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Weak metric approximation properties and nice projections

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ABSTRACT. We prove that a Banach space X has the weak MAP (the weak MCAP) [the very weak MCAP] if and only if there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ ($\mathcal{K}(X,X)$) [$\mathcal{K}(X,X^{**})$] in $\mathcal{L}(X^{**},X^{**})$.

1. Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(Y,X)$ the Banach space of bounded linear operators from Y to X, and by $\mathcal{F}(Y,X)$, $\mathcal{K}(Y,X)$, $\mathcal{W}(Y,X)$ its subspaces of finite rank operators, compact operators, and weakly compact operators, respectively.

We denote by $X\hat{\otimes}_\pi Y$ the (completed) projective tensor product of X and Y. Recall that we may identify the dual of $X\hat{\otimes}_\pi Y$ with $\mathcal{L}(Y,X^*)$ and that the action of an operator $T : Y \to X^*$, as a linear functional on $X\hat{\otimes}_\pi Y$, is given by

$$\left\langle T, \sum_{n=1}^{\infty} x_n \otimes y_n \right\rangle = \sum_{n=1}^{\infty} (Ty_n)(x_n).$$

Let I_X denote the identity operator on X. Recall that X is said to have the *approximation property (AP)* if there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. If the net (S_α) can be chosen such that $\sup_\alpha \|S_\alpha\| \le 1$, then X is said to have the *metric approximation property (MAP)*.

In [8] Lima and Oja introduced and studied the weak metric approximation property. Following Lima and Oja a Banach space X is said to have the *weak metric approximation property (weak MAP)* if, for every Banach space

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Y and every operator $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.

It is immediate from the definitions that MAP $\Rightarrow$ weak MAP $\Rightarrow$ AP. However, the AP does not imply the weak MAP in general as was shown in [8, Proposition 2.2]. Recently it was also shown [10, Corollary 1] that if a Banach space has the weak MAP then it has the MAP if either its dual or its bidual has the Radon-Nikodým property (RNP). It is, however, not known whether the weak MAP and the MAP in general are equivalent properties.

Let X be a subspace of a Banach space Y. A linear operator $\varphi : X^* \to Y^*$ is called a *Hahn-Banach extension operator* if $(\varphi x^*)(x) = x^*(x)$ and $\|\varphi x^*\| = \|x^*\|$ for every $x \in X$ and $x^* \in X^*$. We denote the set of Hahn-Banach extension operators $\varphi : X^* \to Y^*$ by $\mathrm{HB}(X,Y)$. It is easy to show that $\mathrm{HB}(X,Y)$ is non-void if and only if X is an ideal in Y (in the sense of Godefroy, Kalton, and Saphar [2]).

The following result [5, Proposition 2.1 and Proposition 2.5] of Lima establishes a connection between the weak MAP and the existence of a Hahn-Banach extension operator.

Theorem 1.1 (Lima). *Let X be a Banach space. Then X has the weak MAP if and only if there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

Note that we can consider $\mathcal{F}(X,X)$ as a subspace of $\mathcal{L}(X^{**}, X^{**})$ through the embedding operator which maps an operator $T \in \mathcal{F}(X,X)$ to its second adjoint $T^{**} \in \mathcal{L}(X^{**}, X^{**})$.

In Section 2 we improve Theorem 1.1 by showing that we can replace the Hahn-Banach extension operator $\varphi : X^* \to X^{***}$ by a Hahn-Banach extension operator $\varphi_P : X^* \to X^{***}$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}. This result is then thereafter used to improve other characterizations of the weak MAP.

In Section 3 we establish characterizations similar to those in Section 2 for two, recently introduced [6], natural compact companions of the weak MAP.

We will consider Banach spaces over the real scalar field only. We use standard Banach space notation, as can be found e.g. in [9]. The closed unit ball of a Banach space X is denoted by B_X and the unit sphere of X by S_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$, its linear span by $\mathrm{span}A$, and its convex hull by $\mathrm{conv}A$. We will write X^* for the dual of X.

2. The weak MAP

We might ask what more can be said about the Hahn-Banach extension operator in Theorem 1.1? In fact, by using a technique of Godefroy and Kalton from [1], we will prove that we can replace the Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ in Theorem 1.1 by a Hahn-Banach extension operator $\varphi_P \in \mathrm{HB}(X, X^{**})$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}. More explicitly we have the following theorem.

Theorem 2.1. *Let X be a Banach space.*

(a) *If P is a norm one projection on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$, then there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

(b) *If there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$, then there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

Proof. (a) Assume that there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$. Then put $\varphi_P = P^* i_{X^*}$ where $i_{X^*} : X^* \to X^{***}$ is the natural embedding of X^* into X^{***}. Finally observe that $\varphi_P : X^* \to X^{***}$ is a Hahn-Banach extension operator such that $\varphi_P^*|_{X^{**}} = P$.

(b) We use an argument from the proof of [1, Theorem III.1]. Assume that there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$. Now, pick a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha^{**} \to \varphi^*|_{X^{**}}$ weak* in $\mathcal{L}(X^{**}, X^{**})$. Let $\mathfrak{S}$ be the convex semi-group generated by the net (S_α^{**}), i.e. the smallest convex semi-group in $\mathcal{L}(X^{**}, X^{**})$ that contains (S_α^{**}). Let $\mathfrak{S}^*$ denote the weak*-closure of $\mathfrak{S}$. Now $\mathfrak{S}^*$ is a convex semi-group. To see this let U and V be in $\mathfrak{S}^*$ and write

$$U = w^*\text{-}\lim U_\alpha^{**}$$
$$V = w^*\text{-}\lim V_\beta^{**},$$

where U_α^{**} and V_β^{**} are in $\mathfrak{S}$. Choose $u = \sum_{n=1}^\infty x_n^* \otimes x_n^{**} \in X^* \hat{\otimes}_\pi X^{**}$ arbitrarily. Then it follows that

$$\begin{aligned} UV(u) &= \lim_\alpha \sum_{n=1}^\infty \langle x_n^*, U_\alpha^{**} V x_n^{**}\rangle = \lim_\alpha \sum_{n=1}^\infty \langle U_\alpha^* x_n^*, V x_n^{**}\rangle \qquad (1) \\ &= \lim_\alpha \lim_\beta \sum_{n=1}^\infty \langle U_\alpha^* x_n^*, V_\beta^{**} x_n^{**}\rangle = \lim_\alpha \lim_\beta (U_\alpha V_\beta)^{**}(u). \end{aligned}$$

Hence $UV \in \mathfrak{S}^*$. It is obvious that $\mathfrak{S}^*$ is convex.

Now put $\mathfrak{S}_0{}^* = \{T \in \mathfrak{S}^* : T|_X = I_X, \|T\| = 1\}$. Note that $\mathfrak{S}_0{}^* \neq \emptyset$ since $\varphi^*|_{X^{**}} \in \mathfrak{S}_0{}^*$. Since $\mathfrak{S}_0{}^*$ is closed under composition, it is a semi-group. It is straightforward to show that it is convex and weak*-closed.

Equip $\mathfrak{S}_0{}^*$ with the order-relation $\leq$ defined by $S \leq T$ if $\|Sx^{**}\| \leq \|Tx^{**}\|$ for every $x^{**} \in X^{**}$. Now let N be any maximal chain in $(\mathfrak{S}_0{}^*, \leq)$ and for

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$S \in N$ let $N_S = \{T \in N : T \leq S\}$. We can write $N = \bigcup_{S \in N} N_S$. Note that each N_S is weak*-closed. Indeed, choose a net (V_α) in N_S and assume $V_\alpha \underset{\alpha}{\to} V'$ weak*, where $V' \in \mathfrak{S}_0^*$. Then for every $x^{**} \in X^{**}$ we get

$$\|V'x^{**}\| \leq \liminf_{\alpha} \|V_\alpha x^{**}\| \leq \|Sx^{**}\|.$$

By the maximality of N it follows that $V' \in N$ so N_S is weak*-closed. Now choose $(S_i)_{i=1}^n \subset N$ arbitrarily. Then $(N_{S_i})_{i=1}^n$ is a finite family of weak*-closed sets and

$$\bigcap_{i=1}^n N_{S_i} = \{T \in N : T \leq \min_{1 \leq i \leq n} S_i\} \neq \emptyset.$$

Since ${\mathfrak{S}_0}^*$ is weak*-compact, every family of closed sets having the finite intersection property has a non-void intersection. Hence $\bigcap_{S \in N} N_S \neq \emptyset$. By the Hausdorff maximality theorem every chain is contained in a maximal chain. Hence, by the above argument, every chain in ${\mathfrak{S}_0}^*$ has a lower bound. It now follows by Zorn's lemma that ${\mathfrak{S}_0}^*$ has a minimal element. Denote such a minimal element by P.

We now show that P is a projection of norm one. Since P is minimal and $\|S\| = 1$ for all $S \in {\mathfrak{S}_0}^*$ we have $\|SPx^{**}\| = \|Px^{**}\|$ for all $S \in {\mathfrak{S}_0}^*$ and all $x^{**} \in X^{**}$. Applying this observation to

$$S_n = \frac{1}{n}(\sum_{i=1}^n P^i),$$

which by convexity is in ${\mathfrak{S}_0}^*$, gives

$$\begin{aligned} \|(S_nP^2 - S_nP)x^{**}\| &= \|S_nP(Px^{**} - x^{**})\| \\ &= \|P(Px^{**} - x^{**})\| \\ &= \|P^2x^{**} - Px^{**}\|. \end{aligned}$$

Since we have

$$S_nP^2 - S_nP = \frac{1}{n}(P^{n+2} - P^2),$$

we get that $\|P^2x^{**} - Px^{**}\| \leq 2/n$ for all $n \geq 1$. It follows that P is a projection on X^{**} such that P is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$. By the definition of ${\mathfrak{S}_0}^*$, P is of norm one and $X \subset P(X^{**})$. □

In fact we can do slightly better than in Theorem 2.1. The result below tells us that we may assume that the net converging weak* to the projection satisfies some boundedness property.

Proposition 2.2. *Let X be a Banach space with the weak MAP. Then there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that for every reflexive Banach space Y and for every $T \in \mathcal{W}(X, Y)$, there exists a net*

$(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to P$ weak in $\mathcal{L}(X^{**},X^{**})$.*

Proof. Let $\epsilon > 0$, let Y be a reflexive Banach space, and let $T \in \mathcal{W}(X,Y)$ of norm one. Let $u_k = \sum_{n=1}^{\infty} x_{k,n}^* \otimes x_{k,n}^{**} \in X^*\hat{\otimes}_\pi X^{**}$ for $k = 1,...,m$. Assume $\sum_{n=1}^{\infty} \|x_{k,n}^{**}\| < \infty$ and $1 \geq \|x_{k,n}^*\| \to 0$ for each $k = 1,...,m$. Put $K = \overline{\text{conv}}\{\pm x_{k,n}^* : k = 1,...,m; n = 1,2,...\} \subset B_{X^*}$. Let Z be the Banach space constructed from K in the factorization lemma [7, Lemma 1.1], and let $J : Z \to X^*$ be the identity embedding of Z into X^*. Now Z is separable, reflexive and $J \in \mathcal{K}(Z,X^*)$ is of norm one. Define a map $V : X \to Z^* \oplus_\infty Y$ by $Vx = (J^*x, Tx)$. Note that $V \in \mathcal{W}(X, Z^* \oplus_\infty Y)$. By Theorem 1.1 and Theorem 2.1 there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$. Note that $V^{**}P$ is in the weak*-closure of the convex set $\{V^{**}S^{**} : S \in \mathcal{F}(X,X)\}$ in $\mathcal{W}(X^{**}, Z^* \oplus_\infty Y)$. Since $Z^* \oplus_\infty Y$ is reflexive we have, by [3, Theorem 1.5], that $V^{**}P$ is in the weak*-closure of

$$\{V^{**}S^{**} : S \in \mathcal{F}(X,X), \|V^{**}S^{**}\| < \|V^{**}P\| + \epsilon\}$$

in $\mathcal{W}(X^{**}, Z^* \oplus_\infty Y)$, which again is a subset of the weak*-closure of

$$\{V^{**}S^{**} : S \in \mathcal{F}(X,X), \|VS\| < 1 + \epsilon\}$$

in $\mathcal{W}(X^{**}, Z^* \oplus_\infty Y)$. Now choose $z_{k,n} \in B_Z$ such that $Jz_{k,n} = x_{k,n}^*$ for all k and n. Find S in the above set such that

$$\begin{aligned}
\epsilon &> \max_{1\leq k\leq m} |V^{**}S^{**}(\sum_{n=1}^{\infty}(z_{k,n},0) \otimes x_{k,n}^{**}) - V^{**}P(\sum_{n=1}^{\infty}(z_{k,n},0)\otimes x_{k,n}^{**})| \\
&= \max_{1\leq k\leq m} |\sum_{n=1}^{\infty} \langle z_{k,n}, J^*S^{**}x_{k,n}^{**}\rangle - \sum_{n=1}^{\infty}\langle z_{k,n}, J^*Px_{k,n}^{**}\rangle| \\
&= \max_{1\leq k\leq m} |\sum_{n=1}^{\infty} \langle x_{k,n}^*, S^{**}x_{k,n}^{**}\rangle - \sum_{n=1}^{\infty}\langle x_{k,n}^*, Px_{k,n}^{**}\rangle|.
\end{aligned}$$

Since $\|TS\| \leq \|VS\| \leq 1 + \epsilon$, the result follows. □

When the space X is separable and does not contain a copy of l_1, we know even more about the projection.

Corollary 2.3. *Let X be a separable Banach space not containing l_1. Then there exists a Hahn-Banach extension operator $\varphi \in I\!B(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$ if and only if there exists a norm one projection P on X^{**} with weak*-closed kernel and with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

Proof. This follows directly from Theorem 2.1 and [1, Claim III.2]. □

Building on Theorem 2.1, we arrive at the result below. This improves [5, Propositions 2.5 and 3.1] in the way that the Hahn-Banach extension operator $\varphi : X^* \to X^{***}$, in each of these results, is replaced by a Hahn-Banach extension operator $\varphi_P : X^* \to X^{***}$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}.

Theorem 2.4. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the weak-MAP.*
(b) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*
(c) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every operator $T \in \mathcal{W}(Y,X^{**})$, one has $PT \in \mathcal{F}(Y,X)^{**}$.*
(d) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every separable reflexive Banach space Y and every operator $T \in \mathcal{K}(Y,X^{**})$, one has $PT \in \mathcal{F}(Y,X)^{**}$.*

Proof. (a)⇔(b) follows from Theorem 1.1 and Theorem 2.1.

(b)⇒(c) is obtained by the same reasoning as in [5, Proposition 3.1 (a)⇒ (b)].

(c)⇒(d) is trivial.

(d)⇒(a) is obtained by the same reasoning as in [5, Proposition 3.1 (c)⇒ (a)]. □

3. The weak MCAP and the very weak MCAP

Recently Lima and Lima [6] introduced and investigated two approximation properties that are natural compact companions of the weak MAP. Following [6], a Banach space X has the *weak metric compact approximation property (weak MCAP)* if, for every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{K}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Moreover, X is said to have the *very weak metric compact approximation property (very weak MCAP)* if for every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$ there exists a net $(S_\alpha) \subset \mathcal{K}(X,X^{**})$ with $\sup_\alpha \|T^{**}S_\alpha\| \leq \|T\|$ such that $\lim_\alpha \operatorname{tr}(S_\alpha u) = \operatorname{tr}(I_X u)$ for every $u \in X^*\hat{\otimes}_\pi X$. By comparing the definitions, it is immediate that the following implications hold:

$$\text{weak MAP} \Rightarrow \text{weak MCAP} \Rightarrow \text{very weak MCAP}.$$

As pointed out in [6, Remark 5.2], there is a space with the very weak MCAP, but without the weak MCAP. Moreover, the space of Willis [11, Proposition 4] has the weak MCAP, but not the weak MAP.

It should be noted that results similar to Theorem 2.1 also hold for the weak MCAP and the very weak MCAP. The results differ from Theorem 2.1

only in the way that $\mathcal{F}(X,X)$ is replaced by $\mathcal{K}(X,X)$ in the weak MCAP case, and $\mathcal{K}(X,X^{**})$ in the very weak MCAP case. The proofs of these results are verbatim to that of Theorem 2.1, using $\mathcal{K}(X,X)$ and $\mathcal{K}(X,X^{**})$ instead of $\mathcal{F}(X,X)$ respectively. The reason why the arguments work is that the image of the second adjoint of a compact operator is a subspace of the range space of the operator itself. Hence the calculation in (1) holds.

Proposition 3.1. *Let X be a Banach space.*

(a) *If P is a norm one projection on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $K(X,X)$ $[\mathcal{K}(X,X^{**})]$ in $\mathcal{L}(X^{**},X^{**})$, then there exists a Hahn-Banach extension operator $\varphi \in H\!B(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{K}(X,X)$ $[\mathcal{K}(X,X^{**})]$ in $\mathcal{L}(X^{**},X^{**})$.*

(b) *If there exists a Hahn-Banach extension operator $\varphi \in H\!B(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{K}(X,X)$ $[\mathcal{K}(X,X^{**})]$ in $\mathcal{L}(X^{**},X^{**})$, then there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $K(X,X)$ $[\mathcal{K}(X,X^{**})]$ in $\mathcal{L}(X^{**},X^{**})$.*

By applying these results in companion with the proof of [5, Proposition 3.1] and the proofs of [6, Theorem 4.3] and [6, Theorem 5.3], we obtain the following strengthenings of [6, Theorem 4.3] for the weak MCAP case, and [6, Theorem 5.3] for the very weak MCAP case. The results improve [6, Theorem 4.3] and [6, Theorem 5.3] in the way that the Hahn-Banach extension operator $\varphi : X^* \to X^{***}$ in each of these theorems is replaced by a Hahn-Banach extension operator $\varphi_P : X^* \to X^{***}$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}.

Theorem 3.2. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the weak MCAP.*

(b) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{K}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

(c) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{W}(Y,X^{**})$, one has $PT \in \mathfrak{C}^{**}$ where $\mathfrak{C} = \{S^{**}T : S \in \mathcal{K}(X,X)\} \subset \mathcal{K}(Y,X)$.*

(d) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every separable reflexive Banach space Y and every $T \in \mathcal{K}(Y,X^{**})$, one has $PT \in \mathfrak{C}^{**}$ where $\mathfrak{C}$ is as in (c).*

(e) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for all sequences $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ with $\sum_{n=1}^{\infty} \|x_n^*\|\|x_n^{**}\| < \infty$ and $\sum_{n=1}^{\infty} x_n^{**}(S^*x_n^*) = 0$ for all $S \in \mathcal{K}(X,X)$, one has $\sum_{n=1}^{\infty} x_n^{**}(P^*x_n^*) = 0$.*

Proof. (a)⇔(b) follows from [6, Theorem 4.3 (a)⇔(b)] and Proposition 3.1.

(b)⇒(c) is similar to the proof of [5, Proposition 3.1 (a)⇒(b)].

(c)⇒(d) is trivial.

(d)⇒(e) is similar to the proof of [6, Theorem 4.3 (f)⇒(g)].

(e)⇒(b) is trivial. □

Theorem 3.3. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the very weak MCAP.*
(b) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{K}(X, X^{**})$ in $\mathcal{L}(X^{**}, X^{**})$.*
(c) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{W}(X, Y)$, one has $T^{**}P \in \mathfrak{E}^{**}$ where $\mathfrak{E} = \{T^{**}S : S \in \mathcal{K}(X, X^{**})\} \subset \mathcal{K}(X, Y)$.*
(d) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{K}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{K}(X, X^{**})$ with $\sup_\alpha \|T^{**}S_\alpha\| \leq \|T\|$ such that $w^*\text{-}\lim_\alpha S_\alpha^* T^* y = P^* T^* y^*$ in X^{***} for all $y^* \in Y^*$.*
(e) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{K}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{K}(X, X^{**})$ with $\sup_\alpha \|T^{**}S_\alpha\| \leq \|T\|$ such that $T^{**}S_\alpha^{**} \to T^{**}P$ in the strong operator topology.*
(f) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for all sequences $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ with $\sum_{n=1}^\infty \|x_n^*\|\|x_n^{**}\| < \infty$ and $\sum_{n=1}^\infty x_n^{**}(S^* x_n^*) = 0$ for all $S \in \mathcal{K}(X, X^{**})$, one has $\sum_{n=1}^\infty x_n^{**}(P^* x_n^*) = 0$.*

Proof. (a)⇔(b) follows from [6, Theorem 5.3 (a)⇔(b)] and Proposition 3.1.

(b)⇒(c)⇒(d)⇒(e) are similar to the proofs of (b)⇒(c)⇒(d)⇒(e) in [6, Theorem 5.3] respectively.

(e)⇒(f). Let $\epsilon > 0$, let $u = \sum_{n=1}^\infty x_n^* \otimes x_n^{**} \in X^* \hat{\otimes}_\pi X^{**}$, and assume $\sum_{n=1}^\infty \|x_n^{**}\| < \infty$ and $1 \geq \|x_n^*\| \to 0$. Put $K = \overline{\text{conv}}\{\pm x_n^* : n = 1, 2, \dots\} \subset B_{X^*}$. Let Z be the Banach space constructed from K in the factorization lemma [7, Lemma 1.1], and let $J : Z \to X^*$ be the identity embedding of Z into X^*. Now Z is separable, reflexive and $J \in \mathcal{K}(Z, X^*)$ is of norm one. Choose $z_n \in B_Z$ such that $Jz_n = x_n^*$ for every $n \in \mathbb{N}$. From the assumption it follows that there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ and a net $(S_\alpha) \subset \mathcal{K}(X, X^{**})$ with $\sup_\alpha \|(J^*|_X)^{**} S_\alpha\| \leq 1$ such that $(J^*|_X)^{**} S_\alpha^{**} \to (J^*|_X)^{**} P$ in the strong operator topology. Since $((J^*|_X)^{**} S_\alpha)$ is bounded, we may assume that the net converges to $(J^*|_X)^{**} P$ in the topology τ of uniform convergence on compact sets in X^{**}. By the description of $(\mathcal{L}(X^{**}, Z^*), \tau)^*$, due to Grothendieck [4] (see e.g. [9, Proposition

1.e.3]), it now follows that there exists an $S \in \mathcal{K}(X, X^{**})$ such that

$$\begin{aligned}
\epsilon &> |\sum_{n=1}^{\infty} \langle (J^*|_X)^{**} S^{**} x_n^{**}, z_n \rangle - \sum_{n=1}^{\infty} \langle (J^*|_X)^{**} P x_n^{**}, z_n \rangle | \\
&= |\sum_{n=1}^{\infty} \langle S^{**} x_n^{**}, J z_n \rangle - \sum_{n=1}^{\infty} \langle P x_n^{**}, J z_n \rangle | \\
&= |\sum_{n=1}^{\infty} \langle S^{**} x_n^{**}, x_n^* \rangle - \sum_{n=1}^{\infty} \langle P x_n^{**}, x_n^* \rangle |,
\end{aligned}$$

and we are done.

(f)⇒(b) is clear by using the Hahn–Banach theorem. □

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References

[1] G. Godefroy and N. J. Kalton, *Approximating sequences and bidual projections*, Quart. J. Math. Oxford Series (2) **48** (1997), 179–202.

[2] G. Godefroy, N. J. Kalton, and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59.

[3] G. Godefroy and P. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. **32** (1988), 672–695.

[4] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[5] V. Lima, *The weak metric approximation property and ideals of operators*, Preprint.

[6] V. Lima and Å. Lima, *Geometry of spaces of compact operators*, Preprint.

[7] Å. Lima, O. Nygaard, and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. **119** (2000), 325–348.

[8] Å. Lima and E. Oja, *The weak metric approximation property*, Math. Ann. **333** (2005), 471–484.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer, Berlin–Heidelberg–New York, 1977.

[10] E. Oja, *The impact of the Radon-Nikodým property on the weak bounded approximation property*, Rev. R. Acad. Cien. Serie A. Mat. **100** (2006), 325–331.

[11] G. Willis, *The compact approximation property does not imply the approximation property*, Studia Math. **103** (1992), 99–108.

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