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# On strong summability of sequences

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ABSTRACT. We extend the notion of strong summability of sequences by matrix methods. Combining the notions of strong summability given by D. Borwein and I. J. Maddox we come to a more general notion. The properties of the extended strong summability are characterized and relations between strong and ordinary summabilities are described. The defined notion of strong summability is applied to certain families of summability methods, including some families of generalized Nörlund methods. Partial cases are the families of Cesàro and Euler–Knopp methods.

## 1. Introduction and preliminaries

**1.1.** We start with some basics of the summability theory (see [1]). Let us consider sequences $x = (\xi_n)$ with $\xi_n \in \mathbb{C}$ for every $n \in \mathbb{N}^0 = \{0, 1, 2, \dots\}$. Let $A$ be a transformation which transforms a sequence $x$ into the sequence $y = Ax = (\eta_n)$. If the limit $\lim_n \eta_n = \xi$ exists, then we say that $x = (\xi_n)$ is summable to $\xi$ by the summability method $A$ and write $\xi_n \to \xi(A)$. For the set of all $x$ summable by $A$ we use the notation $c_A$. The most common summability method is a matrix method $A$ defined with the help of the matrix $A = (a_{nk})$, where $a_{nk} \in \mathbb{C}$ for any $n, k \in \mathbb{N}^0$, and which transforms $x$ into $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} a_{nk}\xi_k \qquad (n \in \mathbb{N}^0).$$

If

$$\xi_n \to \xi \Longrightarrow \xi_n \to \xi(A)$$

for any $x = (\xi_n) \in c$, then the method $A$ is called regular.

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**Proposition A.** i) *A matrix method* $A = (a_{nk})$ *is regular if and only if the following conditions are satisfied:*

1) $\lim_n a_{nk} = 0 \quad (k \in \mathbb{N}^0)$,
2) $\sum_{k=0}^{\infty} |a_{nk}| = O(1)$,
3) $\lim_n \sum_{k=0}^{\infty} a_{nk} = 1$.

ii) *A matrix method* $A = (a_{nk})$ *is of type* $c_0 \to c_0$ *if and only if the conditions* 1) *and* 2) *are satisfied.*

Let $A$ and $B$ be two summability methods. If

$$\xi_n \to \xi(A) \Longrightarrow \xi_n \to \xi(B)$$

for all $x \in c_A$, then it is said that $B$ is not weaker than $A$ and it is denoted $A \subset B$.

**1.2.** Next we will introduce the families of generalized Nörlund methods (see [10]). Let be given two nonnegative sequences $(p_n)$ and $(q_n)$ with $p_0, q_0 > 0$. Then we define for $\alpha \in \mathbb{R}$ the sequences

$$p_n^\alpha := \sum_{k=0}^{n} c_{n-k}^\alpha \, p_k$$

where $c_n^\alpha$ is either

case $\mathcal{A}$) $\quad c_n^\alpha = A_n^{\alpha-1} := \binom{n+\alpha-1}{n}, \quad n \in \mathbb{N}^0,$

or

case $\mathcal{B}$) $\quad c_n^\alpha = \frac{\alpha^n}{n!}, \quad n \in \mathbb{N}^0,$

and

$$r_n^\alpha = \sum_{k=0}^{n} p_{n-k}^\alpha \, q_k = (p^\alpha * q)_n .$$

Fix some $\alpha_0 \in \mathbb{R}$ such that $r_n^\alpha > 0$ for all $n \in \mathbb{N}^0$ and $\alpha > \alpha_0$ (the choice $\alpha_0 = 0$ is always possible).

Now we consider the family of generalized Nörlund summability methods

$$N_\alpha = (N, p_n^\alpha, q_n) \ (\alpha > \alpha_0).$$

We say that a sequence $x = (\xi_n)$ is summable to $\xi$ by the method $N_\alpha$ $(\alpha > \alpha_0)$ and write $\xi_n \to \xi(N_\alpha)$ if

$$\eta_n^\alpha := \frac{1}{r_n^\alpha} \sum_{k=0}^{n} p_{n-k}^\alpha \, q_k \, \xi_k \longrightarrow \xi \ (n \to \infty).$$

The methods $N_\alpha$ and $N_\beta$ are connected for any $\beta > \alpha > \alpha_0$ through the relation

$$\eta_n^\beta = \frac{1}{r_n^\beta} \sum_{k=0}^{n} c_{n-k}^{\beta-\alpha} \, r_k^\alpha \, \eta_k^\alpha, \quad n \in \mathbb{N}^0. \tag{1}$$

Particular cases of $\mathcal{A}$) are the Cesàro methods $(C,\alpha)$ where $p_n = \delta_{0,n}$ and $q_n \equiv 1$, generalized Cesàro methods $(C,\alpha,\gamma)$ where $p_n = \delta_{0,n}$ and $q_n = \binom{n+\gamma}{n}$ and more generally the methods $(N, A_n^{\alpha-1},\ q_n)$. An example for the case $\mathcal{B}$) is given by the Euler–Knopp family $(E_{1/(\alpha+1)})$ with $p_n = \delta_{0,n}\,, q_n = 1/n!$ and $\alpha > 0$. That is why it is told that the family $(N_\alpha)$ is of Cesàro-type in case $\mathcal{A}$) and of Euler–Knopp-type in case $\mathcal{B}$) (see [10] for more detailed references).

**1.3.** In our paper we consider more general families $(A_\alpha)$ (see [10]) of summability methods than $(N_\alpha)$ are. In order to study the relations between methods from the family of generalized Nörlund methods $N_\alpha$, the specific form of these matrix transforms is less important than the form of the connection matrices from (1). Therefore the family of methods $N_\alpha$ is generalized to a family of more general summability methods $A_\alpha$ $(\alpha > \alpha_1)$ (see [10]).

**Definition A.** Let $(A_\alpha)$ $(\alpha > \alpha_1)$ be a family of summability methods. The family $(A_\alpha)$ is said to be a Cesàro- or an Euler–Knopp-type family if for every $\beta > \alpha > \alpha_1$ the transformed sequences $A_\alpha x = (\eta_n^\alpha)$ and $A_\beta x = (\eta_n^\beta)$ of $x = (\xi_n)$ are related by the connection formula

$$\eta_n^\beta = \frac{1}{b_n^\beta}\sum_{k=0}^{n} c_{n-k}^{\beta-\alpha}\, b_k^\alpha\, \eta_k^\alpha, \quad (n \in \mathbb{N}^0, \quad \beta > \alpha > \alpha_1)\,, \tag{2}$$

where $(b_n^\alpha)$ $(\alpha > \alpha_1)$ are some positive sequences being related by

$$b_n^\beta = \sum_{k=0}^{n} c_{n-k}^{\beta-\alpha}\, b_k^\alpha \quad (n \in \mathbb{N}^0, \quad \beta > \alpha > \alpha_1) \tag{3}$$

and $(c_n^\alpha)$ are defined as in Section 1.2 in case $\mathcal{A}$) or in case $\mathcal{B}$), respectively.

From relations (2) and (3) we obtain the connection formula

$$A_\beta = D_{\alpha,\beta} \circ A_\alpha \quad (\beta > \alpha > \alpha_1)$$

where $D_{\alpha,\beta} = (d_{n,k}^{\alpha,\beta})$ with

$$d_{n,k}^{\alpha,\beta} = \begin{cases} c_{n-k}^{\beta-\alpha}\, b_k^\alpha / b_n^\beta & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n\,. \end{cases}$$

The connection methods $D_{\alpha,\beta}$ can be seen as generalizations of Cesàro methods in case $\mathcal{A}$) and Euler–Knopp methods in case $\mathcal{B}$), that is why $(A_\alpha)$ is named a Cesàro-type family in case $\mathcal{A}$) and an Euler–Knopp-type family in case $\mathcal{B}$) (see [10]). In particular, the methods $A_\alpha = N_\alpha$ $(\alpha > \alpha_0)$ form a Cesàro-type family in case $\mathcal{A}$) and an Euler–Knopp-type family in case $\mathcal{B}$), satisfying (2) and (3) with $(b_n^\alpha) = (r_n^\alpha)$. More examples can be found in [9].

Formula (3) implies that the connection methods in (2) are regular, even if the methods $A_\alpha$ are not (see [10], Lemma 1).

**Proposition B.** *Consider a Cesàro- or an Euler–Knopp-type family* $(A_\alpha)$ $(\alpha > \alpha_1)$. *The methods* $D_{\alpha,\beta} = (d_{n,k}^{\alpha,\beta})$ *are regular for all* $\beta > \alpha > \alpha_1$.

The following inclusion theorem holds for methods in a family $(A_\alpha)$ (see [10], Proposition 1 and Corollary 1).

**Proposition C.** *Let* $(A_\alpha)$ $(\alpha > \alpha_1)$ *be a Cesàro- or an Euler–Knopp-type family. Then we have for a sequence* $(\xi_n)$ *and* $\beta > \alpha > \alpha_1$ *that*

$$\xi_n \to \xi\,(A_\alpha) \Rightarrow \xi_n \to \xi\,(A_\beta).$$

In particular, if $A_\alpha = N_\alpha$ $(\alpha > \alpha_0)$, the statements of Propositions B and C are true with $\alpha_1 = \alpha_0$.

**1.4.** The idea of the present paper is to develop the notion of strong summability of sequences by matrix methods. Combining the definitions of strong summability given in [2] and [5] we come to a more general definition of strong summability which will be formulated and characterized in Section 2. In Section 3 this definition will be applied to the methods in families $(A_\alpha)$ described above.

## 2. A general definition of strong summability

Let be given a summability method $A$ which transforms a sequence $x = (\xi_n)$ into the sequence $Ax = (\eta_n)$. Moreover, let be given also a matrix method $P = (p_{nk})$. We denote

$$\sigma_n := \sum_{k=0}^{\infty} p_{nk}\,\eta_k \quad (n \in \mathbb{N}^0), \tag{4}$$

and say that $x$ is $PA$-summable to $\xi$ if $\sigma_n$ is finite for every $n \in \mathbb{N}^0$ and $\lim_n \sigma_n = \xi$. Next we define the main notion of our paper.

**Definition 1.** Let $r = (r_n)$ be a sequence of positive numbers. We say that a sequence $x = (\xi_n)$ is strongly summable by the method $PA$ with index $r = (r_n)$ (in short, $[P, A]_r$-summable) to $\xi$ and write $\xi_n \to \xi[P, A]_r$, if

$$\mu_n := \sum_{k=0}^{\infty} |p_{nk}|\,|\eta_k - \xi|^{r_k} \to 0 \quad (n \to \infty). \tag{5}$$

Some comments should be added to the formulated definition.

**Remark 1.** i) In particular, if $r_n \equiv r$, matrix $P = (p_{nk})$ is positive and $A$ is a matrix method, this definition was formulated by D. Borwein in 1960 (see [2]). If, in addition, $A = I$, the definition was known already in 1938 (see [4]).

ii) In particular, if $A = I$ and matrix $P = (p_{nk})$ is positive Definition 1 reduces to the definition given for strong $P$-summability by I.J. Maddox in 1967 (see [5]). This definition of Maddox was generalized for strong sequential summability ($\alpha$-summability) by the sequence of methods $P_i = (p_{nik})$ $(n, i, k \in \mathbb{N}^0)$ in [8].

iii) In particular, if $r_n \equiv r$, Definition 1 was given in [12].

**Theorem 1.** *Suppose that $P = (p_{nk})$ satisfies the condition*

$$\sum_{k=1}^{\infty} |p_{nk}| = O(1), \tag{6}$$

*and $r = (r_n)$ and $r' = (r'_n)$ are sequences satisfying the condition*

$$0 < r'_n \leq r_n \leq K\, r'_n \qquad (n \in \mathbb{N}^0)$$

*where $K$ is some positive constant. If a sequence $x = (\xi_n)$ is $[P, A]_r$-summable to $\xi$, then $x$ is also $[P, A]_{r'}$-summable to $\xi$, i. e., the relation $[P, A]_r \subset [P, A]_{r'}$ holds.*

*Proof.* In particular, if $A = I$, this theorem was proved in [5], i. e., in [5] it was proved that

$$\xi_n \to \xi[P, I]_r \Longrightarrow \xi_n \to \xi[P, I]_{r'}. \tag{7}$$

for any $x = (\xi_n)$. We have also that

$$\xi_n \to \xi[P, A]_r \Longleftrightarrow \eta_n \to \xi[P, I]_r$$

and

$$\xi_n \to \xi[P, A]_{r'} \Longleftrightarrow \eta_n \to \xi[P, I]_{r'}$$

for any $x = (\xi_n)$. Thus $[P, A]_r \subset [P, A]_{r'}$ holds by (7), if we apply it to $(\eta_n)$. □

In particular, if the $r_n \equiv r$, matrix $P = (p_{nk})$ is positive and $A$ is a matrix method, Theorem 1 turns into Theorem 1 from [2].

**Remark 2.** If $\frac{r_n}{r_{n'}} \to \infty$, then the inclusion relation $[P, A]_r \subset [P, A]_{r'}$ does not hold in general. For example, suppose that $P = (p_{nk})$ is regular and $A$ is a normal matrix method. Then there exists a sequence $b = (b_n)$ of numbers 0 and 1 such that $b \notin c_P$. Thus also $d = (d_n) = \frac{b+e}{4} \notin c_P$ (where $e = (e_n)$, $e_n \equiv 1$). Let $x = (\xi_n)$ be the sequence such that $Ax = (\eta_n) = (d_n^{1/r'_n})$. Thus we have the sequence $x$ that is $[P, A]_r$-summable but not $[P, A]_{r'}$-summable.

Theorem 1 implies the following corollary.

**Corollary 1.** *Suppose that $0 < \inf_n r_n = m \leq r_n \leq M = \sup_n r_n < \infty$ and condition (6) is satisfied. Then we have:*

$$[P, A]_{\mathbf{M}} \subset [P, A]_r \subset [P, A]_{\mathbf{m}},$$

*where $\mathbf{M} = (M_n)$ and $\mathbf{m} = (m_n)$ with $M_n \equiv M$ and $m_n \equiv m$.*

**Proposition 1.** *Suppose that*

$$\lim_n \sum_{k=0}^{\infty} p_{nk} \neq 0 \tag{8}$$

*and* $0 < r_n \leq H < \infty$. *Then a sequence* $(\xi_n)$ *can have at most one* $[P, A]_r$ *limit.*

*Proof.* a) In particular, if $A = I$, the statement is true by Theorem 2 from [5].

b) In general, $\xi$ is a $[P, A]_r$ limit for $(\xi_n)$ if and only if $\xi$ is a $[P, I]_r$ limit for $(\eta_n)$. As $(\eta_n)$ can have at most one $[P, I]_r$ limit by part a) then also $(\xi_n)$ can have at most one $[P, A]_r$ limit. □

**Remark 3.** Let us note that if

$$\lim_n \sum_{k=0}^{\infty} p_{nk} = 0,$$

then $[P, A]_r$ limit can be not unique (see [5], the proof of Theorem 2).

**Theorem 2.** *Suppose that* $P = (p_{nk})$ *satisfies conditions (6) and*

$$\lim_n \sum_{k=0}^{\infty} p_{nk} = 1, \tag{9}$$

*and* $r = (r_n)$ *satisfies the conditions* $1 \leq r_n \leq H < \infty$ *for every* $n \in \mathbb{N}^0$. *If a sequence* $x = (\xi_n)$ *is* $[P, A]_r$*-summable to* $\xi$, *then* $x$ *is also* $PA$*-summable to* $\xi$, *i. e., the relation* $[P, A]_r \subset PA$ *holds.*

*Proof.* Suppose that $r_n \equiv 1$ and assume that $\lim_n \sum_{k=0}^{\infty} |p_{nk}|\, |\eta_k - \xi| = 0$. Using (4) we get

$$|\sigma_n - \sum_{k=0}^{\infty} p_{nk}\, \xi| = |\sum_{k=0}^{\infty} p_{nk}\, (\eta_k - \xi)| \leq \sum_{k=0}^{\infty} |p_{nk}|\, |\eta_k - \xi|.$$

Thus the relation

$$\lim_n (\sigma_n - \sum_{k=0}^{\infty} p_{nk}\, \xi) = 0$$

holds, i. e., $\xi_n \to \xi(PA)$.

If $r_n \geq 1$, then $[P, A]_r \subset [P, A]_1$ by Theorem 1, and $\xi_n \to \xi(PA)$ again. □

**Remark 4.** If $0 < r_n < 1 \quad (n \in \mathbb{N}^0)$, then the assertion of Theorem 2 is not true. For example, if $P = (C, 1)$, $A$ is a normal matrix method (i. e., $a_{nk} = 0$ for $n > k$ and $a_{nn} \neq 0$) and $0 < r_n \equiv r < 1$, then $[P, A]_r \not\subset PA$ (see [6], p. 202).

The next result is proved in [5] by Theorem 3.

**Lemma 1.** i) *If $P = (p_{nk})$ is of type $c_0 \to c_0$, i. e., $P$ satisfies conditions (6) and*

$$\lim_n p_{nk} = 0 \quad (n \in \mathbb{N}^0), \tag{10}$$

*and if* $\inf_n r_n = m > 0$ $(n \in \mathbb{N}^0)$, *then*

$$\xi_n \to \xi \Longrightarrow \xi_n \to \xi[P, I]_r \tag{11}$$

*for every $x = (\xi_n) \in c$.*

ii) *If* $\sup_n r_n = M < \infty$ *and (11) holds for every $x \in c$, then $P = (p_{nk})$ is of type $c_0 \to c_0$.*

**Proposition 2.** i) *If $P = (p_{nk})$ is of type $c_0 \to c_0$, and* $\inf_n r_n = m > 0$, *then*

$$\xi_n \to \xi(A) \Longrightarrow \xi_n \to \xi[P, A]_r \tag{12}$$

*for any $x \in c_A$, i. e., the relation $A \subset [A, P]_r$ holds .*

ii) *Suppose that* $\sup_n r_n = M < \infty$ *and $A = (a_{nk})$ is a normal matrix method. If (12) holds for any $x \in c_A$, then $P = (p_{nk})$ is of type $c_0 \to c_0$.*

*Proof.* i). We have the equivalence

$$\xi_n \to \xi[P, A]_r \Longleftrightarrow \eta_n \to \xi[P, I]_r.$$

Thus our statement is true by (11) in Lemma 1 i) if we apply it to $(\eta_n)$.

ii). The relation $\xi_n \to \xi(A)$ for every $x \in c_A$ is the same as $\eta_n \to \xi$ for any $(\eta_n) \in c$ and our statement is true by Lemma 1 ii). □

In particular, if $r_n \equiv r$, the matrix $P = (p_{nk})$ is positive and $A$ is any matrix, then Proposition 2 i) and Theorem 2 give us Theorem 3 from [2] in weaker restrictions on method $P$ (in Theorem 3 from [2] $P$ is assumed to be regular).

**Theorem 3.** *Suppose that $P = (p_{nk})$ is regular and $1 \le r_n \le \sup_n r_n = M < \infty$. Then a sequence $x = (\xi_n)$ is $[P, A]_r$-summable to $\xi$ if and only if the following two conditions are satisfied:*

1) *The sequence $x = (\xi_n)$ is $PA$-summable to $\xi$;*
2) $\lim_n \sum_{k=0}^{\infty} |p_{nk}| \, |\sigma_k - \eta_k|^{r_k} = 0$.

*Proof. Necessity.* Assume that $x$ is $[P, A]_r$-summable to $\xi$ and prove that conditions 1) and 2) are satisfied. Notice that 1) is satisfied by Theorem 2. It remains to prove that also 2) is satisfied. Here we use the inequality (see [5], p. 346)

$$|a + b|^{c_n} \le K \left( |a|^{c_n} + |b|^{c_n} \right) \qquad (n \in \mathbb{N}^0), \tag{13}$$

where $0 < c_n \le \sup_n c_n = L < \infty$, $K = \max\{1, 2^{L-1}\}$, and $a$ and $b$ are any numbers from $\mathbb{C}$.

Now we get using inequality (13) (taking $c_n = \frac{r_n}{M}$ in it and realizing that $K = 1$) and Minkowski's inequality that

$$\begin{aligned}[\sum_{k=0}^{\infty} |p_{nk}|\, |\sigma_k - \eta_k|^{r_k}]^{\frac{1}{M}} &= \{\sum_{k=0}^{\infty}[\, |p_{nk}|^{\frac{1}{M}}\, |(\sigma_k - \xi) - (\eta_k - \xi)|^{\frac{r_k}{M}}\,]^M\}^{\frac{1}{M}} \\ &\leq \{\sum_{k=0}^{\infty}[\, |p_{nk}|^{\frac{1}{M}}\, (|\sigma_k - \xi|^{\frac{r_k}{M}} + |\eta_k - \xi)|^{\frac{r_k}{M}})\,]^M\}^{\frac{1}{M}} \\ &\leq (\sum_{k=0}^{\infty} |p_{nk}||\sigma_k - \xi|^{r_k})^{\frac{1}{M}} + (\sum_{k=0}^{\infty} |p_{nk}||\eta_k - \xi|^{r_k})^{\frac{1}{M}}.\end{aligned}$$

It follows now from the proved inequality that condition 2) is satisfied because the last two sums in this inequality both tend to zero as $n \to \infty$. First of the mentioned sums tends to zero by Lemma 1 because the method $P$ is of type $c_0 \to c_0$ and $\sigma_n - \xi \to 0$ by condition 1). The second of the sums mentioned above tends to zero by the assumption.

*Sufficiency.* Assume that conditions 1) and 2) are satisfied and prove that $\xi_n \to \xi[P, A]_r$. Using the same technique as in the proof of necessity of condition 2), we get by (13) and by Minkowski's inequality the following estimation:

$$[\sum_{k=0}^{\infty} |p_{nk}|\, |\eta_k - \xi|^{r_k}\,]^{\frac{1}{M}} \leq (\sum_{k=0}^{\infty} |p_{nk}||\sigma_k - \eta_k|^{r_k})^{\frac{1}{M}} + (\sum_{k=0}^{\infty} |p_{nk}||\sigma_k - \xi|^{r_k})^{\frac{1}{M}}. \quad (14)$$

We realize that the both sums in the right side of inequality (14) tend to zero as $n \to \infty$. First of these sums tends to zero by 2) and second by Lemma 1 because the method $P$ is of type $c_0 \to c_0$ and 1) is satisfied. □

In particular, if $r_n \equiv r$, Theorem 3 can be seen as Theorem 1 from [12]. If, in addition, the matrix $P = (p_{nk})$ is positive and A is a matrix method, Theorem 3 turns into Theorem 7 from [2].

Conditions 1) and 2) are sufficient for $[P, A]_r$-summability of $x$ for wider class of indices $r = (r_n)$.

**Proposition 3.** *Suppose that the method $P = (p_{nk})$ is of type $c_0 \to c_0$ and $0 < \inf_n r_n = m \leq r_n \leq 1$. If a sequence $x$ satisfies conditions 1) and 2) of Theorem 3, then $x$ is $[P, A]_r$-summable to $\xi$.*

*Proof.* We use inequality (13) with $c_n = r_n$ and notice that in our case $K = 1$ in it. Thus we have the inequality

$$\sum_{k=0}^{\infty} |p_{nk}|\, |\eta_k - \xi|^{r_k} \leq (\sum_{k=0}^{\infty} |p_{nk}|\, |\eta_k - \sigma_k|^{r_k} + (\sum_{k=0}^{\infty} |p_{nk}|\, |\sigma_k - \xi|^{r_k}).$$

The $[P, A]_r$-summability of $x$ to $\xi$ can be concluded from the last inequality in the same way as it was concluded from (14) in the proof of Theorem 3. □

## 3. Strong summability in certain families of summability methods

Let $(A_\alpha)$ $(\alpha > \alpha_1)$ be a Cesàro- or an Euler–Knopp-type family (see Definition A). Noticing that $A_{\alpha+1} = D_{\alpha,\alpha+1} \circ A_\alpha$ where the matrix methods $D_{\alpha,\alpha+1}$ are regular by Proposition B we can transfer the results of Section 2 on the strong $[P, A]_r$-summability to the methods $A_{\alpha+1}$, applying them to the methods $P = D_{\alpha,\alpha+1}$ and $A = A_\alpha$. Thus, applying Definition 1 we will define the notion of strong summability for the methods $A_{\alpha+1}$, i. e., we will define the methods of strong summability $[A_{\alpha+1}]_r$.

**Definition 2.** Let $(A_\alpha)$ $(\alpha > \alpha_1)$ be a Cesàro- or an Euler–Knopp-type family and $r = (r_n)$ be a positive sequence. We say that a sequence $x = (\xi_n)$ is strongly summable by the method $A_{\alpha+1}$ with index $r = (r_n)$ (in short, $[A_{\alpha+1}]_r$-summable) to $\xi$ and write $\xi_n \to \xi[A_{\alpha+1}]_r$ if

$$\mu_n^\alpha := \frac{1}{b_n^{\alpha+1}} \sum_{k=0}^{n} c_{n-k}^1 b_k^\alpha |\eta_k^\alpha - \xi|^{r_k} \to 0 \quad (n \to \infty). \tag{15}$$

Notice that we have $c_n^1 \equiv 1$ in case $\mathcal{A}$) and $c_n^1 = 1/n!$ in case $\mathcal{B}$) in (15).

In particular, if $r_n \equiv r$, this definition was formulated in [11]. For methods $A_\alpha = N_\alpha$ of case $\mathcal{A}$) this definition was given in [7] with $\alpha_1 = 0$. The following theorem can be proved with the help of results from Section 2.

**Theorem 4.** *Let $(A_\alpha)$ $(\alpha > \alpha_1)$ be a Cesàro- or an Euler–Knopp-type family. Then the following statements are true for every $\alpha > \alpha_1$:*

i) *If $\xi_n \to \xi[A_{\alpha+1}]_r$, then $\xi_n \to \xi[A_{\beta+1}]_r$ for every $\beta > \alpha$, provided that $r = (r_n)$ is nonincreasing and $r_n \geq 1$ $(n \in \mathbb{N}^0)$;*

ii) *If $\xi_n \to \xi(A_\alpha)$, then $\xi_n \to \xi[A_{\alpha+1}]_r$, provided that $\inf_n r_n = m > 0$;*

iii) *If $\xi_n \to \xi[A_{\alpha+1}]_r$, then $\xi_n \to \xi[A_{\alpha+1}]_{r'}$, provided that $(r_n)$ and $(r'_n)$ satisfy the conditions $0 < r'_n \leq r_n \leq K\, r'_n$ $(n \in \mathbb{N}^0)$ where $K$ is some positive constant;*

iv) *If $\xi_n \to \xi[A_{\alpha+1}]_r$, then $\xi_n \to \xi(A_{\alpha+1})$, provided that $1 \leq r_n \leq M < \infty$ $(n \in \mathbb{N}^0)$.*

*Proof.* Notice that ii) follows directly from Proposition 2, and conditions iii) and iv) are immediate consequences from Theorems 1 and 2, respectively. So, it remains to prove i). Take $\xi = 0$ and consider

$$\mu_n^\alpha = \frac{1}{b_n^{\alpha+1}} \sum_{k=0}^{n} c_{n-k}^1\, b_k^\alpha\, |\eta_k^\alpha|^{r_k}.$$

Using (2), (3) and Hölder's inequality we get:

$$\begin{aligned}
\mu_n^\beta &= \frac{1}{b_n^{\beta+1}} \sum_{k=0}^{n} c_{n-k}^1 \, b_k^\beta \, \Big| \frac{1}{b_k^\beta} \sum_{\nu=0}^{k} c_{k-\nu}^{\beta-\alpha} \, b_\nu^\alpha \, |\eta_\nu^\alpha|^{r_k} \\
&\le \frac{1}{b_n^{\beta+1}} \sum_{k=0}^{n} c_{n-k}^1 \, b_k^\beta \, \Big(\frac{1}{b_k^\beta} \sum_{\nu=0}^{k} c_{k-\nu}^{\beta-\alpha} \, b_\nu^\alpha \, |\eta_\nu^\alpha|^{r_k}\Big)\Big(\frac{1}{b_k^\beta} \sum_{\nu=0}^{k} c_{k-\nu}^{\beta-\alpha} \, b_\nu^\alpha\Big)^{r_k-1} \\
&= \frac{1}{b_n^{\beta+1}} \sum_{k=0}^{n} c_{n-k}^1 \sum_{\nu=0}^{k} c_{k-\nu}^{\beta-\alpha} \, b_\nu^\alpha \, |\eta_\nu^\alpha|^{r_k} \\
&= \frac{1}{b_n^{\beta+1}} \sum_{k=0}^{n} c_{n-k}^1 \sum_{\nu=0}^{k} c_{\nu}^{\beta-\alpha} \, b_{k-\nu}^\alpha \, |\eta_{k-\nu}^\alpha|^{r_k} \\
&= \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_{\nu}^{\beta-\alpha} \sum_{k=0}^{n-\nu} c_{n-\nu-k}^1 \, b_k^\alpha \, |\eta_k^\alpha|^{r_{k+\nu}}.
\end{aligned}$$

Let us denote

$$u_k^\alpha(\nu) = \begin{cases} |\eta_k^\alpha|^{r_{k+\nu}}, & \text{if } |\eta_k^\alpha| \ge 1, \\ 0, & \text{if } |\eta_k^\alpha| < 1, \end{cases}$$

and

$$v_k^\alpha(\nu) = \begin{cases} |\eta_k^\alpha|^{r_{k+\nu}}, & \text{if } |\eta_k^\alpha| < 1, \\ 0, & \text{if } |\eta_k^\alpha| \ge 1. \end{cases}$$

Now we have the relations

$$|\eta_k^\alpha|^{r_{k+\nu}} = u_k^\alpha(\nu) + v_k^\alpha(\nu), \;\; u_k^\alpha(\nu) \le |\eta_k^\alpha|^{r_k}, \;\; v_k^\alpha(\nu) \le |\eta_k^\alpha|.$$

Denoting

$$\varphi_n^\alpha = \frac{1}{b_n^{\alpha+1}} \sum_{k=0}^{n} c_{n-k}^1 \, b_k^\alpha \, |\eta_k^\alpha|$$

we can continue the evaluations for $\mu_n^\beta$ in the following way:

$$\begin{aligned}
\mu_n^\beta &\le \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_\nu^{\beta-\alpha} \sum_{k=0}^{n-\nu} c_{n-\nu-k}^1 b_k^\alpha u_k^\alpha(\nu) + \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_\nu^{\beta-\alpha} \sum_{k=0}^{n-\nu} c_{n-\nu-k}^1 b_k^\alpha v_k^\alpha(\nu) \\
&\le \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_\nu^{\beta-\alpha} \sum_{k=0}^{n-\nu} c_{n-\nu-k}^1 b_k^\alpha |\eta_k^\alpha|^{r_k} + \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_\nu^{\beta-\alpha} \sum_{k=0}^{n-\nu} c_{n-\nu-k}^1 b_k^\alpha \, |\eta_k^\alpha| \\
&= \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_\nu^{\beta-\alpha} \, b_{n-\nu}^{\alpha+1} \, \mu_{n-\nu}^\alpha + \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_\nu^{\beta-\alpha} \, b_{n-\nu}^{\alpha+1} \, \varphi_{n-\nu}^\alpha \\
&= \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_{n-\nu}^{\beta-\alpha} \, b_\nu^{\alpha+1} \, \mu_\nu^\alpha + \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_{n-\nu}^{\beta-\alpha} \, b_\nu^{\alpha+1} \, \varphi_\nu^\alpha.
\end{aligned}$$

As a result, we proved the inequality

$$\mu_n^\beta \le \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_{n-\nu}^{\beta-\alpha} b_\nu^{\alpha+1} \mu_\nu^\alpha + \frac{1}{b_n^{\beta+1}} \sum_{\nu=0}^{n} c_{n-\nu}^{\beta-\alpha} b_\nu^{\alpha+1} \varphi_\nu^\alpha. \tag{16}$$

Suppose now that $\mu_n^\alpha \to 0$. We know that the method $D_{\alpha+1,\beta+1}$ is regular. Therefore the first sum in the right side of inequality (16) tends to zero as $n \to \infty$. As the implication

$$\lim_n \mu_n^\alpha = 0 \Longrightarrow \lim_n \varphi_n^\alpha = 0$$

is true by Theorem 1 also the second sum in the right side of (16) tends to zero. Thus it follows from (16) that also $\mu_n^\beta \to 0$ as $n \to \infty$. □

In particular, if $r_n \equiv r \ge 1$, the statements of Theorem 4 are proved by Theorem 1 from [11].

The following theorem can be seen as an immediate consequence from Theorem 3, taking in this theorem $P = D_{\alpha,\alpha+1}$ and $A = A_\alpha$ again.

**Theorem 5.** *Let $(A_\alpha)$ $(\alpha > \alpha_1)$ be a Cesàro- or an Euler–Knopp-type family and $1 \le r_n \le \sup_n r_n = M < \infty$. Then we have for every $\alpha > \alpha_1$ that $\xi_n \to \xi[A_{\alpha+1}]_r$ if and only if $\xi_n \to \xi(A_{\alpha+1})$ and $|\eta_n^{\alpha+1} - \eta_n^\alpha|^{r_n} \to 0(D_{\alpha,\alpha+1})$.*

In particular, if $r_n \equiv r \ge 1$, Theorem 5 reduces to Theorem 2 i) from [11].

**Remark 5.** It follows from Proposition 3 that if $0 < \inf r_n = m \le r_n \le 1$, then the conditions $\xi_n \to \xi(A_{\alpha+1})$ and $|\eta_n^{\alpha+1} - \eta_n^\alpha|^{r_n} \to 0(D_{\alpha,\alpha+1})$ imply that $\xi_n \to \xi[A_{\alpha+1}]_r$.

We will finish our paper with two examples.

**Example 1.** Consider the family of Cesàro methods $A_\alpha = (C, \alpha)$ $(\alpha > -1)$. Cesàro methods are generalized Nörlund methods of case $\mathcal{A}$) (see Section 1.3), more precisely, $A_\alpha = N_\alpha$ with $r_n^\alpha = A_n^\alpha$. Therefore our family is a Cesàro-type family with $b_n^\alpha = A_n^\alpha$. Hence the results of this section are applicable to the family $(A_\alpha)$. We note that a sequence $x = (\xi_n)$ is $[A_{\alpha+1}]_r$-summable to $\xi$ by Definition 2 if

$$\mu_n^\alpha = \frac{1}{A_n^{\alpha+1}} \sum_{k=0}^{n} A_k^\alpha |\eta_k^\alpha - \xi|^{r_k} \to 0 \quad (n \to \infty) \tag{17}$$

where $(\eta_n^\alpha) = A_\alpha x$ (see (15)). It should be noticed that condition (17) is equivalent to the condition

$$\mu_n^\alpha = \frac{1}{n+1} \sum_{k=0}^{n} |\eta_k^\alpha - \xi|^{r_k} \to 0 \qquad (n \to \infty)$$

because the Riesz means $(\overline{N}, A_n^\alpha)$ are equivalent to the arithmetical means $(C, 1)$ by Proposition 3 in [3].

**Example 2.** Consider the family of Euler–Knopp methods $A_\alpha = E_{1/(\alpha+1)}$ $(\alpha > 0)$. Euler–Knopp methods are generalized Nörlund methods of case $\mathcal{B}$) (see Section 1.3), more precisely, $A_\alpha = N_\alpha$ with $r_n^\alpha = (\alpha+1)^n/n!$. Therefore our family is a Cesàro-type family with $b_n^\alpha = (\alpha+1)^n/n!$. Hence the results of this section are applicable to the family $(A_\alpha)$. Thus, by Definition 2 a sequence $x = (\xi_n)$ is $[A_{\alpha+1}]_r$-summable to $\xi$ if

$$\mu_n^\alpha = \frac{n!}{(\alpha+2)^n} \sum_{k=0}^{n} \frac{(\alpha+1)^k}{(n-k)!\,k!} |\eta_k^\alpha - \xi|^{r_k} \to 0 \quad (n \to \infty)$$

where $(\eta_n^\alpha) = A_\alpha x$ (see (15)), which is the same as

$$\mu_n^\alpha = \frac{1}{(\alpha+2)^n} \sum_{k=0}^{n} \binom{n}{k} (\alpha+1)^k |\eta_k^\alpha - \xi|^{r_k} \to 0 \quad (n \to \infty).$$

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