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Hardy–Hilbert's integral inequalities via homogeneous functions and some other generalizations

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Abstract. Inequalities similar to Hardy–Hilbert's integral inequality in which the weight function is homogeneous are given. As application, some recent results by B. Yang and D. Xin follow.

1. Introduction

Let $f,\, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)dt < \infty \; and \; 0 < \int_0^\infty g^2(t)dt < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{1/2}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [3]). Inequality (1) is well known as Hilbert's integral inequality. This inequality has been extended by Hardy [1] as follows.

If $p > 1,\ 1/p + 1/q = 1,\ f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)dt < \infty \; and \; \int_0^\infty g^q(t)dt < \infty,$$

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then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)dt \right)^{1/p} \left(\int_0^\infty g^q(t)dt \right)^{1/q}, \tag{2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Inequality (2) is called Hardy–Hilbert's integral inequality and is important in analysis and its applications (cf. Mitrinović et al. [4]).

B. Yang gave the following extension of (2).

Theorem A (see [5]). *If* $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$, *satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty \text{ and } \int_0^\infty t^{1-\lambda} g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)dt \right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)dt \right)^{1/q}, \tag{3}$$

where the constant factor $k_\lambda(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ *is the best possible (here* B *is the beta function).*

Another type of such inequalities was given by D. Xin as follows.

Theorem B (see [7]). *If* $p > 1$, $1/p + 1/q = 1$, $r > 1$, $1/s + 1/r = 1$, $\lambda > 0$, $f, g \geq 0$ *are such that*

$$0 < \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x)dx < \infty \text{ and } \int_0^\infty x^{q(1-\lambda/s)-1} g^q(x)dx < \infty,$$

then we have

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y)f(x)g(y)}{x^\lambda - y^\lambda}dxdy < \left(\frac{\pi}{\lambda \sin(\pi/r)} \right)^2$$
$$\times \left(\int_0^\infty x^{p(1-\lambda/r)-1} f^p(x)dx \right)^{1/p} \left(\int_0^\infty x^{q(1-\lambda/s)-1} g^q(x)dx \right)^{1/q}, \tag{4}$$

where the constant factor $(\pi/(\lambda \sin(\pi/r)))^2$ *is the best possible.*

A function $H = H(x, y)$ is said to be *homogeneous of degree* λ if $H(tx, ty) = t^\lambda H(x, y)$, for every $t > 0$.

We need the following result for our aim.

Theorem C (see [2]). *Let f be a nonnegative integrable function. Define*

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p > 1,$$

where the constant factor $(p/(p-1))^p$ is the best possible.

The object of this paper is to give some inequalities similar to Hardy–Hilbert's inequality. It may be mentioned here that one of our results (Theorem 3) gives a generalization of Theorem A and Theorem B.

2. Main Results

Theorem 1. *Let $F, G, f, g, \geq 0$ be defined on $\mathbb{R}, f', g' > 0$, $-\infty \leq t < T \leq \infty$, $f(t) = g(t) = 0$, $f(T) = g(T) = \infty$, $a, b, \lambda < 1 < \lambda + a$, $\lambda + b$,*

$$0 < \int_t^T \frac{[f(t)]^{1+ap/q-\lambda-b}F^p(t)}{[f'(t)]^{p/q}}dt < \infty, \quad 0 < \int_t^T \frac{[g(t)]^{1+bq/p-\lambda-a}G^q(t)}{[g'(t)]^{q/p}}dt < \infty.$$

Then, we have

$$\int_0^T\int_0^T \frac{F(x)G(y)}{|f(x)-g(y)|^\lambda}dxdy \leq K_{\lambda,b}^{1/p}K_{\lambda,a}^{1/q}\left(\int_0^T \frac{[f(x)]^{1+ap/q-\lambda-b}F^p(x)}{[f'(x)]^{p/q}}dx\right)^{1/p}$$
$$\times\left(\int_0^T \frac{[g(y)]^{1+bq/p-\lambda-a}G^q(y)}{[g'(y)]^{q/p}}dy\right)^{1/q}, \tag{5}$$

where

$$K_{\lambda,b} = B(1-\lambda, 1-b) + B(1-\lambda, \lambda+b-1),$$
$$K_{\lambda,a} = B(1-\lambda, 1-a) + B(1-\lambda, \lambda+a-1).$$

Proof. We have

$$\int_t^T\int_t^T \frac{F(x)G(y)}{|f(x)-g(y)|^\lambda}dxdy$$
$$= \int_t^T\int_t^T \frac{F(x)[g(y)]^{-b/p}[g'(y)]^{1/p}}{[f(x)]^{-a/q}[f'(x)]^{1/q}|f(x)-g(y)|^{\lambda/p}}$$
$$\times\frac{G(y)[f(x)]^{-a/q}[f'(x)]^{1/q}}{[g(y)]^{-b/p}[g'(y)]^{1/p}|f(x)-g(y)|^{\lambda/q}}dxdy$$

$$
\begin{aligned}
&\leq \left(\int_t^T\int_t^T \frac{F^p(x)[g(y)]^{-b}g'(y)}{[f(x)]^{-ap/q}[f'(x)]^{p/q}|f(x)-g(y)|^{\lambda}}dxdy\right)^{1/p}\\
&\quad\times\left(\int_t^T\int_t^T \frac{G^q(y)[f(x)]^{-a}f'(x)}{[g(y)]^{-bq/p}[g'(y)]^{q/p}|f(x)-g(y)|^{\lambda}}dxdy\right)^{1/q}\\
&= M^{1/p}N^{1/q}.
\end{aligned}
$$

We first consider

$$
M=\int_t^T \frac{[f(x)]^{1-\lambda-b+ap/q}F^p(x)}{[f'(x)]^{p/q}}dx \int_t^T \frac{[g(y)/f(x)]^{-b}g'(y)}{f(x)|1-g(y)/f(x)|^{\lambda}}dy.
$$

Observe that

$$
\begin{aligned}
\int_t^T \frac{[g(y)/f(x)]^{-b}g'(y)}{f(x)|1-g(y)/f(x)|^{\lambda}}dy &= \int_0^{\infty}\frac{u^{-b}}{|1-u|^{\lambda}}du\\
&= \int_0^1 \frac{u^{-b}}{(1-u)^{\lambda}}du+\int_1^{\infty}\frac{u^{-b}}{(u-1)^{\lambda}}du\\
&= \int_0^1 \frac{u^{-b}}{(1-u)^{\lambda}}du+\int_0^1\frac{v^{\lambda+b-2}}{(1-v)^{\lambda}}dv\\
&= B(1-\lambda,1-b)+B(1-\lambda,\lambda+b-1)\\
&= K_{\lambda,b}.
\end{aligned}
$$

Therefore, we have

$$
M=K_{\lambda,b}\int_t^T \frac{[f(x)]^{1-\lambda-b+ap/q}F^p(x)}{[f'(x)]^{p/q}}dx.
$$

Similarly, we can show that

$$
N=K_{\lambda,a}\int_t^T \frac{[g(y)]^{1-\lambda-a+bq/p}G^q(y)}{[g'(y)]^{q/p}}dy.
$$

This implies (5). □

Theorem 2. *Let $f, g \geq 0$, let $H > 0$ be homogeneous of degree λ, and let $p > 1$, $1/p + 1/q = 1$, $\lambda = 2(1 + \alpha - 1/p) = 2(1 + \beta - 1/q) > 2$. Define*

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt.$$

Then

$$\int_0^\infty \int_0^\infty \frac{F^\alpha(x)G^\beta(y)}{H(x,y)} dxdy \leq C \left(\int_0^\infty f^{\alpha p}(x)dx \right)^{1/p} \left(\int_0^\infty g^{\beta q}(y)dy \right)^{1/q}, \tag{6}$$

where

$$C = K_\lambda \left(\frac{\alpha p}{\alpha p - 1} \right)^\alpha \left(\frac{\beta q}{\beta q - 1} \right)^\beta, \quad K_\lambda = \int_0^\infty \frac{t^{\lambda/2-1}}{H(1,t)} dt,$$

provided the integrals on the right-hand side exist.

Proof. The hypothesis implies that

$$1 + (\lambda/2 - 1)p = \alpha p > 1, \quad 1 + (\lambda/2 - 1)q = \beta q > 1.$$

We have

$$\begin{aligned}
\int_0^\infty \int_0^\infty \frac{F^\alpha(x)G^\beta(y)}{H(x,y)} dxdy &= \int_0^\infty \int_0^\infty \frac{y^{(\lambda/2-1)1/p} F^\alpha(x)}{x^{(\lambda/2-1)1/q} H^{1/p}(x,y)} \\
&\quad \times \frac{x^{(\lambda/2-1)1/q} G^q(y)}{y^{(\lambda/2-1)1/p} H^{1/q}(x,y)} dxdy \\
&\leq \left(\int_0^\infty \int_0^\infty \frac{y^{\lambda/2-1} F^{\alpha p}(x)}{x^{(\lambda/2-1)p/q} H(x,y)} dx \right)^{1/p} \\
&\quad \times \left(\int_0^\infty \int_0^\infty \frac{x^{\lambda/2-1} G^{\beta q}(y)}{y^{(\lambda/2-1)q/p} H(x,y)} dy \right)^{1/q} \\
&= M^{1/p} N^{1/q}.
\end{aligned}$$

Observe that

$$\begin{aligned}
M &= \int_0^\infty \frac{F^{\alpha p}(x)}{x^{(\lambda/2-1)(p-1)}} dx \int_0^\infty \frac{y^{\lambda/2-1}}{H(x,y)} dy \\
&= \int_0^\infty \frac{F^{\alpha p}(x)}{x^{(\lambda/2-1)(p-1)+\lambda/2}} dx \int_0^\infty \frac{u^{\lambda/2-1}}{H(1,u)} du
\end{aligned}$$

$$= K_\lambda \int_0^\infty \left(\frac{F(x)}{x}\right)^{\alpha p} dx < K_\lambda \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p} \int_0^\infty f^{\alpha p}(x)dx,$$

by Theorem C. Similarly,

$$N = \int_0^\infty \left(\frac{G(y)}{y}\right)^{\beta q} dy \int_0^\infty \frac{v^{\lambda/2-1}}{H(v,1)} dv.$$

But since

$$\int_0^\infty \frac{v^{\lambda/2-1}}{H(v,1)} dv = \int_0^\infty \frac{v^{\lambda/2-1}}{H(v,vv^{-1})} dv = \int_0^\infty \frac{v^{-\lambda/2-1}}{H(1,v^{-1})} dv = \int_0^\infty \frac{u^{\lambda/2-1}}{H(1,u)} du,$$

we get

$$N = K_\lambda \int_0^\infty \left(\frac{G(y)}{y}\right)^{\beta q} dy < K_\lambda \left(\frac{\beta q}{\beta q-1}\right)^{\beta q} \int_0^\infty g^{\beta q}(y)dy.$$

Therefore, we have inequality (6). □

Theorem 3. *Let $f, g \geq 0$, let $h > 0$ be homogeneous of degree $\lambda > 0$, and let $p > 1$, $1/p + 1/q = 1$, $r > 1$, $1/r + 1/s = 1$. Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{h(x,y)} dxdy \leq K_\lambda \Big(\int_0^\infty x^{p(1-\lambda/r)-1} f^p(x)dx\Big)^{1/p} \times \Big(\int_0^\infty y^{q(1-\lambda/s)-1} g^q(y)dy\Big)^{1/q} \tag{7}$$

and

$$\int_0^\infty y^{\frac{p\lambda}{s}-1} \Big(\int_0^\infty \frac{f(x)}{h(x,y)} dx\Big)^p dy \leq (K_\lambda)^p \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x)dx, \tag{8}$$

where

$$K_\lambda = \int_0^\infty \frac{u^{\lambda/s-1}}{h(1,u)} du = \int_0^\infty \frac{u^{\lambda/r-1}}{h(u,1)} du,$$

provided the integrals on the right-hand side exist. Moreover, inequalities (7) and (8) are equivalent.

Proof. We have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{h(x,y)} dxdy = \int_0^\infty \int_0^\infty \frac{y^{(\lambda/s-1)1/p} f(x)}{x^{(\lambda/r-1)1/q} h^{1/p}(x,y)}$$

$$
\begin{aligned}
& \times \frac{x^{(\lambda/r-1)1/q} g(y)}{y^{(\lambda/s-1)1/p} h^{1/q}(x,y)} dxdy \\
\leq & \left(\int_0^\infty \int_0^\infty \frac{y^{\lambda/s-1} f^p(x)}{x^{(\lambda/r-1)(p-1)} h(x,y)} dx \right)^{1/p} \\
& \times \left(\int_0^\infty \int_0^\infty \frac{x^{(\lambda/r-1)} g^q(y)}{y^{(\lambda/s-1)(q-1)} h(x,y)} dy \right)^{1/q} \\
= & P^{1/p} Q^{1/q}.
\end{aligned}
$$

We first consider

$$
\begin{aligned}
P &= \int_0^\infty \frac{f^p(x)}{x^{(\lambda/r-1)(p-1)}} dx \int_0^\infty \frac{y^{\lambda/s-1}}{h(x,y)} dy \\
&= \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x) dx \int_0^\infty \frac{u^{\lambda/s-1}}{h(1,u)} du \\
&= K_\lambda \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x) dx.
\end{aligned}
$$

Similarly,

$$
Q = K_\lambda \int_0^\infty y^{q(1-\lambda/s)-1} g^q(y) dy.
$$

Therefore, we have inequality (7).

In order to show that inequalities (7) and (8) are equivalent, suppose first that (7) is satisfied. Then we have

$$
\begin{aligned}
& \int_0^\infty y^{\frac{p\lambda}{s}-1} \left(\int_0^\infty \frac{f(x)}{h(x,y)} dx \right)^p dy \\
= & \int_0^\infty \int_0^\infty \frac{f(x) y^{\frac{p\lambda}{s}-1}}{h(x,y)} \left(\int_0^\infty \frac{f(z)}{h(z,y)} dz \right)^{p-1} dxdy \\
\leq & K_\lambda \left(\int_0^\infty x^{p(1-\lambda/r)-1} f^p(x) dx \right)^{1/p} \\
& \times \left(\int_0^\infty y^{q(1-\lambda/s)-1} y^{q(p\lambda/s-1)} \left(\int_0^\infty \frac{f(z)}{h(z,y)} dz \right)^{q(p-1)} dy \right)^{1/q}
\end{aligned}
$$

$$= K_\lambda\left(\int_0^\infty x^{p(1-\lambda/r)-1}f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{p\lambda/s-1}\left(\int_0^\infty \frac{f(z)}{h(z,y)}dz\right)^p dy\right)^{1/q},$$

which implies inequality (8). Now suppose that (8) is satisfied. Then

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{h(x,y)}dxdy\\
= &\int_0^\infty y^{(1-\frac{p\lambda}{s})\frac{1}{p}}g(y)\; y^{(\frac{p\lambda}{s}-1)\frac{1}{p}}\int_0^\infty \frac{f(x)}{h(x,y)}dxdy\\
\le &\left(\int_0^\infty y^{(1-\frac{p\lambda}{s})\frac{q}{p}}g^q(y)dy\right)^{1/q}\left(\int_0^\infty y^{\frac{p\lambda}{s}-1}\left(\int_0^\infty \frac{f(x)}{h(x,y)}dx\right)^p dy\right)^{1/p}\\
\le &K_\lambda\left(\int_0^\infty x^{p(1-\lambda/r)-1}f^p(x)dx\right)^{1/p}\left(\int_0^\infty y^{q(1-\lambda/s)-1}g^q(y)dy\right)^{1/q}.
\end{aligned}$$

□

3. Applications

The following is obvious from Theorem 1.

Corollary 1. *Let $f,g\ge 0$, $0<a,b,\lambda<1$, $\lambda+a$, $\lambda+b>1$. Then*

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda}dxdy \;\le\; &K_{\lambda,b}^{1/p}K_{\lambda,a}^{1/q}\left(\int_0^\infty x^{1+ap/q-\lambda-b}f^p(x)dx\right)^{1/p}\\
&\times\left(\int_0^\infty y^{1+bq/p-\lambda-a}g^q(y)dy\right)^{1/q},\end{aligned}\tag{9}$$

in particular, for $\frac{1}{2}<\lambda<1$,

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda}dxdy \le C_\lambda\left(\int_0^\infty t^{1-\lambda}f^2(t)dt\right)^{1/2}\left(\int_0^\infty t^{1-\lambda}g^2(t)dt\right)^{1/2},$$

where

$$C_\lambda=\left(1-\frac{1}{2\cos\pi\lambda}\right)B(1-\lambda,\ 1-\lambda);$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{f(x)g(y)}{|\tan x-\tan y|^\lambda}dxdy \le K_{\lambda,b}^{1/p}K_{\lambda,a}^{1/q}\left(\int_0^{\pi/2}\frac{[\tan x]^{1+ap/q-\lambda-b}f^p(x)}{[\sec x]^{2p/q}}dx\right)^{1/p}$$

$$\times \left(\int_0^{\pi/2} \frac{[\tan y]^{1+bq/p-\lambda-a} g^q(y)}{[\sec y]^{2q/p}} dy \right)^{1/q}; \quad (10)$$

$$\int_{-\infty}^{\infty} \int_{\infty}^{\infty} \frac{f(x)g(y)}{|e^x - e^y|} dx dy \leq K_{\lambda,b}^{1/p} K_{\lambda,a}^{1/q} \left(\int_{-\infty}^{\infty} e^{x(1-\lambda-b-(1-a)p/q)} f^p(x) dx \right)^{1/p}$$

$$\times \left(\int_{-\infty}^{\infty} e^{y(1-\lambda-a-(1-b)q/p)} g^q(y) dy \right)^{1/q}, \quad (11)$$

provided the integrals on the right-hand side exist.

Corollary 2. *Theorem 2 holds for* $H(x,y) = (x+y)^\lambda$ *with* $K_\lambda = B(\lambda/2, \lambda/2)$ *and for* $H(x,y) = x^\lambda + y^\lambda$ *with* $K_\lambda = \pi/\lambda$.

Proof. It suffices to notice that

$$\int_0^{\infty} \frac{u^{\lambda/2-1}}{1+u^\lambda} du = \frac{1}{\lambda} \int_0^{\infty} \frac{v^{-1/2}}{1+v} dv = \frac{1}{\lambda} B(\frac{1}{2}, \frac{1}{2}) = \frac{\pi}{\lambda}.$$

□

Corollary 3. *Theorem 3 holds for* $h(x,y) = |x-y|^\lambda$, $0 < \lambda < 1$, $s = r = 2$, *with* $K_\lambda = 2B(1-\lambda, \lambda/2)$ *and for* $h(x,y) = (x+y)^\lambda$ *with* $K_\lambda = B(\lambda/r, \lambda/s)$.

Proof. It suffices to notice that

$$\begin{aligned} \int_0^{\infty} \frac{u^{\lambda/2-1}}{|1-u|^\lambda} du & \\ &= \int_0^1 \frac{u^{\lambda/2-1}}{(1-u)^\lambda} du + \int_1^{\infty} \frac{u^{\lambda/2-1}}{(u-1)^\lambda} du \\ &= 2 \int_0^1 \frac{u^{\lambda/2-1}}{(1-u)^\lambda} du \\ &= 2B(1-\lambda, \lambda/2). \end{aligned}$$

□

Remark 1. The second part of Corollary 3 contains inequality (3) of Theorem A as a special case when $r = \frac{\lambda p}{p+\lambda-2}$.

Remark 2. Inequality (4) of Theorem B can be obtained from Theorem 3 by putting $h(x,y) = (x^\lambda - y^\lambda)/\ln(x/y)$, which is homogeneous of degree λ, since

$$K_\lambda = \int_0^\infty \frac{x^{\lambda/r-1}}{x^\lambda - 1}dx = \frac{1}{\lambda^2}\int_0^\infty \frac{u^{-1/s}\ln u}{u-1}du = \left(\frac{\pi}{\lambda \sin(\pi/s)}\right)^2$$

(see [3, Theorem 342]).

In particular, the result of [6] also follows by putting $r = q, \ s = p$.

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