), Nato Adv. Sci. Inst. Ser. C Math. Phys. Sci. **497**, 167–224, Kluwer Acad. Publ., Dordrecht, 1997.

[5] W. Imrich and S. Klavžar, *Product Graphs. Structure and Recognition*, Wiley-Interscience, New York, 2000.

[6] M. Kilp and U. Knauer, *Graph operations and categorical constructions*, Acta Comment. Univ. Tartuensis Math. **5** (2001), 43–57.
[7] M. Kilp, U. Knauer, and A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, 2000.
[8] S. Panma, U. Knauer, and Sr. Arworn, *On transitive Cayley graphs of right (left) groups and of Clifford semigroups*, Thai J. Math. **2** (2004), 183–195.
[9] M. Petrich and N. R. Reilly, *Completely Regular Semigroups*, John Wiley & Sons, New York, 1999.

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