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$M(r, s)$-inequality for $\mathcal{K}(X, Y)$ in $\mathcal{L}(X, Y)$

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Abstract. We study Banach spaces X and Y for which the subspace of all compact operators $\mathcal{K}(X, Y)$ forms an ideal satisfying the $M(r, s)$-inequality in the space of all continuous linear operators $\mathcal{L}(X, Y)$. We prove that $\mathcal{K}(X, Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$- and an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathcal{L}(X, Y)$ whenever $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are $M(r_1, s_1)$- and $M(r_2, s_2)$-ideals in $\operatorname{span}(\mathcal{K}(X) \cup I_X)$ and $\operatorname{span}(\mathcal{K}(Y) \cup I_Y)$, respectively, with $r_1 + s_1/2 > 1$ and $r_2 + s_2/2 > 1$. Our results extend some well-known results on M-ideals.

Introduction

According to the terminology in [3], a closed subspace $\mathcal{K} \neq \{0\}$ of a Banach space $\mathcal{L}$ is said to be an *ideal* in $\mathcal{L}$ if there exists a norm one projection P on $\mathcal{L}^*$ with $\ker P = \mathcal{K}^{\perp}$. If moreover, there are $r, s \in (0, 1]$ so that $\|f\| \geq r\|Pf\| + s\|f - Pf\|$ for all $f \in \mathcal{L}^*$, then we say that $\mathcal{K}$ is an *$M(r, s)$-ideal* in $\mathcal{L}$. (In [2] and subsequent works such a $\mathcal{K}$ was called an ideal satisfying the $M(r, s)$-inequality in $\mathcal{L}$.) Well-studied M-ideals (see [4] for results and references) are precisely $M(1, 1)$-ideals.

If $\mathcal{K}$ is an ideal in $\mathcal{L}$, then it is well known and straightforward to verify that for every $f \in \mathcal{L}^*$, $Pf \in \mathcal{L}^*$ is a norm-preserving extension of the restriction $f|_{\mathcal{K}} \in \mathcal{K}^*$. Therefore, $\operatorname{ran} P$ is canonically isometric to $\mathcal{K}^*$ and we shall identify them whenever convenient, identifying Pf and $f|_{\mathcal{K}}$ for all $f \in \mathcal{L}^*$.

In this paper we study Banach spaces X and Y for which the subspace of all compact operators $\mathcal{K}(X, Y)$ forms an $M(r, s)$-ideal in the space of all continuous linear operators $\mathcal{L}(X, Y)$ from X to Y. Instead of $\mathcal{K}(X, X)$ and $\mathcal{L}(X, X)$ we write $\mathcal{K}(X)$ and $\mathcal{L}(X)$, respectively. Our results assume (sometimes implicitly) that X or Y has a (shrinking) metric compact approximation of the identity.

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Recall that a net (K_α) of compact operators on a Banach space X is a *metric compact approximation of the identity* (*MCAI*) provided $\|K_\alpha\| \le 1$, for any α, and $K_\alpha \longrightarrow I_X$ strongly (where I_X denotes the identity operator on X). If, moreover, $K_\alpha^* \longrightarrow I_{X^*}$ strongly, then (K_α) is called *shrinking.*

Our main theorem (see Theorem 11 and Corollary 12) asserts that $\mathcal{K}(X,Y)$ is an $M(r_1r_2^2, s_1s_2^2)$-ideal in $\mathcal{L}(X,Y)$ whenever $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are $M(r_1,s_1)$- and $M(r_2,s_2)$-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, respectively. This theorem contains, as a special case of $r_1 = s_1 = r_2 = s_2 = 1$, its prototype from [9] (see also [4, p. 301]): if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are M-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. The theorem will be proven in Section 3 relying on results of [11], on conditions expressed in terms of $\mathcal{L}(X,Y)$ for $\mathcal{K}(X,Y)$ to be an $M(r,s)$-ideal established in the next Section 1, and on Section 2 where M-ideals results and methods from [9] are extended and developed.

Let us fix some more notation. The closed unit ball of a Banach space X is denoted by B_X. The linear span of a set $A \subset X$ is denoted by $\operatorname{span} A$.

1. The $M(r,s)$-inequality in terms of $\mathcal{L}(X,Y)$

Let X and Y be Banach spaces. By the proof of Lemma 1 in [5], if (K_α) is a weak* convergent (in $\mathcal{K}(X)^{**}$) shrinking MCAI of X (respectively, a weak* convergent (in $\mathcal{K}(Y)^{**}$) MCAI of Y), then $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to the projection P on $\mathcal{L}(X,Y)^*$ defined by

$$Pf(T) = \lim_\alpha f(TK_\alpha), \quad f \in \mathcal{L}(X,Y)^*, \quad T \in \mathcal{L}(X,Y)$$

(respectively,

$$Pf(T) = \lim_\alpha f(K_\alpha T), \quad f \in \mathcal{L}(X,Y)^*, \quad T \in \mathcal{L}(X,Y)).$$

Following [12] we call P the *Johnson projection.* The following result holds by the proof of Theorem 2.4 in [8, p. 36]. We present a self-contained proof for completeness.

Proposition 1. *Let X and Y be Banach spaces. Then $\mathcal{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$ with respect to some Johnson projection whenever there is an MCAI of Y (respectively, a shrinking MCAI of X) with*

$$\limsup_\alpha \|rS + s(T - K_\alpha T)\| \le 1$$

(respectively,

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \le 1)$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$.

Proof. Let (K_α) be an MCAI of Y (the proof is almost verbatim with obvious changes if we assume that there exists a shrinking MCAI of X). By the weak* compactness of $B_{\mathcal{K}(X)^{**}}$, passing to a subnet if necessary, we can assume that $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to the Johnson projection defined by

$$(Pf)(T) = \lim_\alpha f(K_\alpha T), \quad f \in \mathcal{L}(X,Y)^*, \quad T \in \mathcal{L}(X,Y).$$

Let us fix $f \in \mathcal{L}(X,Y)^*$ and $\epsilon > 0$. Recalling that $\|Pf\| = \|f|_\mathcal{K}\|$, we choose $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$ so that

$$r\|Pf\| + s\|f - Pf\| - \epsilon \le rf(S) + s(f - Pf)(T).$$

Therefore, by definition of P, we have

$$\begin{aligned} r\|Pf\| + s\|f - Pf\| - \epsilon &\le rf(S) + sf(T) - s\lim_\alpha f(K_\alpha T) \\ &= \lim_\alpha f(rS + s(T - K_\alpha T)) \\ &\le \|f\| \limsup_\alpha \|rS + s(T - K_\alpha T)\| \\ &\le \|f\|, \end{aligned}$$

whenever $\limsup_\alpha \|rS + s(T - K_\alpha T)\| \le 1$. □

Remark. From [2, Theorem 3.1] it easily follows that Proposition 1 is invertible in the case when $X = Y$ and $r + s/2 > 1$: if $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then X admits a shrinking MCAI (K_α) such that

$$\limsup_\alpha \|rS + sT(I_X - K_\alpha)\| \le 1$$

for any $S \in B_{\mathcal{K}(X)}$ and $T \in B_{\mathcal{L}(X)}$.

2. Properties $M(r,s)$ and $M^*(r,s)$

Let $r, s \in (0,1]$. According to [1], we shall say that a Banach space X has *property* $M(r,s)$ if

$$\limsup_\alpha \|ru + sx_\alpha\| \le \limsup_\alpha \|v + x_\alpha\|$$

whenever $u, v \in X$ satisfy $\|u\| \le \|v\|$, and (x_α) is a bounded net converging weakly to null in X. We shall say that X has *property* $M^*(r,s)$ if

$$\limsup_\alpha \|ru^* + sx^*_\alpha\| \le \limsup_\alpha \|v^* + x^*_\alpha\|$$

whenever $u^*, v^* \in X^*$ satisfy $\|u^*\| \le \|v^*\|$, and (x^*_α) is a bounded net converging weak* to null in X^*.

An impulse for investigating properties $M(r,s)$ and $M^*(r,s)$ came from the study of M-ideals where the prototypical properties (M) and (M^*), introduced in [7] (see also [6]) (where the sequential version was used; see [9] for the general version), have turned out to be the key structure conditions for X in order for $\mathcal{K}(X)$ to be an M-ideal in $\mathcal{L}(X)$. A much more general version of property (M^*), namely property $M^*(a,B,c)$, was introduced and studied in [11] (see also [10]). It can easily be seen that property $M^*(s,\{-s\},r)$ is precisely property $M^*(r,s)$ and property $M^*(1,1)$ is property (M^*).

Analogously to [7, Proposition 2.3] (see also [9, Proposition 2] or [4, Proposition 4.15] and [11, Proposition 1.3]), one can prove that property $M^*(r,s)$ implies property $M(r,s)$ and, moreover, it implies that X is an $M(r,s)$-ideal in X^{**} with respect to the canonical ideal projection on X^{***}.

Similarly to [7, Lemmas 2.1 and 2.2] (see also [9, Lemma 4] or [4, Lemma 4.14]) one can prove the following lemma. For the sake of completeness, we present its proof here.

Lemma 2. *Let X and Y be Banach spaces with properties $M(r_1,s_1)$ and $M(r_2,s_2)$, respectively. If $(u_\alpha)\subset X$ and $(v_\alpha)\subset Y$ are relatively norm-compact nets with $\|v_\alpha\|\le\|u_\alpha\|$ for every α, and (x_α) is a bounded weakly null net in X, then*

$$\limsup_\alpha \|r_1r_2v_\alpha + s_1s_2Tx_\alpha\| \le \limsup_\alpha \|u_\alpha + x_\alpha\|$$

for any $T\in B_{\mathcal{L}(X,Y)}$.

Let X and Y be Banach spaces with properties $M^*(r_1,s_1)$ and $M^*(r_2,s_2)$, respectively. If $(u_\alpha^*)\subset X^*$ and $(v_\alpha^*)\subset Y^*$ are relatively norm-compact nets with $\|u_\alpha^*\|\le\|v_\alpha^*\|$ for every α, and (y_α^*) is a bounded weak-null net in Y^*, then*

$$\limsup_\alpha \|r_1r_2u_\alpha^* + s_1s_2T^*y_\alpha^*\| \le \limsup_\alpha \|v_\alpha^* + y_\alpha^*\|$$

for any $T\in B_{\mathcal{L}(X,Y)}$.

Proof. We only give the proof of the first half of the lemma; the other half is a matter of similarity. We first do the case $\|T\|=1$. Suppose that, contrary to our claim,

$$\lim_\alpha \|r_1r_2v_\alpha + s_1s_2Tx_\alpha\| > \lim_\alpha \|u_\alpha + x_\alpha\|$$

for some relatively compact nets $(u_\alpha)\subset X$ and $(v_\alpha)\subset Y$ with $\|v_\alpha\|\le\|u_\alpha\|$ for every α, and for some bounded weakly null net $(x_\alpha)\subset X$. By passing to subnets, we may assume that $u_\alpha\to u$ in X and $v_\alpha\to v$ in Y. Consequently,

$$\lim_\alpha \|r_1r_2v + s_1s_2Tx_\alpha\| > \lim_\alpha \|u + x_\alpha\|.$$

For any ϵ choose $x\in B_X$ so that $(1+\epsilon)\|Tx\|>1$. Note that (Tx_α) is a bounded weakly null net in Y. Applying property $M(r_2,s_2)$ we have

$$\begin{aligned}\lim_\alpha \|r_1r_2v + s_1s_2Tx_\alpha\| &\le \limsup_\alpha \|r_1(1+\epsilon)\|v\|Tx + s_1Tx_\alpha\|\\ &\le \limsup_\alpha \|r_1\|v\|x + s_1x_\alpha\| + \epsilon\|v\|,\end{aligned}$$

and applying property $M(r_1,s_1)$ we have

$$\limsup_\alpha \|r_1\|v\|x + s_1x_\alpha\| \le \lim_\alpha \|u + x_\alpha\|.$$

This leads to

$$\lim_\alpha \|r_1r_2v + s_1s_2Tx_\alpha\| \le \lim_\alpha \|u + x_\alpha\|,$$

which is a contradiction.

The general case follows now by writing $T \in B_{\mathcal{L}(X,Y)}$ in the form $T = \lambda T' + (1-\lambda)T''$ for some $\lambda \in [0,1]$ and T', T'' with $\|T'\| = \|T''\| = 1$. □

Remark. In the special case of $r_1 = s_1 = r_2 = s_2 = 1$ Lemma 2 reduces to [9, Lemma 4].

The following lemma (inspired by [9, Theorem 5, (d)⇒(e)]) shows how to fulfill the lim sup assumptions of Proposition 1.

Lemma 3. *Let X and Y be Banach spaces with properties $M(r_1,s_1)$ and $M(r_2,s_2)$, respectively. If there exists a shrinking MCAI (K_α) of X such that*

$$\limsup_\beta \limsup_\alpha \|\tilde{r}K_\beta + \tilde{s}(I_X - K_\alpha)\| \le 1$$

for some $\tilde{r}, \tilde{s} \ge 0$, then

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \le 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$, where $r = r_1r_2\tilde{r}$ and $s = s_1s_2\tilde{s}$.

Let X and Y be Banach spaces with properties $M^*(r_1,s_1)$ and $M^*(r_2,s_2)$, respectively. If there exists an MCAI (K_α) of Y such that

$$\limsup_\beta \limsup_\alpha \|\tilde{r}K_\beta + \tilde{s}(I_Y - K_\alpha)\| \le 1$$

for some $\tilde{r}, \tilde{s} \ge 0$, then

$$\limsup_\alpha \|rS + s(T - K_\alpha T)\| \le 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$, where $r = r_1r_2\tilde{r}$ and $s = s_1s_2\tilde{s}$.

Proof. Assume that (K_α) is a shrinking MCAI of X. Fix $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. Since $K_\alpha^* \to I_{X^*}$ uniformly on compact sets, $SK_\alpha \to S$. Therefore

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \le \limsup_\beta \limsup_\alpha \|rSK_\beta + s(T - TK_\alpha)\|.$$

Fix β. We may assume that there is a net $(x_\alpha) \subset B_X$ such that

$$\limsup_\alpha \|rSK_\beta + s(T - TK_\alpha)\| = \limsup_\alpha \|rSK_\beta x_\alpha + s(T - TK_\alpha)x_\alpha\|.$$

Note that $(SK_\beta x_\alpha)_\alpha \subset Y$ and $(K_\beta x_\alpha)_\alpha \subset X$ are relatively norm-compact nets with $\|SK_\beta x_\alpha\| \le \|K_\beta x_\alpha\|$ for any α, and $((I_X - K_\alpha)x_\alpha)$ is a bounded weakly null net in X. Hence, by Lemma 2,

$$\begin{aligned}\limsup_\alpha \|rSK_\beta x_\alpha + s(T - TK_\alpha)x_\alpha\| &\le \limsup_\alpha \|\tilde{r}K_\beta x_\alpha + \tilde{s}(I_X - K_\alpha)x_\alpha\| \\ &\le \limsup_\alpha \|\tilde{r}K_\beta + \tilde{s}(I_X - K_\alpha)\| \le 1,\end{aligned}$$

and the claim follows.

Assume now that (K_α) is an MCAI of Y. Fix $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. Since $K_\alpha \to I_X$ uniformly on compact sets, $K_\alpha S \to S$. Therefore

$$\begin{aligned}\limsup_\alpha \|rS + s(T - K_\alpha T)\| &\le \limsup_\beta \limsup_\alpha \|rK_\beta S + s(T - K_\alpha T)\| \\ &= \limsup_\beta \limsup_\alpha \|rS^* K_\beta^* + s(T^* - T^* K_\alpha^*)\|.\end{aligned}$$

Fix β. We may assume that there is a net $(y_\alpha^*) \subset B_{Y^*}$ such that

$$\limsup_\alpha \|rS^* K_\beta^* + s(T^* - T^* K_\alpha^*)\| = \limsup \|rS^* K_\beta^* y_\alpha^* + s(T^* - T^* K_\alpha^*) y_\alpha^*\|.$$

Note that $(S^* K_\beta^* y_\alpha^*)_\alpha \subset X^*$ and $(K_\beta^* y_\alpha^*)_\alpha \subset Y^*$ are relatively norm-compact nets with $\|S^* K_\beta^* y_\alpha^*\| \le \|K_\beta^* y_\alpha^*\|$ for any α, and $((I_Y - K_\alpha)^* y_\alpha^*)$ is a bounded weak*-null net in Y^*. Hence, by Lemma 2,

$$\begin{aligned}\limsup_\alpha \|rS^* K_\beta^* y_\alpha^* + s(T^* - T^* K_\alpha^*) y_\alpha^*\| &\le \limsup_\alpha \|\tilde{r} K_\beta^* y_\alpha^* + \tilde{s}(I_Y - K_\alpha)^* y_\alpha^*\| \\ &\le \limsup_\alpha \|\tilde{r} K_\beta^* + \tilde{s}(I_Y - K_\alpha)^*\| \le 1,\end{aligned}$$

and the claim follows. □

3. Main results

As auxiliary results, we shall need two more lemmas together with their obvious corollaries. We first introduce the special notation $\mathcal{I}(X)$ for $\operatorname{span}(\mathcal{K}(X) \cup I_X)$ while X is a Banach space.

Lemma 4 (see [11, Corollary 4.4]). *Let X be a Banach space. If $r, s \in (0,1]$ satisfy $r + s/2 > 1$, then the following assertions are equivalent.*

1° *$\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$.*

2° *X has an MCAI and property $M^*(r,s)$.*

Lemma 5 (see [2, Theorem 3.1]). *Let X be a Banach space and let $\mathcal{L} \subset \mathcal{L}(X)$ be a closed subspace containing $\mathcal{I}(X)$. If $r, s \in (0,1]$ satisfy $r+s/2 > 1$, then the following assertions are equivalent.*

1° *$\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}$.*

2° *There exists a shrinking MCAI (K_α) such that*

$$\limsup_\alpha \|rSK_\alpha + s(T - TK_\alpha)\| \quad \forall S, T \in B_{\mathcal{L}}.$$

Corollary 6. *Let $r, s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$, then X has property $M^*(r,s)$ and there is a shrinking MCAI (K_α) of X with*

$$\limsup_\alpha \|rS + s(I_X - K_\alpha)\| \le 1 \quad \forall S \in B_{\mathcal{K}(X)}.$$

Corollary 7. *Let $r,s\in(0,1]$ satisfy $r+s/2>1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$.*

The M-ideal prototype (that is the case when $r_1=s_1=r_2=s_2=1$) of the following Theorems 8 and 9 and their Corollary 10 is [9, Theorem 8].

Theorem 8. *Let X and Y be Banach spaces. Assume that $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{I}(X)$ with $r_1+s_1/2>1$, and Y has property $M(r_2,s_2)$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2r_2,s_1^2s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. By Corollary 6, X has property $M^*(r_1,s_1)$, recall that this implies property $M(r_1,s_1)$, and there is a shrinking MCAI $(K_\alpha)_{\alpha\in\mathcal{A}}$ of X with

$$\limsup_\alpha \|r_1K_\beta+s_1(I_X-K_\alpha)\|\le 1 \quad \forall\beta\in\mathcal{A}.$$

By the first part of Lemma 3,

$$\limsup_\alpha \|r_1^2r_2S+s_1^2s_2(T-TK_\alpha)\|\le 1$$

for any $S\in B_{\mathcal{K}(X,Y)}$ and $T\in B_{\mathcal{L}(X,Y)}$. Now the claim follows from Proposition 1. □

Theorem 9. *Let X and Y be Banach spaces. Assume that X has property $M^*(r_1,s_1)$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{I}(Y)$ with $r_2+s_2/2>1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1r_2^2,s_1s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. Analogously to Theorem 8, the claim follows from Proposition 1 by Corollary 6 and the second part of Lemma 3. □

Recall that $M(1,1)$-ideals are just M-ideals. Hence, the following Corollary 10 is immediate from Theorems 8 and 9 by Corollary 7.

Corollary 10. *Let X be a Banach space such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Then $\mathcal{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$ for all Banach spaces Y with property $M(r,s)$, and $\mathcal{K}(Y,X)$ is an $M(r,s)$-ideal in $\mathcal{L}(Y,X)$ for all Banach spaces Y with property $M^*(r,s)$.*

Gathering the assumptions of Theorems 8 and 9 together and using Lemma 4 yield our main result.

Theorem 11. *Let X and Y be Banach spaces. Assume that $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{I}(X)$ with $r_1+s_1/2>1$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{I}(Y)$ with $r_2+s_2/2>1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2r_2,s_1^2s_2)$- and an $M(r_1r_2^2,s_1s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Using Corollary 7 this immediately implies

Corollary 12. *Let X and Y be Banach spaces. Assume that $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{L}(X)$ with $r_1+s_1/2>1$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{L}(Y)$ with $r_2+s_2/2>1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2r_2,s_1^2s_2)$- and an $M(r_1r_2^2,s_1s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Remark. Corollary 12 extends [9, Corollary 9] (which is [4, Corollary 4.18]) from M-ideals to $M(r, s)$-inequalities.

The following is immediate from Corollary 7 and Theorem 11.

Corollary 13. *Let $r+s/2 > 1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$, then $\mathcal{K}(X)$ is an $M(r^3, s^3)$-ideal in $\mathcal{L}(X)$*

Problem. In the special case of $r = s = 1$ Corollary 13 reduces to Kalton's theorem [7, Theorem 2.6] (see [9, Theorem 5] or [4, Theorem 4.17] for its non-separable case): $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if $\mathcal{K}(X)$ is an M-ideal in $\mathcal{I}(X)$. It is not known whether Corollary 13 could be improved to yield the desirable result: $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$ if and only if $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$.

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