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On shift permutation invariance, covariance structures, and Toeplitz matrices

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Abstract. The objective of this paper is to present a comprehensive study of shift permutation invariant covariance structures in linear models. It follows that the corresponding covariance matrices belong to the class of symmetric circular Toeplitz matrices. Results for the spectrum of shift permutation invariant covariance matrices of random factors in linear models are given. Among others the results are of use when reparametrization conditions are imposed in order to perform inference based on a unique set of parameters.

1. Introduction

In this paper the spectrum of covariance matrices in linear models with random factors which are shift permutation invariant are studied. It is a follow up paper of Nahtman (2006) where, among others, the spectrum of covariance matrices of permutation invariant linear models was considered. The class of shift permutations provide more structure on the covariance matrices than the class of permutations. The main idea is to deepen the knowledge about designs which are invariant under some kind of permutations. In particular, we want to have parametrizations which give unique set of parameters. Usually this is achieved by putting constraints on the parameters in the model and our main result is of help to find these constraints so that the model is still shift permutation invariant. In this work we are only interested in model formulations and not how to obtain explicit expressions for the estimators. However, results concerning estimation of maximum likelihood estimators (MLEs) are long time available (e.g. see Andersson, 1975; Marin and Dhorne, 2002, 2003).

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The best way to describe invariance properties of random factors, including interactions, is via their covariance matrices. Because of invariance it appears that the natural quantities to study are the eigenvalues and eigenvectors of covariance matrices. It is easy to imagine that restrictions on the factor levels will lead to singular covariance matrices with eigenvalues equal to 0. The corresponding eigenvectors then tell us what kind of restrictions can be put on the factors. For example, the restriction which puts the sum of factor levels to 0 and which does not violate the assumption of exchangeability.

In the present paper we are going to study shift invariance in $K-$way tables. This leads to covariance matrices with Toeplitz-structures. It is interesting to note that in practice shift invariance is natural but that this property is not always taken into account when modelling data. Since eigenvalues and eigenvectors of Toeplitz matrices are known, we start with this observation and shall extend the results to the study of higher order interactions of factors. The approach implies that covariance matrices are build up with the help of Kronecker products. In Section 2 spectral properties of symmetric Toeplitz matrices are given, Section 3 connects shift invariance with Toeplitz covariance matrices and Section 4 considers the spectrum of shift permutation invariant covariance matrices.

2. Preliminaries and definitions

An $n \times n$ matrix T of the form

$$T = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_1 \\ t_1 & t_0 & t_1 & \cdots & t_2 \\ t_2 & t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_1 \\ t_1 & t_2 & \cdots & t_1 & t_0 \end{pmatrix} = Toep(t_0, t_1, t_2, \ldots, t_1) \tag{2.1}$$

is called a *symmetric circular Toeplitz matrix.* The matrix T depends on $[n/2]+1$ parameters, where $[\bullet]$ stands for the integer part, and $t_{i,j} = t_{|i-j|}$, $i,j = 1,\ldots,n$.

A symmetric circular matrix $SC(n,k)$ is defined in the following way:

$$SC(n,k) \;=\; Toep(\overbrace{\underbrace{0,\ldots,0}_{k},1,0,\ldots,0,1,\underbrace{0,\ldots,0}_{k-1}}^{n}), \tag{2.2}$$

where $k \in \{1,\ldots,[n/2]\}$. Note that $SC(n,k)$ can be written as

$$SC(n,k) = \sum_{\substack{i,j \\ |i-j|= \\ k,n-k}} e_i e_j', \tag{2.3}$$

where e_h is the h-th column of the identity matrix I_n, $h = 1, \ldots, n$. For notational convenience denote $SC(n, 0) = I_n$. It is easy to see that

$$Toep(t_0, t_1, t_2, \ldots, t_1) = t_0 I_n + \sum_{i=1}^{[n/2]} t_i SC(n, i) = \sum_{i=0}^{[n/2]} t_i SC(n, i).$$

Since

$$a_0 I_n + \sum_{j=1}^{[n/2]} a_i SC(n, i) = 0,$$

it follows that $a_0 = \ldots = a_l = 0$, i.e. $I_n, SC(n, 1), \ldots, SC(n, l)$ are linearly independent.

The spectral properties of symmetric Toeplitz matrices can be found in Davis (1979) or Basilevsky (1983). We present some additional results concerning multiplicities of the eigenvalues of such matrices.

Let λ_k, $k = 1, \ldots, n$, be eigenvalues of the matrix $T : n \times n$. The following lemma gives the spectral property of the matrix T.

Lemma 2.1. *Let $T : n \times n$ be a symmetric Toeplitz matrix with elements as in (2.1). If n is odd*

$$\lambda_k = t_0 + 2 \sum_{j=1}^{[n/2]} t_j \cos(2\pi kj/n).$$

There is only one eigenvalue λ_n which has multiplicity 1 *and all other eigenvalues are of multiplicity* 2.

If n is even

$$\lambda_k = t_0 + 2 \sum_{j=1}^{n/2-1} t_j \cos(2\pi kj/n) + t_{n/2} \cos(\pi k).$$

There are only two eigenvalues $\lambda_n, \lambda_{n/2}$ which have multiplicity 1 *and all other eigenvalues are of multiplicity* 2.

The eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ are

$$v_k = n^{-1/2}(v_{k1}, \ldots, v_{kn})' \tag{2.4}$$

with

$$v_{ki} = \cos(2\pi ik/n) + \sin(2\pi ik/n), \ i = 1, \ldots, n.$$

Proof. For derivation of the eigenvalues and eigenvectors we refer the reader to Basilevsky (1983).

If n is odd, we can see that $\lambda_k = \lambda_{n-k}$, $k = 1, \ldots, n-1$, and $\lambda_n = t_0 + 2\sum_{j=1}^{(n-1)/2} t_j$.

If n is even, then for $k \neq n, n/2 : \lambda_k = \lambda_{n-k}$. However, the eigenvalues

$$\lambda_n = t_0 + 2\sum_{j=1}^{n/2-1} t_j \cos(2\pi j) + t_{n/2}\cos(\pi n),$$

$$\lambda_{n/2} = t_0 + 2\sum_{j=1}^{n/2-1} t_j \cos(\pi j) + t_{n/2}\cos(\pi n/2)$$

are distinct. □

It is worth observing that from (2.2) it follows that Lemma 2.1 immediately gives eigenvalues and eigenvalues for $SC(n,k)$. Moreover, eigenvectors given in (2.4) do not depend on the elements in (2.1). One important consequence of this result is that any two symmetric circular Toeplitz matrices always commute.

3. Shift permutation invariance and Toeplitz covariance matrices

An orthogonal matrix $P: n \times n$ is a *shift permutation matrix* if

$$p_{ij} = \begin{cases} 1, & \text{if } j = i + 1 - nI_{(i>n-1)} \\ 0, & \text{otherwise} \end{cases},$$

where

$$I_{(a>b)} = \begin{cases} 1, & \text{if } a > b \\ 0, & \text{otherwise} \end{cases}.$$

Suppose that we have observations $Y_{i_1,i_2,\ldots,i_k}$ for which we assume a model consisting of k random factors, i.e., the observations $Y_{i_1,i_2,\ldots,i_k}$ form a so-called *K-way* table. A crucial assumption will be that if we permute the levels of one factor, the others will not be affected. This leads to the concept of marginal permutations (see Nahtman, 2006). For $k = 2$ we have $Y_{i_1 i_2}$, i.e., a matrix $Y = (Y_{ij})$. Invariance implies that we can post-multiply Y by a shift matrix $P^{(1)'}$ and $YP^{(1)'}$ will have the same distribution as Y. Observe that $P^{(1)'}$ affects the index j in Y_{ij} and if $E(Y) = 0$, invariance means $D(YP^{(1)'}) = D(Y)$. If we want to permute the i-index, we look at $P^{(2)}Y$. Furthermore, if we intend to permute the indices i and j independently of each other, we study $P^{(2)}YP^{(1)'}$, or, equivalently, $(P^{(2)} \otimes P^{(1)})y$, where y is the vector of lexicographically ordered observations. In the case of several factors we can repeat the arguments and obtain the next theorem.

Theorem 3.1. *In a K-way table the structure of the shift permutation matrix P_k equals*

$$P_k = P^{(k)} \otimes \cdots \otimes P^{(1)}, \tag{3.1}$$

where $P^{(h)}$ *are shift permutation matrices,* $h = 1, \dots, k$.

The matrix P_k defined in Theorem 3.1 is called a *marginal shift permutation matrix of order k*.

Definition 3.1. Let Q be an arbitrary element of a group $\mathcal{G}$ of one-to-one transformations. The covariance matrix $D(\xi)$ of a factor ξ is called *invariant with respect to* $\mathcal{G}$ if $D(\xi) = D(Q\xi)$ or, equivalently, $D(\xi) = QD(\xi)Q'$.

In this paper we are not going to discuss the covariance structure of the vector of observations y. Instead we solely focus on the underlying random factors. Based on our results for these factors we can immediately consider observations but this is a straightforward exercise. For example, for

$$y_{ijk} = \xi_i^1 + \xi_j^2 + \gamma_{ij}^{(2)} + \varepsilon_{ijk},$$

where $\xi^1=(\xi_i^1) \sim N(0, \Sigma_{\xi^1})$, $\xi^2=(\xi_j^2) \sim N(0, \Sigma_{\xi^2})$, $\gamma^{(2)}=(\gamma_{ij}^{(2)}) \sim N(0, \Sigma_{\gamma^{(2)}})$, $\varepsilon = (\varepsilon_{ijk}) \sim N(0, \Sigma_\varepsilon)$ are independent, results are obtained when we have knowledge about the factors ξ^1, ξ^2, $\gamma^{(2)}$, and ε. Here $\gamma^{(2)}$ is a second order interaction factor.

In the subsequent we are going to study an s-order interaction factor $\gamma^{(s)}$ with $D(\gamma^{(s)}) = \Sigma_s$. The next theorems show that the invariance has strong implications on the structure of the covariance matrix. We first present two special cases which are of interest in applications but then also serve as a basis in a proof by induction which will be used for proving the general statement.

Theorem 3.2. *The covariance matrix* Σ: $n_1 \times n_1$ *of the factor* ξ *is shift permutation invariant if and only if it is a symmetric circular Toeplitz matrix:*

$$\Sigma = Toep(\tau_0, \tau_1, \tau_2, \dots, \tau_1) = \sum_{i=0}^{[n_1/2]} \tau_i SC(n_1, i),$$

where the matrices $SC(n_1, i)$, $i = 1, \dots, [n_1/2]$, *are given by* (2.2).

Proof. Let e_h be the h-th column of the identity matrix I_{n_1}, $h = 1, \dots, n_1$. Then we can express $\Sigma = (\sigma_{ij})$ in the following way:

$$\Sigma = \sum_{i=1}^{n_1}\sum_{j=1}^{n_1} \sigma_{ij} e_i e_j' = \sum_{i}^{n_1} \sigma_{ii} e_i e_i' + \sum_{k=1}^{[n_1/2]} \sum_{\substack{i,j \\ |i-j|= \\ k, n_1-k}} \sigma_{ij} e_i e_j'.$$

Since by invariance, i.e., $P_1\gamma^{(1)}$ and $\gamma^{(1)}$ have the same covariance matrix, we study when $P_1\Sigma P_1' = \Sigma$. Now,

$$P_1\Sigma P_1' = \sum_{i=1}^{n_1} \sigma_{ii}(P_1 e_i e_i' P_1') + \sum_{k=1}^{[n_1/2]} \sum_{\substack{i,j \\ |i-j|= \\ k, n_1-k}} \sigma_{ij}(P_1 e_i e_j' P_1')$$

equals Σ for all P_1, if and only if

$$\begin{aligned} \sigma_{11} &= \sigma_{22} = \ldots = \sigma_{n_1 n_1}, \\ \sigma_{12} &= \sigma_{23} = \ldots = \sigma_{n_1-1,n_1} = \sigma_{n_1,1}, \\ \sigma_{13} &= \sigma_{24} = \ldots = \sigma_{n_1-2,n_1} = \sigma_{n_1,2}, \\ &\vdots \end{aligned}$$

By using (2.3) and τ_k instead of σ_{ij}, where $k = |i-j|$, we obtain

$$\begin{aligned} \Sigma &= \sum_{i=1}^{n_1} \tau_0 e_i e_i' + \sum_{k=1}^{[n_1/2]} \tau_k \sum_{\substack{i,j \\ |i-j|= \\ k, n_1-k}} e_i e_j' = \tau_0 I_{n_1} + \sum_{k=1}^{[n_1/2]} \tau_k SC(n_1, k) \\ &= \sum_{k=0}^{[n_1/2]} \tau_k SC(n_1, k) = Toep(\tau_0, \tau_1, \tau_2, \ldots, \tau_1). \end{aligned}$$

□

Before stating the general theorem we will also consider the second order interactions.

Theorem 3.3. *The covariance matrix $\Sigma_2 : n \times n$ of $\gamma^{(2)}$ is shift permutation invariant if and only if*

$$\Sigma_2 = \sum_{k_2=0}^{[n_2/2]} \sum_{k_1=0}^{[n_1/2]} \tau_k SC(n_2, k_2) \otimes SC(n_1, k_1),$$

where $\gamma^{(2)}$ represents the interaction between a factor with n_1 levels and a factor with n_2 levels, $n = n_1 n_2$, the matrices $SC(i,j)$ are given by (2.2), and

$$k = ([\tfrac{n_1}{2}] + 1)k_2 + k_1. \tag{3.2}$$

Proof. Observe that we may write

$$\Sigma_2 = \sum_{r,s} \sigma_{rs} e_r e_s' = \sum_{i_2,j_2=1}^{n_2} \sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2 i_1)(j_2 j_1)} (e_{2i_2} e_{2j_2}') \otimes (e_{1i_1} e_{1j_1}'),$$

where e_r, e_s are the r-th and the s-th columns of the identity matrix I_n, respectively, e_{hi_h} is the i_h-th column of the identity matrix I_{n_h}, $h = 1, 2$,

$\sigma_{(i_2 i_1)(j_2 j_1)} = Cov(\gamma^{(2)}_{i_2 i_1}, \gamma^{(2)}_{j_2 j_1})$ is the element of Σ_2 in the r-th row and the s-th column,

$$r = (i_2 - 1)n_1 + i_1,$$
$$s = (j_2 - 1)n_1 + j_1,$$

and

$$e_k = e_{2i_2} \otimes e_{1i_1},$$
$$e_l = e'_{2j_2} \otimes e'_{1j_1}.$$

Now, by applying the proof of Theorem 3.2, i.e., inquiring the condition

$$P_2 \Sigma_2 P_2' = \Sigma_2,$$

for all P_2, where $P_2 = P^{(1)} \otimes P^{(2)}$ is the marginally shift permutation matrix of order 2 defined in (3.1). It follows that

$$\Sigma_2 = \sum_{k_2=0}^{[n_2/2]} \sum_{i_1,j_1=1}^{n_1} \sigma_{(k_2 i_1)(k_2 j_1)} SC(n_2, k_2) \otimes P^{(2)}(e_{1i_1} e'_{1j_1}) P^{(2)}$$

which implies that the theorem is true. □

We can formulate the result in the case of s-order interactions which is one of the main results in this paper.

Theorem 3.4. *The covariance matrix $\Sigma_s : n \times n$ of $\gamma^{(s)}$ is shift permutation invariant if and only if*

$$\Sigma_s = \sum_{k_s=0}^{[n_s/2]} \cdots \sum_{k_1=0}^{[n_1/2]} \tau_k SC(n_s, k_s) \otimes \cdots \otimes SC(n_1, k_1), \tag{3.3}$$

where, if $s = 1$, $k = k_1$ and otherwise

$$k = \sum_{h=2}^{s} \prod_{i=1}^{h-1} \left(\left[\tfrac{n_i}{2}\right] + 1 \right) k_h + k_1. \tag{3.4}$$

The factor $\gamma^{(s)}$ represents interaction effects between s factors, the matrices $SC(n_i, k_i)$, $i = 1, \ldots, s$, are given by (2.2) *and τ_k are constants.*

Proof. We only prove (3.4) since (3.3) is a straightforward consequence of (3.4) and the proof of Theorem 3.3. From Theorem 3.2 and Theorem 3.3 it follows that (3.4) is true for $s = 1, 2$. Suppose that (3.4) holds for $s - 1$, i.e., holds for $\Sigma_{s-1} : N_{s-1} \times N_{s-1}$, where $N_{s-1} = n_1 \times \cdots \times n_{s-1}$. However the $s - 1$ factors can be viewed as one factor with an index defined via $k_1, \ldots, k_{s-1}$:

$$\sum_{h=2}^{s-1} \prod_{i=1}^{h-1} \left(\left[\tfrac{n_i}{2}\right] + 1 \right) k_h + k_1.$$

Let the index of the s factor be given by k_s. Then, by using (3.2) in Theorem 3.3 for two factors

$$\begin{aligned} k &= \prod_{i=1}^{s-1}\left([\tfrac{n_i}{2}]+1\right)k_s + \left(\sum_{h=2}^{s-1}\prod_{i=1}^{h-1}\left([\tfrac{n_i}{2}]+1\right)k_h + k_1\right) \\ &= \sum_{h=2}^{s}\prod_{i=1}^{h-1}\left([\tfrac{n_i}{2}]+1\right)k_h + k_1, \end{aligned}$$

and thus (3.4) has been proved. □

4. Spectrum of Σ_2

Suppose now that in our model, besides main effects, there are also second order interactions

$$\begin{aligned} y &= (\mathbf{1}_{n_1}\otimes\mathbf{1}_{n_2}\otimes\mathbf{1}_n)\mu + (I_{n_1}\otimes\mathbf{1}_{n_2}\otimes\mathbf{1}_n)\xi^1 \\ &+ (\mathbf{1}_{n_1}\otimes I_{n_2}\otimes\mathbf{1}_n)\xi^2 + (I_{n_1}\otimes I_{n_2}\otimes\mathbf{1}_n)\gamma^{(2)} \\ &+ (I_{n_1}\otimes I_{n_2}\otimes I_n)\varepsilon, \end{aligned}$$

where $\gamma^{(2)}$ represent second order interaction effects between factor ξ^1 with n_1 levels and factor ξ^2 with n_2 levels, and ε is a random error. We number the levels of the factor $\gamma^{(2)}$ lexicographically.

Let Σ_2 denote the covariance matrix of $\gamma^{(2)}$. Due to marginal shift permutation invariance Σ_2 has a specific structure which can be described by a block symmetric circular Toeplitz matrix:

$$\Sigma_2 = Toep(A_0, A_1, A_2, \dots, A_2, A_1), \tag{4.1}$$

where every block A_k is a symmetric circular Toeplitz matrix with $[n_2/2]+1$ different parameters, $k = 0, \dots, [n_1/2]$. Hence, the matrix Σ_2: $n_1n_2 \times n_1n_2$ is defined by $([\frac{n_1}{2}]+1)([\frac{n_2}{2}]+1)$ different parameters. For example,

(i) $n_1 = n_2 = 4$ (ii) $n_1 = 3,\ n_2 = 4$

$$\Sigma_2 = \begin{pmatrix} A_0 & A_1 & A_2 & A_1 \\ A_1 & A_0 & A_1 & A_2 \\ A_2 & A_1 & A_0 & A_1 \\ A_1 & A_2 & A_1 & A_0 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} A_0 & A_1 & A_1 \\ A_1 & A_0 & A_1 \\ A_1 & A_0 & A_1 \end{pmatrix}.$$

In (4.1) all blocks $A_k : n_2 \times n_2$ are symmetric circular Toeplitz matrices defined in (2.1):

$$A_k = Toep(\tau_0^{(k)}, \tau_1^{(k)}, \tau_2^{(k)}, \dots, \tau_2^{(k)}, \tau_1^{(k)}),\ k = 0, \dots, [n_1/2].$$

For example, if $n_2 = 4$,

$$A_k = \begin{pmatrix} \tau_0^{(k)} & \tau_1^{(k)} & \tau_2^{(k)} & \tau_1^{(k)} \\ \tau_1^{(k)} & \tau_0^{(k)} & \tau_1^{(k)} & \tau_2^{(k)} \\ \tau_2^{(k)} & \tau_1^{(k)} & \tau_0^{(k)} & \tau_1^{(k)} \\ \tau_1^{(k)} & \tau_2^{(k)} & \tau_1^{(k)} & \tau_0^{(k)} \end{pmatrix}. \tag{4.2}$$

Next we shall show that the eigenvalues of Σ_2 can be directly obtained using the eigenvalues of blocks A_k, $k = 0, \ldots, [n_1/2]$.

Theorem 4.1. *Let $\lambda_i^{(k)}$, $i = 1, \ldots, [n_2/2] + 1$, $k = 0, \ldots, [n_1/2]$, be the distinct eigenvalues of block A_k in (4.2) of multiplicities m_i, respectively. Then the eigenvalues of Σ_2 in (4.1) are the following:*

If n_1 is odd,

$$\lambda_{h,i} = \lambda_i^{(0)} + 2\sum_{k=1}^{[\frac{n_1}{2}]} \lambda_i^{(k)} \cos(2\pi hk/n_1), \quad h = 1, \ldots, n_1;\ i = 1, \ldots, [n_2/2] + 1.$$

The multiplicity of $\lambda_{n_1,i}$ is m_i and all other are of multiplicity $2m_i$.

If n_1 is even,

$$\lambda_{h,i} = \lambda_i^{(0)} + 2\sum_{k=1}^{\frac{n_1}{2}-1} \lambda_i^{(k)} \cos(2\pi hk/n_1) \ + \ \lambda_i^{(\frac{n_1}{2})} \cos(\pi h),$$

$$h = 1, \ldots, n_1;\ i = 1, \ldots, [n_2/2] + 1.$$

Only the eigenvalues $\lambda_{n_1,i}$ and $\lambda_{\frac{n_1}{2},i}$ are of multiplicity m_i and all other eigenvalues are of multiplicity $2m_i$.

Proof. According to Theorem 3.4, Σ_2 in (4.1) can be written as

$$\Sigma_2 = \sum_{k_2=0}^{[\frac{n_2}{2}]} \sum_{k_1=0}^{[\frac{n_1}{2}]} \tau_k SC(n_2, k_2) \otimes SC(n_1, k_1),$$

where

$$k = k_2([\tfrac{n_1}{2}] + 1) + k_1.$$

We can rewrite Σ_2 in the following way:

$$\Sigma_2 = \sum_{k_2=0}^{[\frac{n_2}{2}]} SC(n_2, k_2) \otimes A_{k_2},$$

where

$$\begin{aligned} A_{k_2} &= \sum_{k_1=0}^{[\frac{n_1}{2}]} \tau_k SC(n_1, k_1), \\ k &= k_2([\tfrac{n_1}{2}] + 1) + k_1. \end{aligned}$$

Since $SC(n_i, k_i)$ are symmetric circular matrices, $k_i = 0, \dots, [n_i/2]$, we know that they commute. Thus, there exists an orthogonal matrix V_i, $i = 1, 2$, such that

$$V_i' SC(n_i, k_i) V_i = \Lambda_{k_i},$$

where Λ_{k_i} is a diagonal matrix, where the diagonal elements are the eigenvalues of $SC(n_i, k_i)$ given in Lemma 2.1. Let $V = V_2 \otimes V_1$. Then

$$V' \Sigma_2 V = \sum_{k_2=0}^{[\frac{n_2}{2}]} \Lambda_{k_2} \otimes \Lambda_{A,k_2},$$

where Λ_{A,k_2} is a diagonal matrix with diagonal elements equal to the eigenvalues of A_{k_2} given in Lemma 2.1. By straightforward calculations of the Kronecker product $\Lambda_{k_2} \otimes \Lambda_{A,k_2}$ and using the eigenvalues of $SC(n_2, k_2)$, which follow from Lemma 2.1 as a special case, the statements of the theorem are established. □

The similarity in structure between Lemma 2.1 and Theorem 4.1 is worth observing. Basically it stems from the fact that $SC(n, k)$ are commuting symmetric Toeplitz matrices generated by one non-zero element.

Let us now illustrate the results obtained in Theorem 4.1 for (i) $n_1 = n_2 = 4$ and (ii) $n_1 = 3$, $n_2 = 4$.

Example 1. Let $n_1 = n_2 = 4$. Since $n_1 = 4$, from Theorem 4.1 we obtain

$$\begin{aligned}
\lambda_{1,i} &= \lambda_i^{(0)} - \lambda_i^{(2)},\\
\lambda_{2,i} &= \lambda_i^{(0)} - 2\lambda_i^{(1)} + \lambda_i^{(2)},\\
\lambda_{3,i} &= \lambda_i^{(0)} - \lambda_i^{(2)},\\
\lambda_{4,i} &= \lambda_i^{(0)} + 2\lambda_i^{(1)} + \lambda_i^{(2)},
\end{aligned}$$

where $i = 1, 2, 3$. Moreover, since each block $A_k : 4 \times 4$, $k = 0, 1, 2$, is a symmetric Toeplitz matrix, its distinct eigenvalues are the following (see Lemma 2.1):

$$\begin{aligned}
\lambda_1^{(k)} &= \tau_0^{(k)} - \tau_2^{(k)},\\
\lambda_2^{(k)} &= \tau_0^{(k)} - 2\tau_1^{(k)} + \tau_2^{(k)},\\
\lambda_3^{(k)} &= \tau_0^{(k)} + 2\tau_1^{(k)} + \tau_2^{(k)}.
\end{aligned}$$

Example 2. Let $n_1 = 3$ and $n_2 = 4$. According to Theorem 4.1 the eigenvalues of Σ_2 are the following:

$$\begin{aligned} \lambda_{1,i} &= \lambda_i^{(0)} - \lambda_i^{(1)}, \\ \lambda_{2,i} &= \lambda_i^{(0)} - \lambda_i^{(1)}, \\ \lambda_{3,i} &= \lambda_i^{(0)} + 2\lambda_i^{(1)}. \end{aligned}$$

The eigenvalues $\lambda_i^{(0)}$ and $\lambda_i^{(1)}$, $i = 1, 2, 3$, of blocks $A_0 : 4 \times 4$ and $A_1 : 4 \times 4$, respectively, could be easily obtained from Lemma 2.1. Thus, using Theorem 4.1 we can obtain all eigenvalues of $\Sigma_2 : 12 \times 12$ directly using the eigenvalues of smaller blocks A_i.

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