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Core theorems concerning Riesz method and Abel type method of summability

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ABSTRACT. Relations between cores that are defined by Abel type summability method J_p and by Riesz method R_p are investigated. Two Tauberian theorems are proved for these cores.

1. Introduction and background

Let ω be the set of all sequences $x = (\xi_k)$ where $\xi_k \in \mathbb{C}$, $k \in \mathbb{N}^0$ and $\mathbb{N}^0 := \{0, 1, 2, ...\}$. Each linear subspace of ω is called a sequence space. The following sequence spaces are well known:

1) the space of all bounded sequences l_∞,
2) the space of all convergent sequences c,
3) the space of all null sequences c_0.

Let A be the matrix method that is determined by an infinite matrix $A = (a_{nk})$ and let ω_A denote the application domain of A and let c_A denote the space of A-convergent sequences, i.e.,

$$c_A := \left\{ x \in \omega_A \,|\, \exists \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\xi_k =: A\text{-}\lim x \right\}.$$

The sequence space

$$c_{A0} := \{x \in c_A \,|\, A\text{-}\lim x = 0\}$$

is called the space of A-null sequences. A method A is called regular, if $c \subset c_A$ and A-$\lim x = \lim x$ for every $x \in c$.

Let X be a sequence space and let π be an arbitrarily fixed functional on X with range $[-\infty, \infty]$ such that

1) $\pi(0) = 0$,

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2) $\pi(\alpha x) = \alpha\pi(x) \quad \forall\alpha > 0$,

3) $\pi(x+y) \le \pi(x) + \pi(y) \quad \forall x : |\pi(x)| < \infty$.

With the usual conventions for the manipulation of ∞ and $-\infty$, these conditions are always meaningful. This functional π is called *Bonsall functional* and the set

$$K_\pi(x) := \{t \in \mathbb{C} \mid \operatorname{Re}(\alpha t) \le \pi(\alpha x) \quad \forall\alpha \in \mathbb{C}\}$$

is called *Bonsall core* of the $x \in X$ (defined by π). If $\pi(\alpha x) = -\infty$ holds for a certain α, then $K_\pi(x)$ is empty. It is easy to check that

$$K_\pi(x) = \{t \in \mathbb{C} \mid -\pi(-\alpha x) \le \operatorname{Re}(\alpha t) \le \pi(\alpha x) \quad \forall\alpha \in \mathbb{C}\}.$$

Let π_1 and π_2 be two Bonsall functionals. An immediate consequence of the definition of core is that if

$$\pi_1(\alpha x) \le \pi_2(\alpha y) \quad \forall\alpha \in \mathbb{C}, \tag{1}$$

then

$$K_{\pi_1}(x) \subset K_{\pi_2}(y).$$

Due to the possibility of empty cores the converse implication is not always true. Denote the set of all π-*convergent elements* by

$$c_\pi := \{x \in X \mid K_\pi(x) \text{ is a singleton and } |\pi(\alpha x)| < \infty\ \forall\alpha \in \mathbb{C}\}$$

and the set of all π-*null elements* by

$$c_{\pi 0} := \{x \in c_\pi \mid \pi(x) = 0\}.$$

The sets c_π and $c_{\pi 0}$ are sequence spaces (see [1]).

The concept of the core of a sequence $x = (\xi_k)$ of complex numbers has been defined by Knopp in 1930 (see [3], Chapter VI). The Bonsall functional that defines the Knopp core $K^\circ(x)$ is

$$\pi^\circ(x) := \limsup_{k\to\infty} \operatorname{Re}\xi_k$$

(cf. [1], [3]). It is obvious that the set $c_{\pi^\circ 0}$ of π°-null elements is c_0 and the set c_{π° of π°-convergent elements is c. The following is the well-known Knopp core theorem (cf. [4], Theorem 9):

Theorem 1. *If a matrix method A is positive and regular, then*

$$\pi^\circ(Ax) \le \pi^\circ(x) \quad \forall x \in \omega_A$$

and

$$K^\circ(Ax) \subset K^\circ(x) \quad \forall x \in \omega_A.$$

2. Auxiliary results

In this section we give some definitions and propositions which are needed in the proofs of main results.

We assume throughout that (p_k) is a sequence of reals satisfying

$$\left.\begin{array}{l} p_0 > 0, \quad p_k \geq 0 \ (k \in \mathbb{N}), \quad P_n := \sum\limits_{k=0}^{n} p_k \to \infty \text{ as } n \to \infty \\ \text{and} \\ p(t) := \sum_{k=0}^{\infty} p_k t^k \text{ has the radius of convergence } R = 1. \end{array}\right\} \qquad (2)$$

Definition 1. A sequence $x = (\xi_k) \in \omega$ is said to be *summable by the Riesz method* R_p to a number a (or R_p-summable to a) if

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=0}^{n} p_k \xi_k = a.$$

Riesz method R_p is regular under (2) (see [2], p. 113).

Let for $x = (\xi_k) \in \omega$

$$p_x(t) := \sum_{k=0}^{\infty} p_k \xi_k t^k$$

and let

$$\omega_p := \{x \in \omega \,|\, \text{radius of convergence of } p_x(t) \text{ is equal or greater than } 1\}.$$

It is obvious that $l_\infty \subset \omega_p$.

Definition 2. A sequence $x \in \omega_p$ is said to be *summable by power series method* J_p to a number a (or J_p-summable to a) if

$$\lim_{t\to 1-} \frac{p_x(t)}{p(t)} = a =: J_p\text{-}\lim x.$$

The set of all sequences x, that are summable by a power series method J_p is denoted by c_{J_p}. The set ω_p is called the *application domain of* J_p.

Remark 1. The well-known Abel summability method J_1 is the power series method J_p defined by $p = (p_k)$ where $p_k = 1$ $(k \in \mathbb{N}^0)$. Then $R = 1$ and $p(t) = \dfrac{1}{1-t}$ for $t \in (-1, 1)$. For that reason the power series summability method J_p defined above is called *Abel type method.*

Abel type method J_p is regular under (2), that is, $c \subset c_{J_p}$ and J_p-$\lim x = \lim x$ (see [2], p. 160). Moreover $c_{R_p} \subset c_{J_p}$ and J_p-$\lim x = R_p$-$\lim x$ for every $x \in c_{R_p}$ (see [5]).

Let

$$W := \{w = (t_k) \mid 0 < t_k \to 1-\}.$$

The matrix method corresponding to the infinite matrix $A_w = (a_{nk})$, where

$$a_{nk} = \frac{p_k t_n^k}{p(t_n)},$$

is called a *discrete J_p-method* (with respect to $p = (p_k)$ and $w = (t_n) \in W$). An immediate consequence of the sequential criterion for the existence of a limit is

$$c_{J_p} = \bigcap_{w \in W} c_{A_w},$$

where c_{A_w} is the set of A_w-convergent sequences.

Proposition 1. *The following statements are equivalent:*

(a) *J_p is regular,*

(b) *for each $w \in W$ the discrete J_p-method A_w is regular.*

Proof. For the proof see [2], p. 160. □

Corollary 1. *Assume that* (2) *holds and $w \in W$. Then for every discrete J_p-method A_w*

$$\pi^\circ(A_w x) \leq \pi^\circ(x) \text{ for each } x \in \omega_p$$

and

$$K^\circ(A_w x) \subset K^\circ(x) \text{ for each } x \in \omega_p.$$

Proof. As by Proposition 1 the method A_w is positive and regular, the proof of this corollary follows directly from Theorem 1. □

The notion of a core for a power series method J_p is as follows.

Definition 3. The core $K_p(x)$ of $x \in \omega_p$ defined by Bonsall functional

$$\pi_p(x) = \limsup_{t \to R-} \frac{\operatorname{Re} p_x(t)}{p(t)}$$

on ω_p is called a *power series Knopp core induced by $p = (p_k)$*.

It is easy to see that π_p is a Bonsall functional and

$$\pi_p(x) = \sup_{w \in W} \pi^\circ(A_w x) \tag{3}$$

for every $x \in \omega_p$ (see [9]).

Proposition 2. *Assume that* (2) *holds. Then*

(a) *$K^\circ(A_w x) \subset K_p(x)$ for every $x \in \omega_p$ and $w \in W$,*

(b) *$K_p(x) \subset K^\circ(x)$ for every $x \in \omega_p$,*

(c) $c_{\pi_p} = \bigcap_{w \in W} c_{A_w} = c_{J_p}$.

Proof. These properties are immediate consequences of the definition of core $K_p(x)$ and of Corollary 1. □

3. Inclusion between the cores concerning weighted means and power series

The main result of this section is as follows.

Theorem 2. *Assume that $p = (p_k)$ satisfies (2) and $p_k > 0$ $(k \in \mathbb{N}^0)$. Let R_p be the Riesz method and let $A_w = (a_{nk})$ be the discrete J_p-method that is defined with respect to $p = (p_k)$ and $w = (t_n) \in W$. Then*

$$\pi^\circ(A_w x) \leq \pi^\circ(R_p x) \quad \forall x \in \omega_p \tag{4}$$

and

$$\pi_p(x) \leq \pi^\circ(R_p x) \quad \forall x \in \omega_p. \tag{5}$$

Proof. If $p_k > 0$ for each $k \in \mathbb{N}^0$, then the inverse $R_p^{-1} = (r_{nk})$ of R_p is given by

$$r_{nk} = \begin{cases} \dfrac{P_n}{p_n} & \text{if } k = n \\ -\dfrac{P_{n-1}}{p_n} & \text{if } k = n-1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall n, k \in \mathbb{N}^0. \tag{6}$$

(see [2], p. 113). Let $x = (\xi_k) \in \omega_p$ and $y = (\eta_n) = R_p x$, i.e., $x = R_p^{-1} y$. Put $G = (g_{nk}) := A_w R_p^{-1}$. It means that

$$g_{nk} = \frac{p_k t_n^k P_k}{p(t_n) p_k} - \frac{p_{k+1} t_n^{k+1} P_k}{p(t_n) p_{k+1}} = \frac{P_k}{p(t_n)} t_n^k (1 - t_n).$$

We will prove that we can calculate associatively for each $x \in \omega_p$, that is

$$A_w x = A_w\left(R_p^{-1} y\right) = \left(A_w R_p^{-1}\right) y = Gy. \tag{7}$$

According to (6)

$$\xi_k = \frac{1}{p_k}\left(P_k \eta_k - P_{k-1}\eta_{k-1}\right) \tag{8}$$

(we put $\eta_{-1} = 0 = P_{-1}$). Let $A_w x =: (\zeta_n)$, and $Gy =: (\vartheta_n)$, then

$$\zeta_n = \frac{1}{p(t_n)} \sum_{k=0}^{\infty} p_k \xi_k t_n^k = \frac{1}{p(t_n)} \lim_{m\to\infty} \sum_{k=0}^{m} p_k \xi_k t_n^k.$$

By (8) we have

$$\sum_{k=0}^{m} p_k \xi_k t_n^k = \sum_{k=0}^{m} \left(P_k \eta_k - P_{k-1}\eta_{k-1}\right) t_n^k.$$

Using Abel's partial summation formula we get that

$$\begin{aligned}\sum_{k=0}^{m}(P_k\eta_k - P_{k-1}\eta_{k-1})\,t_n^k &= \sum_{k=0}^{m}\left(t_n^k - t_n^{k+1}\right)\sum_{j=0}^{k}(P_j\eta_j - P_{j-1}\eta_{j-1}) + \\ +P_m\eta_m t_n^{m+1} &= \sum_{k=0}^{m}(1-t_n)\,t_n^k P_k\eta_k + P_m\eta_m t_n^{m+1}.\end{aligned}$$

Now, if we can show that for every $x \in \omega_p$ the last term converges to zero when $m \to \infty$, then

$$\zeta_n = \vartheta_n \quad \forall n \in \mathbb{N}^0$$

and (7) holds for every $x \in \omega_p$. To start, let us evaluate the following:

$$\begin{aligned}\left|P_m\eta_m t_n^{m+1}\right| &= \left(\sqrt{t_n}\right)^{m+3}\left|\sum_{k=0}^{m} p_k\xi_k\left(\sqrt{t_n}\right)^m\right| \leq \\ &\leq \left(\sqrt{t_n}\right)^{m+3}\sum_{k=0}^{m} p_k\left|\xi_k\right|\left(\sqrt{t_n}\right)^k \leq \\ &\leq \left(\sqrt{t_n}\right)^{m+3} p_{|x|}\left(\sqrt{t_n}\right)\end{aligned}$$

(here $|x| = (|\xi_k|)$).

Due to $0 < t_n < 1$ and $|x| \in \omega_p$ we get that

$$\left|P_m\eta_m t_n^{m+1}\right| \leq \left(\sqrt{t_n}\right)^{m+3} p_{|x|}\left(\sqrt{t_n}\right) \to 0 \text{ as } m \to \infty,$$

i.e., (7) is proved and this gives

$$\pi^\circ(A_w x) = \pi^\circ(Gy) \quad \forall x \in \omega_p.$$

According to the fact that

$$J_p\text{-}\lim x = R_p\text{-}\lim x \quad \forall x \in c_{R_p},$$

method G is regular and therefore by Theorem 1

$$\pi^\circ(Gz) \leq \pi^\circ(z) \quad \forall z \in \omega_G \tag{9}$$

holds. Since $y = R_p x \in \omega_G$, for every $x \in \omega_p$, equality (4) holds. The sequence $w \in W$ was arbitrarily fixed, consequently (4) is true for every $w \in W$ and by (3) inequality (5) follows. □

Due to the definition of Bonsall core the next corollary follows directly from Theorem 2.

Corollary 2. *Assume that* $p = (p_k)$ *satisfies* (2) *and* $p_k > 0$ $(k \in \mathbb{N}^0)$. *Then for every* $w \in W$

$$K^\circ(A_w x) \subset K^\circ(R_p x) \quad \forall x \in \omega_p$$

and

$$K_p(x) \subset K^\circ(R_p x) \quad \forall x \in \omega_p. \tag{10}$$

Remark 2. The inclusion (10) is proved in [8] for those $x \in \omega_p$ that have the property $R_p x \in l_\infty$.

4. Tauberian core theorems for J_p

Let A and B be two different matrix methods with $c_B \subset c_A$. The problem to determine the subset L of ω, such that $x \in L \cap c_A$ implies $x \in c_B$, has been studied extensively. In summability theory the theorem which gives the description of certain L is called a Tauberian theorem. The condition which determines L is called a Tauberian condition.

Let π_1 and π_2 be two different Bonsall functionals on a sequence space X with $c_{\pi_1} \subset c_{\pi_2}$. Naturally, there arises a question of how to give the description of certain subsets L of X having one of the following properties:

(a) $x \in L \Longrightarrow K_{\pi_1}(x) = K_{\pi_2}(x)$,

(b) $x \in L \Longrightarrow K_{\pi_1}(x) = K^\circ(x)$,

(c) $x \in L \Longrightarrow K_{\pi_2}(x) \subset K_{\pi_1}(x)$.

We call the theorem which states (a), (b) or (c) *a Tauberian core theorem.* The condition which determines L is called a *Tauberian condition* (see [9]). Our main tool to prove Tauberian theorems is the following proposition (for the proof see [9]).

Proposition 3. *If $x - y \in c_{\pi 0}$, then $\pi(x) = \pi(y)$ and $K_\pi(x) = K_\pi(y)$.*

We now state a general Tauberian theorem for a discrete Abel type method.

Theorem 3. *Assume that $p = (p_k)$ satisfies (2) and $p_k > 0$ $(k \in \mathbb{N}^0)$ and let $\lambda \in \mathbb{C}$. Let A_w and R_p be a discrete Abel type method and Riesz method respectively. Let $G = A_w R_p^{-1}$ and $H = G - \lambda I$, where I is the identity matrix. Then*

$$K^\circ(A_w x) = K^\circ(\lambda R_p x) \quad \forall x \in L$$

where

$$L = \{x \in \omega_p \mid R_p x \in c_{0H}\}.$$

Proof. As (7) holds for each $x \in \omega_p$, the proof follows directly from Proposition 3. □

Let J_p be an Abel type power series method and let $\Delta_k := \inf_{0<t<R} p(t)\, t^{-k}$. For every regular method $J_p$there exists a sequence $w^* = (t_k^*) \in W$ with the property

$$p(t_k^*)(t_k^*)^{-k} = \Delta_k \text{ for all } k \in \mathbb{N}^0. \tag{11}$$

Each sequence $w^* = (t_k^*)$ $(0 < t_k^* < 1,\ k \in \mathbb{N}^0)$ with property (11) has the following properties (see [2], p. 187):

1) $0 \le t_n^* \le t_{n+1}^* \to 1\ (n \to \infty)$,

2) $(t_r^*)^{n-r} \le \dfrac{\Delta_r}{\Delta_n} \le (t_r^*)^{n-r}$ for all $n, r \in \mathbb{N}^0$,

3) $\Delta_n \ge P_n \quad \forall n \in \mathbb{N}^0$.

In Tauberian theorems the Tauberian conditions are frequently connected with the sequence (Δ_k). These quantities Δ_k play an important roll in Tauberian core theorems also. The key to what follows is the next lemma due to W. Kratz and U. Stadtmüller (see [7], p. 148, the first part of proof of Theorem 1).

Lemma 1. *Let J_p be an Abel type method where $p = (p_k)$ satisfies* (2) *and let $w^* = (t_k^*) \in W$ have property* (11)*. Each $x = (\xi_k) \in \omega_p$ which satisfies the Tauberian condition*

$$|\xi_n - \xi_m| \le \delta_m \left(1 + \sum_{k=m}^{n-1} \frac{p_k}{\Delta_k}\right) \quad \text{for} \quad 1 \le m < n-1 \quad \text{with} \quad \lim_{m \to \infty} \delta_m = 0 \tag{12}$$

has the following property

$$\lim_k \left(\frac{p_x(t_k^*)}{p(t_k^*)} - \xi_k \right) = 0.$$

Theorem 4. *Let J_p be a regular Abel type method where $p = (p_k)$ satisfies* (2) *and let $w^* = (t_k^*) \in W$ have property* (11)*. Then*

$$K_p(x) = K^\circ(x) \tag{13}$$

for each $x = (\xi_k) \in \omega_p$ which satisfies Tauberian condition (12)*.*

Proof. Let $x = (\xi_k) \in \omega_p$ satisfy Tauberian condition (12) and let $w^* = (t_k^*) \in W$ have property (11). By Proposition 3 and Lemma 1 we get that

$$\pi^\circ(A_{w^*}(x)) = \pi^\circ(x) \text{ and } K^\circ(A_{w^*}x) = K^\circ(x).$$

Due to Proposition 2 we get

$$K^\circ(A_{w^*}x) \subset K_p(x) \subset K^\circ(x),$$

i.e., (13) holds. □

Remark 3. The condition

$$\max_{P_n \le P_m \le \lambda P_n} |\xi_{m+1} - \xi_n| = o\left(\frac{P_n}{\Delta_n}\right) \qquad (\text{for some } \lambda > 1)$$

implies (12). If the sequence of positive numbers (l_n) is defined by

$$\sum_{k=n}^{n+l_n-1} \frac{p_k}{\Delta_k} < 1 \leq \sum_{k=n}^{n+l_n} \frac{p_k}{\Delta_k},$$

then the condition

$$\max_{n \leq m \leq n+l_n} |\xi_{m+1} - \xi_n| = o\,(1) \quad (\text{as } n \to \infty) \tag{14}$$

is equivalent to the Tauberian condition (12) (see [7]).

Example. Now we specify the sequences (Δ_k) and (l_k) that are needed in Tauberian conditions (12) and (14) for some well-known J_p-methods. The information about the sequence (Δ_k) that corresponds to the given J_p-method is available in [2], p. 191. For the suitable choice of (l_k) see [6]. In what follows $c > 0$ stands for some constant and $[\cdot]$ denotes the greatest integer function.

1. If $p_k = 1$ for each $k \in \mathbb{N}^0$, then $\Delta_k \sim ek$ and $l_k = [ck]$.
2. If $p = \left(\frac{1}{k+1}\right)$, then $\Delta_k \sim \log k$ and $l_k = [ck \log k]$.
3. If $\alpha \in (0,1)$ and $p = \left(e^{k^\alpha}\right)$, then $\Delta_k \sim p_k \sqrt{\frac{2\pi}{\beta\,(1-\beta)}} k^{1-\frac{\alpha}{2}}$ and $l_k = \left[ck^{1-\frac{\alpha}{2}}\right]$.

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On strong summability of sequences

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ABSTRACT. We extend the notion of strong summability of sequences by matrix methods. Combining the notions of strong summability given by D. Borwein and I. J. Maddox we come to a more general notion. The properties of the extended strong summability are characterized and relations between strong and ordinary summabilities are described. The defined notion of strong summability is applied to certain families of summability methods, including some families of generalized Nörlund methods. Partial cases are the families of Cesàro and Euler–Knopp methods.

1. Introduction and preliminaries

1.1. We start with some basics of the summability theory (see [1]). Let us consider sequences $x = (\xi_n)$ with $\xi_n \in \mathbb{C}$ for every $n \in \mathbb{N}^0 = \{0, 1, 2, \dots\}$. Let A be a transformation which transforms a sequence x into the sequence $y = Ax = (\eta_n)$. If the limit $\lim_n \eta_n = \xi$ exists, then we say that $x = (\xi_n)$ is summable to ξ by the summability method A and write $\xi_n \to \xi(A)$. For the set of all x summable by A we use the notation c_A. The most common summability method is a matrix method A defined with the help of the matrix $A = (a_{nk})$, where $a_{nk} \in \mathbb{C}$ for any $n, k \in \mathbb{N}^0$, and which transforms x into $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} a_{nk}\xi_k \qquad (n \in \mathbb{N}^0).$$

If

$$\xi_n \to \xi \Longrightarrow \xi_n \to \xi(A)$$

for any $x = (\xi_n) \in c$, then the method A is called regular.

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