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Timelike parallel p_i-equidistant ruled surfaces by a timelike base curve in the Minkowski 3-space R_1^3

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Abstract. In this paper the timelike parallel p_i-equidistant ruled surfaces are introduced and relations about polar planes, natural curvatures and natural torsions have been given. In addition, relations between distribution parameters, shape operators, Gaussian and mean curvatures of these ruled surfaces have been obtained. After all, an example related to the parallel timelike p_2-equidistant ruled surfaces is given.

1. Introduction

A kinematically generated surface is a surface that is defined by the envelope of a moving object. This object can be a point, a plane, a line or any arbitrary shape primitive. For the case of a line, the kinematically generated surface is a ruled surface. In a spatial motion, the trajectories of the oriented lines embedded in a moving rigid body are generally ruled surfaces. Thus, the spatial geometry of ruled surfaces is important in the study of rational design problems in spatial mechanisms. As an example, A. T. Yang et al. [15] applied some characteristic invariants of ruled surfaces to the mechanism theory. Also, using the geometry of curves and developable surfaces, some spatial design problems were studied by H. Pottmann et al. [11].

In the literature (see, e. g., [3, 4, 6]) there are many studies related to ruled surfaces and their invariants (distribution parameter, apex angle, pithes, etc.) in 3-dimensional Euclidean space E^3.

Some results related to the parallel p-equidistant ruled surfaces and striction curves of these ruled surfaces have been given first by I. E. Valeontis [14] in E^3. Later on, some characteristic properties of integral invariants, shape operators, and Gaussian curvatures of parallel p-equidistant ruled surfaces have been defined by M. Masal and N. Kuruoğlu in [7, 8, 9].

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In the theory of relativity, the geometry of indefinite metric is crucial. Hence, the theory of surfaces in the Minkowski space R_1^3 which has the metric $ds^2 = dx^2 + dy^2 - dz^2$ attracted much attention. The situation is much more complicated than the Euclidean case, since the surfaces may have a definite metric (spacelike surfaces), Lorentz metric (timelike surfaces) or mixed metric. Recently, the timelike and spacelike ruled surfaces have been studied systematically in [1, 12, 13].

This paper concerns timelike parallel p_i-equidistant ruled surfaces by a timelike base curve in the Minkowski 3-space R_1^3. Firstly, timelike parallel p_i-equidistant ruled surfaces by a timelike base curve are defined. Then, curvatures, dralls, matrices of shape operators, Gaussian curvatures, mean curvatures of these surfaces and some relations between these curvatures are found. Finally, an example for the timelike parallel p_2-equidistant ruled surfaces is given. It is hoped that these results will contribute to the study of line geometry and rational design of space mechanisms and physics applications.

2. Preliminaries

Let R_1^3 denote the three-dimensional Minkowski space, i. e. a three-dimensional vector space R^3 equipped with the flat metric $g = -dx_1^2 + dx_2^2 + dx_3^2$ where (x_1, x_2, x_3) is rectangular coordinate system of R_1^3. Since g is indefinite metric, recall that a vector v in R_1^3 can have one of three casual characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \mp 1$. Furthermore, vectors v and w are said to be orthogonal if $g(v,w) = 0$.

For any vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in R_1^3$, the Lorentzian product $v \wedge w$ of v and w is defined (see [2]) as

$$v \wedge w = (v_2 w_3 - v_3 w_2 \,,\, v_1 w_3 - v_3 w_1 \,,\, v_2 w_1 - v_1 w_2)\,.$$

A regular curve $\alpha : I \to R_1^3$, $I \subset R$ in R_1^3 is said to be spacelike, timelike or null curve if the velocity vector $\alpha'(t)$ is a spacelike, timelike or null vector, respectively (see [5]).

Let M be a semi-Riemannian hypersurface in R_1^3 and let D and N represent Levi-Civita connection and unit normal vector field of M, respectively. For all $X \in \chi(M)$ the transformation

$$S(X) = -D_X N \tag{2.1}$$

is called a shape operator of M, where $\chi(M)$ is the space of vector fields of M (see [10]).

Let $S(P)$ be a shape operator of M at a point P, then $K : M \to R$, $K(P) = \det S(P)$, is called the Gaussian curvature function of M. In this case the value of $K(P)$ is defined to be the Gaussian curvature of M at the point P. Similarly, the function $H : M \to R$, $H(P) = \frac{traceS(P)}{\dim M}$, is called the mean curvature of M at the point P.

Let us suppose that α is a curve in M. If

$$S(T) = \lambda T \tag{2.2}$$

then the curve α is called the curvature line in M, where T is the tangential vector field of α and λ is a scalar being not equal to zero.

If the following equation holds

$$g(S(T), T) = 0 \tag{2.3}$$

then α is called an asymptotic curve. If the induced metric on M is the Lorentz metric, then M is called the timelike surface.

The family of lines with one parameter in R_1^3 is called the ruled surface and each of these lines of this family is named as the rulings of the ruled surface. Thus the parametrization of the ruled surface is given by

$$\varphi(t, v) = \alpha(t) + vX(t),$$

where α and X are the base curve and unit vector in the direction of the rulings of the ruled surface, respectively. If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a striction point. The set of striction points on a ruled surface defines the striction curve [1]. For the striction curve of the ruled surface $\varphi(t, v)$ we can write

$$\overline{\alpha} = \alpha - \frac{g(\alpha', X')}{g(X', X')} X. \tag{2.4}$$

For the drall (distribution parameter) of the ruled surface $\varphi(t, v)$ we can write

$$P_X = -\frac{\det(\alpha', X, X')}{g(X', X')}, \quad g(X', X') \neq 0. \tag{2.5}$$

3. Timelike Parallel p_i-Equidistant Ruled Surfaces by a Timelike Base Curve in the Minkowski 3-Space R_1^3

Let $\alpha = \alpha(t)$ be a differentiable timelike curve with arc-length in the 3-dimensional Minkowski space R_1^3. Suppose that $D_{V_1}V_1$ is a spacelike vector while the tangent vector of α is $V_1 = \alpha'$. Therefore, if V_1 moves along the curve α, then a 2-dimensional timelike ruled surface is generated in the 3-dimensional Minkowski space R_1^3. This 2-dimensional ruled surface is parametrically given by

$$\varphi(t, v) = \alpha(t) + vV_1(t) \tag{3.1}$$

and denoted by M, where timelike curve α and $V_1(t)$ are called base curve and direction vector, respectively. Let us consider a Frenet frame $\{V_1, V_2, V_3\}$

attached to the timelike curve α. The structural equations of this frame (or Frenet Formulae) are given as

$$V_1' = k_1 V_2\,, \quad V_2' = k_1 V_1 - k_2 V_3\,, \quad V_3' = k_2 V_2, \tag{3.2}$$

where "$\,'\,$" means the derivative with respect to time t. Considering equation (3.1) yields

$$\varphi_t = V_1 + v k_1 V_2\,, \quad \varphi_v = V_1.$$

From the last equations we find

$$\varphi_t \wedge \varphi_v = v k_1 V_3.$$

It is obvious that $\varphi_t \wedge \varphi_v \in \chi^{\perp}(M)$. This means that M is really a timelike ruled surface. The planes corresponding to subspaces $Sp\,\{V_1, V_2\}$, $Sp\,\{V_2, V_3\}$ and $Sp\,\{V_3, V_1\}$ along striction curve of the timelike ruled surface M are called *asymptotic plane, polar plane* and *central plane*, respectively.

Let us suppose that $\alpha^* = \alpha^*(t^*)$ is another differentiable timelike curve with arc-length and $\{V_1^*, V_2^*, V_3^*\}$ is a Frenet frame of this curve in the three-dimensional Minkowski space R_1^3. Hence, we define the timelike ruled surface M^* parametrically as follows:

$$\varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1(t^*)\,, \quad (t^*, v^*) \in I \times R.$$

Definition 3.1. Let M and M^* be two timelike ruled surfaces and let p_1, p_2 and p_3 be the distances between the polar planes, central planes and asymptotic planes, respectively. If the directions of M and M^* are parallel and the distances p_i, $1 \leq i \leq 3$, of M and M^* are constant, then the pair of ruled surfaces M and M^* is called *timelike parallel p_i-equidistant ruled surfaces with a timelike base curve.* If specifically $p_i = 0$, then this pair of ruled surfaces is named as *timelike parallel p_i-equivalent ruled surfaces with a timelike base curve*, where the base curves of ruled surfaces M and M^* are of class C^2.

Therefore the pair of timelike parallel p_i-equidistant ruled surfaces is defined parametrically as

$$\begin{aligned} M\ &: \varphi(t, v) = \alpha(t) + v V_1(t)\,, \quad (t, v) \in I \times R, \\ M^* &: \varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1(t^*)\,, \quad (t^*, v^*) \in I \times R, \end{aligned} \tag{3.3}$$

where t and t^* are the arc parameters of curves α and α^*, respectively. Let the striction curve of M be the base curve of M and let α^* be a base curve of M^*. In this case we can write

$$\alpha^* = \gamma + p_1 V_1 + p_2 V_2 + p_3 V_3, \tag{3.4}$$

where $p_1(t)$, $p_2(t)$ and $p_3(t)$ are of class C^2.

Considering equations [striction curve equation], (3.2) and (3.4) we can easily see that the striction curve γ^* of M^* is

$$\gamma^* = \gamma + \left(\frac{p_3 k_2 + p_2'}{-k_1}\right) V_1 + p_2 V_2 + p_3 V_3. \tag{3.5}$$

If we take α^* as the striction line of M^*, from equations (3.4) and (3.5) we can deduce the following theorem.

Theorem 3.1. *Let M and M^* be timelike parallel p_i-equidistant (parallel p_i-equivalent) ruled surfaces in R_1^3. Then the distance between the polar planes of M and M^* is*

$$p_1 = \frac{p_3 k_2 + p_2'}{-k_1} = \text{constant} \quad (\text{or } p_1 = \frac{p_3 k_2 + p_2'}{-k_1} = 0).$$

Now we consider the Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ of ruled surfaces M and M^*. From Definition 2.1 it is obvious that $V_1^*(t^*) = V_1(t)$. Furthermore, from $\frac{dV_i}{dt} = \frac{dV_i^*}{dt^*}\frac{dt^*}{dt}$, $1 \le i \le 3$, and equations (3.2) we find that $V_2^*(t^*) = V_2(t)$ and $V_3^*(t^*) = V_3(t)$, for $\frac{dt^*}{dt} > 0$. This yields the following theorem.

Theorem 3.2. *The Frenet vectors of timelike parallel p_i-equidistant ruled surfaces M and M^* at $\alpha(t)$ and $\alpha^*(t^*)$ points are equivalent for $\frac{dt^*}{dt} > 0$.*

From the last theorem and equations (3.2) we obtain the following corollary.

Corollary 3.1. *Let M and M^* be timelike parallel p_i-equidistant ruled surfaces.*

i) *There is a relation between natural curvatures $k_1(t)$ and $k_1^*(t^*)$ of base curves and torsions $k_2(t)$, $k_2^*(t^*)$ of M and M^* as follows:*

$$k_i^* = k_i \frac{dt}{dt^*}, \quad 1 \le i \le 2. \tag{3.6}$$

ii) *Base curve of M is an inclined curve iff base curve of M^* is an inclined curve.*

iii) *Base curves of M and M^* are striction lines.*

Keeping in mind equations (3.2) and (3.6) we examine the distribution parameters of ruled surfaces formed by Frenet vectors V_1, V_2, V_3 of M and find

$$P_{V_1} = 0, \quad P_{V_2} = \frac{k_2}{k_2^2 - k_1^2}, \quad P_{V_3} = \frac{1}{k_2}. \tag{3.7}$$

The distribution parameters of ruled surfaces formed by Frenet vectors V_1^*, V_2^*, V_3^* of M^* are obtained as follows:

$$P_{V_1^*} = 0, \quad P_{V_2^*} = \frac{k_2^*}{(k_2^*)^2 - (k_1^*)^2}, \quad P_{V_3^*} = \frac{1}{k_2^*}. \tag{3.8}$$

From equations (3.7), (3.8) and Corollary 3.1 we obtain the following theorem.

Theorem 3.3. *Let M and M^* be timelike parallel p_i-equidistant ruled surfaces in R_1^3. There exists a relation between the distribution parameters of ruled surfaces formed by orthonormal frames of M and M^* as follows:*

$$P_{V_i^*} = P_{V_i}\frac{dt^*}{dt}, \quad 1 \leq i \leq 3.$$

We now calculate the matrices S and S^* corresponding to the shape operators of the ruled surfaces M and M^*. From equations (3.3) we write

$$\varphi_t = V_1 + vk_1V_2\,, \quad \varphi_v = V_1.$$

It is clear that $g(\varphi_t, \varphi_v) \neq 0$. From the Gram–Schmidt method, we get

$$X = \varphi_v = V_1\,, \quad Y = \varphi_t - \varphi_v = vk_1V_2, \tag{3.9}$$

where $X, Y \in \chi(M)$ form an orthogonal basis $\{X(\alpha(t)), Y(\alpha(t))\}$ of a tangent space at each point $\alpha(t)$ of M. So the normal vector field and the unit normal vector field of M are

$$N = X \wedge Y = -vk_1V_3$$

and

$$N_0 = \frac{N}{\|N\|} = \begin{cases} -V_3, & \text{for } v > 0, \\ V_3, & \text{for } v < 0, \end{cases} \tag{3.10}$$

respectively. Similarly, from equations (3.3), we find

$$X^* = V_1^*\,, \quad Y^* = v^*k_1^*V_2^*, \tag{3.11}$$

where $X^*, Y^* \in \chi(M^*)$ form an orthogonal basis $\{X^*(\alpha^*(t^*)), Y^*(\alpha^*(t^*))\}$ of a tangent space at each point $\alpha^*(t^*)$ of M^*. We can write the unit normal vector field of M^* as

$$N_0^* = \begin{cases} -V_3^*, & \text{for } v^* > 0, \\ V_3^*, & \text{for } v^* < 0. \end{cases} \tag{3.12}$$

The shape operator S of M can be written as

$$S(X) = aX + bY, \quad S(Y) = cX + dY.$$

Therefore, the matrix corresponding to the shape operator is

$$S = \begin{bmatrix} \frac{g(S(X),X)}{g(X,X)} & \frac{g(S(X),Y)}{g(Y,Y)} \\ \frac{g(S(Y),X)}{g(X,X)} & \frac{g(S(Y),Y)}{g(Y,Y)} \end{bmatrix}. \tag{3.13}$$

From equation (3.10), there are two special cases for the shape operator S: $v > 0$ and $v < 0$. First, let as suppose that $v > 0$. In this case from equations (3.9), (3.10), (3.13) and (2.1) we find

$$S = \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_2}{vk_1} \end{bmatrix}. \tag{3.14}$$

For $v < 0$, considering the same equations, we find that

$$S = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{k_2}{vk_1} \end{bmatrix}. \tag{3.15}$$

In a similar way the shape operator matrices S^* of the ruled surface M^* are found to be

$$S^* = \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_2^*}{v^* k_1^*} \end{bmatrix} \quad (v^* > 0) \tag{3.16}$$

and

$$S^* = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{k_2^*}{v^* k_1^*} \end{bmatrix} \quad (v^* < 0). \tag{3.17}$$

From equations (3.14), (3.17) and Corollary 3.1 we obtain the following theorem.

Theorem 3.4. *Let M and M^* be timelike parallel p_i-equidistant ruled surfaces. There is a relation between the shape operators S and S^* of M and M^* as follows:*

$$S^* = S,$$

where $v = v^*$.

From equations (3.14), (3.15), (3.16), (3.17) and the last theorem we deduce the following corollary.

Corollary 3.2. *Let M and M^* be timelike parallel p_i-equidistant ruled surfaces.*

i) *If the Gaussian curvatures of M and M^* are K and K^*, respectively, then*

$$K^* = K = 0.$$

ii) *If the mean curvatures of M and M^* are H and H^*, respectively, then*

$$H^* = H = \begin{cases} \frac{k_2}{2vk_1}, & \text{for } v > 0, \\ -\frac{k_2}{2vk_1}, & \text{for } v < 0, \end{cases}$$

where $v = v^*$.

iii) *A curvature line of M is also a curvature line of M^* and vice-versa.*

iv) *An asymptotic curve of M is also an asymptotic curve of M^* and vice-versa.*

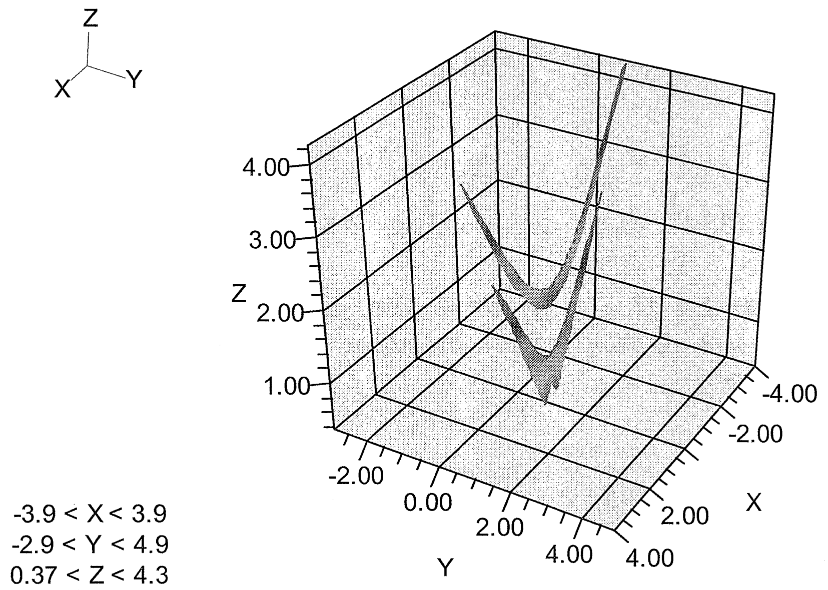
Example 3.1. M and M^* are timelike parallel p_2-equidistant ruled surfaces in the three-dimensional Minkowski space R_1^3 if defined by the following parametric equations:

$$M: \ \varphi(t,v) = (\sinh t + v\cosh t,\ 1,\ \cosh t + v\sinh t)$$

and

$$M^*: \ \varphi^*(t^*,v^*) = (2\sinh t^* + v^*\cosh t^*,\ 1,\ 2\cosh t^* + v^*\sinh t^*),$$

where the curves $\alpha(t) = (\sinh t, 1, \cosh t)$ and $\alpha^*(t^*) = (2\sinh t^*, 1, 2\cosh t^*)$ are the timelike base curves of M and M^*, respectively (see Figure).



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