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# Inference in normal models with commutative orthogonal block structure

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Abstract. Linear mixed normal models are studied in this paper. Using commutative Jordan algebras, the algebraic properties of these models are studied, as well as optimal estimators, hypothesis tests and confidence regions for fixed and random effects. Model crossing and nesting is then presented and analyzed.

## 1. Introduction

Linear mixed models with Orthogonal Block Structure (OBS) play an important role in the design and analysis of experiments (see [1] and [2]). OBS was introduced by J.A. Nelder in 1965 (see [7] and [8]) in the framework of the design of experiments in agricultural trials. Now these designs are also applied in biology, medicine, engineering and social sciences. A model with OBS can be characterized by having a covariance matrix of the form

$$\mathbf{V} = \sum_{i=1}^{w} \lambda_i \mathbf{Q}_i,$$

where the $\mathbf{Q}_i$, $i = 1, \ldots, w$, are known orthogonal projection matrices that are mutually orthogonal, i. e.,

$$\mathbf{Q}_i \mathbf{Q}_j = \mathbf{0}, \ i \neq j.$$

These models allow optimal estimation for variance components of blocks and contrasts of treatments, as we shall see.

We intend to consider a special class of such models. Let the mean vector of the model be $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, belonging to the subspace $\Omega$. If the orthogonal projection matrix $\mathbf{T}$ on $\Omega$, the range space of $\mathbf{X}$, commutes with $\mathbf{Q}_i$, $i = 1, \ldots, w$, the model will have commutative orthogonal block structure,

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COBS. Then $\mathbf{T}$ and $\mathbf{V}$ will commute and the least squares estimators, LSE, will give the best linear unbiased estimators, BLUE, for estimable vectors (see [5]).

We start by studying the algebraic structure of models with COBS using commutative Jordan algebras, CJA. Next we show how, crossing and nesting models with COBS, we obtain models with COBS. Finally, assuming normality we use our results to examine inference problems.

## 2. Algebraic structure

**2.1. Commutative Jordan algebras.** Commutative Jordan Algebras (CJA) are linear spaces constituted by symmetric matrices that commute and contain the squares of their elements. It was shown in [10] that any CJA has one and only one basis constituted by orthogonal projection matrices that are pairwise orthogonal. This basis is called the principal basis: $\mathrm{pb}\,(\mathscr{A}) = Q = \{\mathbf{Q}_1, \dots, \mathbf{Q}_w\}$. These algebras have been widely used in the study of classes of models, for instance see [4], [12] and [11].

Let $\mathscr{A}(\mathbf{K})$ be the family of all symmetric matrices diagonalized by an orthogonal matrix $\mathbf{K}$. It is easy to see that $\mathscr{A}(\mathbf{K})$ is a CJA. Moreover, the symmetric matrices $\mathbf{G}_1, \dots, \mathbf{G}_m$ commute if and only if (see [9], p. 157) they are diagonalized by the same matrix $\mathbf{K}$, thus belonging to $\mathscr{A}(\mathbf{K})$. The intersection of CJA is still a CJA, so, if the $\mathbf{G}_1, \dots, \mathbf{G}_m$ commute, the intersection of all CJA containing $G = \{\mathbf{G}_1, \dots, \mathbf{G}_m\}$ will be a CJA containing $G$. This minimal CJA $\mathscr{A}(G)$ that contains $G$ is called the CJA generated by $G$.

When $G$ is contained in $\mathscr{A}(\mathbf{K})$, the column vectors $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n$ of $\mathbf{K}$ are the eigenvectors of the matrices $\mathbf{G}_1, \dots, \mathbf{G}_n$. We now define the equivalence relation in $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n\}$ writing $\boldsymbol{\alpha}_j \tau \boldsymbol{\alpha}_l$ when

$$\boldsymbol{\alpha}_j' \mathbf{G}_k \boldsymbol{\alpha}_j = \boldsymbol{\alpha}_l' \mathbf{G}_k \boldsymbol{\alpha}_l;\ k = 1, \dots, m.$$

The equivalence classes may be of two types. A $\tau$ equivalence class is of first type if for at least one of the matrices in $G$ its vectors do not have null eigenvalue. Otherwise we have a second type $\tau$ equivalence class. Whilst there are always first type $\tau$ equivalence classes, there exists at most one second type $\tau$ equivalence class. Let $\mathcal{C}_1, \dots \mathcal{C}_w$ be the sets of indexes of the vectors in the first type equivalence classes and $\mathcal{C}_{w+1}$ the set of indexes of the second type equivalence class. If there is no second type $\tau$ equivalence class we will have $\mathcal{C}_{w+1} = \emptyset$. Let

$$\mathbf{Q}_j = \sum_{i \in \mathcal{C}_j} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i';\ j = 1, \dots, w+1.$$

Put $Q = \{\mathbf{Q}_1, \ldots, \mathbf{Q}_w\}$ and, if $\mathcal{C}_{w+1} \neq \emptyset$, $Q_+ = \{\mathbf{Q}_1, \ldots, \mathbf{Q}_w, \mathbf{Q}_{w+1}\}$. $Q$ and $Q_+$ will be the families of orthogonal projection matrices, pairwise orthogonal. Thus $Q$ and, if $\mathcal{C}_{w+1} \neq \emptyset$, $Q_+$ will be the principal basis of the CJA $\mathscr{A}(Q)$ and $\mathscr{A}(Q_+)$, respectively. We now establish

**Proposition 1.** $\mathscr{A}(G) = \mathscr{A}(Q)$.

*Proof.* Let $\gamma_{kj}$ be the eigenvalues of $\mathbf{G}_k$, with $k = 1, \ldots, m$. For the eigenvectors corresponding to the sets $\mathcal{C}_j$, $j = 1, \ldots, w$, we have $\mathbf{G}_k = \sum_{j=1}^{w} \gamma_{kj} \mathbf{Q}_j$ in $G \subseteq \mathscr{A}(Q)$ and so $\mathscr{A}(G) \subseteq \mathscr{A}(Q)$.

Let now $\mathrm{pb}\,(\mathscr{A}(G)) = L = \{\mathbf{L}_1, \ldots, \mathbf{L}_s\}$. Since $\mathscr{A}(G) \subseteq \mathscr{A}(Q)$, we have $L \subseteq \mathscr{A}(Q)$, and so

$$\mathbf{L}_u = \sum_{v=1}^{w} c_{uv} \mathbf{Q}_v;\ u = 1, \ldots, s.$$

Since $\mathbf{L}_u$, $u = 1, \ldots, s$, and $\mathbf{Q}_v$, $v = 1, \ldots, w$, are idempotent and pairwise orthogonal, $c_{uv} = 0$ or $c_{uv} = 1$. The sets

$$\mathcal{D}_u = \{v : c_{uv} = 1\};\ u = 1, \ldots, s,$$

are singletons since, if $v_1 \neq v_2$ belong to $\mathcal{D}_u$, the eigenvalues of the matrices $\mathbf{G}_1, \ldots, \mathbf{G}_n$ with indexes in $\mathcal{C}_{v_1} \cup \mathcal{C}_{v_2}$ have to be equal, which is impossible since $\mathcal{C}_{v_1}$ and $\mathcal{C}_{v_2}$ belong to distinct $\tau$ equivalence classes. Thus $s = w$ since for every $v = 1, \ldots, w$ there must be a $u$ such that $c_{uv} = 1$, otherwise the corresponding $\tau$ equivalence class would not be of the first type. $\square$

The eigenindex of a family $M = \{\mathbf{M}_1, \ldots, \mathbf{M}_l\}$ of symmetric matrices that commute is the number of first type $\tau$ equivalence classes defined for the eigenvectors of the matrices in $M$. This index is the dimension of $\mathscr{A}(M)$. We now establish

**Proposition 2.** *The family $M = \{\mathbf{M}_1, \ldots, \mathbf{M}_l\}$ of symmetric matrices that commute is a basis for $\mathscr{A}(M)$ if and only if it has eigenindex $l$ and if its matrices are linearly independent.*

*Proof.* The condition is necessary since $\dim\big(\mathscr{A}(M)\big)$, the dimension of $\mathscr{A}(M)$, equals the eigenidex of $\mathscr{A}(M)$. To establish sufficiency we represent the principal basis of $\mathscr{A}(M)$ by $\{\mathbf{Q}_1, \ldots, \mathbf{Q}_l\}$. Then $\mathbf{M}_i = \sum_{j=1}^{l} b_{i,j} \mathbf{Q}_j$, $i = 1, \ldots, l$. Since $\sum_{i=1}^{l} c_i \mathbf{M}_i = \sum_{j=1}^{l} \sum_{i=1}^{l} b_{i,j} c_i \mathbf{Q}_j$, and $\mathbf{M}_1, \ldots, \mathbf{M}_l$ are linearly independent, $\mathbf{B} = [b_{i,j}]$ and $\mathbf{B}'$ will be non-singular. Then the matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_l$ are linear combinations of the $\mathbf{M}_1, \ldots, \mathbf{M}_l$ and the proof is complete. $\square$

The arguments used in this proof show that an orthogonal projection matrix belonging to a CJA is the sum of all or part of the matrices in the principal basis. It is also easy to see that the rank of $\mathbf{Q}$ is the sum of the ranks of those matrices. Thus if $\mathrm{rank}(\mathbf{Q}) = 1$, where $\mathrm{rank}(\mathbf{Q})$ denotes the rank of $\mathbf{Q}$, it must belong to the principal basis. Now $\mathbf{J}_n = \frac{1}{n}\mathbf{1}_n{}'\mathbf{1}_n$, where $\mathbf{1}_n$ is a vector of ones, is an orthogonal projection matrix with rank one so it belongs to the principal basis of any CJA it belongs to. These CJA are said to be regular.

The unity $\mathbf{E}$ of a CJA will be the sum of the matrices of the principal basis. If $\mathbf{E} = \mathbf{I}_n$ we say that the CJA is complete. If $Q = \{\mathbf{Q}_1, \ldots, \mathbf{Q}_w\}$ is the principal basis of a non-complete CJA $\mathscr{A}$, $Q_+ = \{\mathbf{Q}_1, \ldots, \mathbf{Q}_w, \mathbf{Q}_{w+1}\}$ with $\mathbf{Q}_{w+1} = \mathbf{I}_n - \mathbf{E}$ will be the principal basis of a complete CJA $\mathscr{A}_+$. If there exists a second type $\tau$ equivalence class, $\mathcal{A}(G)$ will not be complete. Actually, $\mathscr{A}(G)$ is complete if and only if there is no second type $\tau$ equivalence class.

If $\mathbf{M}$ belongs to a CJA with principal basis $Q = \{\mathbf{Q}_1, \ldots, \mathbf{Q}_w\}$ we will have

$$\mathbf{M} = \sum_{i=1}^{w} a_i \mathbf{Q}_i,$$

and the ortogonal projection matrix $\mathbf{T}(\mathbf{M})$ on the range space of $\mathbf{M}$, $\mathrm{R}(\mathbf{M})$, will be

$$\mathbf{T}(\mathbf{M}) = \sum_{i \in \mathcal{C}(\mathbf{M})} \mathbf{Q}_i,$$

where $\mathcal{C}(\mathbf{M}) = \{i : a_i \neq 0\}$. If $\mathbf{M}$ is non-singular, $\mathrm{R}(\mathbf{M}) = \mathbb{R}^n$ and $\mathbf{T}(\mathbf{M}) = \mathbf{I}_n$, so that if $\mathbf{M}$ belongs to a CJA it has to be complete. Vice-versa, if a CJA is complete, it contains $\mathbf{I}_n$, which is non-singular. Thus, for a CJA to contain non-singular matrices, it is necessary and sufficient that it is complete.

Putting

$$a_i^+ = \begin{cases} a_i^{-1} & a_i \neq 0 \\ 0 & a_i = 0 \end{cases}$$

it is easy to check that the Moore-Penrose inverse of $\mathbf{M}$ will be

$$\mathbf{M}^+ = \sum_{i=1}^{w} a_i^+ \mathbf{Q}_i.$$

Namely, when $\mathbf{M}$ is non-singular we will have

$$\mathbf{M}^{-1} = \sum_{i=1}^{w} a_i^{-1} \mathbf{Q}_i.$$

Moreover, the $a_i$ will be the eigenvalues of $\mathbf{M}$ with multiplicity $g_i = \mathrm{rank}(\mathbf{Q}_i)$, $i = 1, \ldots, w$. Thus, if $\mathbf{M}$ is non-singular,

$$\det(\mathbf{M}) = \prod_{i=1}^{w} a_i^{g_i}.$$

Pairs of complete CJA will be instrumental for the purpose of this research. Given a model with COBS we must first consider CJA $\mathscr{A}(Q)$ with principal basis $Q = \{\mathbf{Q}_1, \dots, \mathbf{Q}_w\}$ to which $\mathbf{V}$ belongs to, and CJA $\mathscr{A}(\mathbf{T})$, with principal basis $\{\mathbf{T}, \mathbf{U}\}$, where $\mathbf{U} = \mathbf{I}_n - \mathbf{T}$, the orthogonal projection matrix on the orthogonal complement of the mean vector space. We now establish

**Proposition 3.** *If $Q = \{\mathbf{Q}_1, \dots, \mathbf{Q}_w\}$ and $L = \{\mathbf{L}_1, \dots, \mathbf{L}_s\}$ are the principal bases of complete CJA and all the matrices in both of these sets commute, the principal basis of $\mathscr{A}\big(\mathscr{A}(Q) \cup \mathscr{A}(L)\big)$ will be constituted by the non null products $\mathbf{Q}_i\mathbf{L}_j$, $i = 1, \dots, w$, $j = 1, \dots, s$.*

*Proof.* Since $\mathbf{Q}_i\mathbf{L}_j = \mathbf{L}_j\mathbf{Q}_i$, it is easy to check that these matrices are symmetric and idempotent. Thus, if non-null, they are orthogonal projection matrices, pairwise orthogonal, since

$$\mathbf{Q}_{i_1}\mathbf{L}_{j_1}\mathbf{Q}_{i_2}\mathbf{L}_{j_2} = \mathbf{Q}_{i_1}\mathbf{L}_{j_1}\mathbf{L}_{j_2}\mathbf{Q}_{i_2} = \mathbf{0}_{n\times n}$$

whenever $i_1 \neq i_2$ or $j_1 \neq j_2$. Then the set of non-null matrices $\mathbf{Q}_i\mathbf{L}_j$ will be the principal basis of a CJA $\mathscr{A}_*$. Now, $\mathbf{Q}_i = \sum_{j=1}^{s}\mathbf{Q}_i\mathbf{L}_j$ so $Q \subset \mathscr{A}_*$ and analogously $L \subset \mathscr{A}_*$. Consequently $\mathscr{A}(Q) \cup \mathscr{A}(L) \subseteq \mathscr{A}_*$ and $\mathscr{A}\big(\mathscr{A}(Q) \cup \mathscr{A}(L)\big) \subseteq \mathscr{A}_*$.

To establish the inverse inclusion we point out that $Q \cup L \subset \mathscr{A}\big(\mathscr{A}(Q) \cup \mathscr{A}(L)\big)$ so that, since any CJA contains the products of its matrices, $\mathbf{Q}_i\mathbf{L}_j \in \mathscr{A}\big(\mathscr{A}(Q)\cup\mathscr{A}(L)\big)$, $i = 1, \dots, w$, $j = 1, \dots, s$, it implies that $\mathscr{A}_* \subseteq \mathscr{A}\big(\mathscr{A}(Q)\cup \mathscr{A}(L)\big)$. □

**2.2. Models.** A COBS model encompasses two CJA: $\mathscr{A}(Q)$ and $\mathscr{A}(\mathbf{T})$. The matrices in the principal basis of these CJA commute so their matrices will commute. Thus, according to Proposition 3, the principal basis of the CJA generated by $\mathscr{A}(Q)$ and $\mathscr{A}(\mathbf{T})$ will be constituted by the non-null matrices $\mathbf{Q}_i\mathbf{T}$ and $\mathbf{Q}_i\mathbf{U}$, $i = 1, \dots, w$. We also see that

$$\mathbf{T} = \sum_{i=1}^{w}\mathbf{Q}_i\mathbf{T},$$
$$\mathbf{U} = \sum_{i=1}^{w}\mathbf{Q}_i\mathbf{U}.$$

Let us put

$$\mathcal{D}_1 = \{i : \mathbf{Q}_i\mathbf{T} \neq \mathbf{0}_{n\times n}\},$$
$$\mathcal{D}_2 = \{i : \mathbf{Q}_i\mathbf{U} \neq \mathbf{0}_{n\times n}\}.$$

Since $\mathbf{Q}_i\mathbf{T} + \mathbf{Q}_i\mathbf{U} = \mathbf{Q}_i$, $i = 1, \dots, w$, we have $\mathcal{D}_1 \cup \mathcal{D}_2 = W = \{1, \dots, w\}$. With $k_l = \#(\mathcal{D}_l)$, $l = 1, 2$, and $k = \#(\mathcal{D}_1 \cap \mathcal{D}_2)$, where $\#(A)$ is the cardinality of the set $A$, we have

$$k_1 + k_2 + k = n.$$

Let us denote by $\mathbf{P}_1, \dots, \mathbf{P}_{k_1}$ the non-null matrices $\mathbf{Q}_i\mathbf{T}$ and by $\mathbf{P}_{k_1+1}, \dots,$ $\mathbf{P}_{k_1+k_2}$ the non-null matrices $\mathbf{Q}_i\mathbf{U}$. Let the column vectors of $\mathbf{A}_h$ constitute an orthonormal basis for $\nabla_h = \mathrm{R}(\mathbf{P}_h)$, $h = 1, \dots, k_1 + k_2$. Then

$$\mathbf{P}_h = \mathbf{A}_h\mathbf{A}_h';\ h = 1, \dots, k_1 + k_2.$$

Further,

$$\mathbf{T} = \sum_{i=1}^{w} \mathbf{Q}_i\mathbf{T} = \sum_{h=1}^{k_1} \mathbf{P}_h = \sum_{h=1}^{k_1} \mathbf{A}_h\mathbf{A}_h',$$

thus,

$$\boldsymbol{\mu} = \mathbf{T}\boldsymbol{\mu} = \sum_{h=1}^{k_1} \mathbf{A}_h\boldsymbol{\zeta}_h,$$

with $\boldsymbol{\zeta}_h = \mathbf{A}'\boldsymbol{\mu}$, $h = 1, \dots, k_1 + k_2$. We point out that

$$\boldsymbol{\zeta}_h = \mathbb{E}[\boldsymbol{\eta}_h]\,;\ h = 1, \dots, k_1 + k_2,$$

where $\boldsymbol{\eta}_h = \mathbf{A}_h'\mathbf{y}$, $h = 1, \dots, k_1 + k_2$, and that $\boldsymbol{\zeta}_h = \mathbf{0}$, $h = 1, \dots, k_1$. Moreover,

$$\mathbf{V} = \sum_{i=1}^{w} \sigma_i\mathbf{Q}_i = \sum_{h=1}^{k_1+k_2} \gamma_i\mathbf{P}_h = \sum_{h=1}^{k_1+k_2} \gamma_i\mathbf{A}_h\mathbf{A}_h',$$

where $\sigma_i = \gamma_h$ if either $\mathbf{P}_h = \mathbf{A}_h'\mathbf{T}$ or $\mathbf{P}_h = \mathbf{A}_h'\mathbf{U}$, $h = 1, \dots, k_1 + k_2$. If the $\boldsymbol{\eta}_h$ have mean vectors $\boldsymbol{\zeta}_h$ and covariance matrices $\gamma_i\mathbf{I}_{g_h}$ with $g_h = \mathrm{rank}(\mathbf{P}_h)$, $h = 1, \dots, k_1 + k_2$, and are independent, then

$$\mathbf{y} = \sum_{h=1}^{k_1+k_2} \mathbf{A}_h\boldsymbol{\zeta}_h$$

will have mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$. This may be summarized into the following

**Proposition 4.** *Whatever the orthogonal projection matrices pairwise orthogonal $\mathbf{Q}_1, \dots, \mathbf{Q}_w$ that commute with the orthogonal projection matrix $\mathbf{T}$, there is a model with COBS*

$$\mathbf{y} = \sum_{h=1}^{k_1+k_2} \mathbf{A}_h\boldsymbol{\zeta}_h$$

*with covariance matrix*

$$\mathbf{V} = \sum_{i=1}^{w} \sigma_i\mathbf{Q}_i$$

*and a mean vector $\boldsymbol{\mu}$ that spans $\Omega = \mathrm{R}(\mathbf{T})$.*

*Remark.* Since $\mathbf{V}$ has to be positive semi-definite, the variance components $\sigma_1, \dots, \sigma_w$ cannot be negative. If we require $\mathbf{V}$ to be positive definite we must have $\sigma_i > 0$, $i = 1, \dots, w$.

**Corollary 1.** *There is a normal model with COBS whose mean vector spans $\Omega$ and has covariance matrix $\mathbf{V}$ whenever the conditions of Proposition 4 are met.*

*Proof.* It suffices to require the $\boldsymbol{\eta}_h$, $h = 1, \ldots, k_1 + k_2$, to be normal with mean vectors $\boldsymbol{\zeta}_h$ and covariance matrices $\gamma_h \mathbf{I}_{g_h}$, $h = 1, \ldots, k_1 + k_2$, besides being independent. □

In what follows we write $\mathbf{z} \sim \mathscr{N}(\boldsymbol{\lambda}, \mathbf{C})$ to indicate that $\mathbf{z}$ is normal with mean vector $\boldsymbol{\lambda}$ and covariance matrix $\mathbf{C}$.

These results show that a model with COBS may always be written as a sum of independent terms. This will be the canonical form of the models while $k_1$ and $k_2$ will be the structure parameters of the models. While $k_1$ gives the number in the fixed effects terms part, $k_2$ will give the corresponding number for the random effects of the model.

**2.3. Functional forms.** Let

$$\mathbf{y} = \sum_{j=1}^{m} \mathbf{X}_j \boldsymbol{\beta}_j + \sum_{j=m+1}^{l} \mathbf{X}_j \boldsymbol{\tau}_j + \mathbf{e},$$

where the $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m$ are fixed while $\boldsymbol{\tau}_{m+1}, \ldots, \boldsymbol{\tau}_l$ and $\mathbf{e}$ are independent, with null mean vectors and covariance matrices $\sigma_j^2 \mathbf{I}_{c_j}$, $j = m+1, \ldots, l$, and $\sigma_e^2 \mathbf{I}_n$, with $c_j = \operatorname{rank}(\mathbf{X}_j)$, $j = 1, \ldots, l$. The mean vector and covariance matrix of $\mathbf{y}$ are

$$\boldsymbol{\mu} = \sum_{j=1}^{m} \mathbf{X}_j \boldsymbol{\beta}_j,$$

$$\mathbf{V} = \sum_{j=1}^{l} \sigma_j^2 \mathbf{M}_j + \sigma_e^2 \mathbf{I}_n,$$

where $\mathbf{M}_j = \mathbf{X}_j \mathbf{X}_j'$, $j = 1, \ldots, l$. We now establish

**Proposition 5.** *If the $\mathbf{M}_{m+1}, \ldots, \mathbf{M}_l$ commute, the model has OBS. If matrices $\mathbf{M}_1, \ldots, \mathbf{M}_l$ commute, the model has COBS.*

*Proof.* If $\mathbf{M}_{m+1}, \ldots, \mathbf{M}_l$ commute, they will belong – as well as $\mathbf{I}_n$ – to a CJA with principal basis $\{\mathbf{R}_1, \ldots, \mathbf{R}_d\}$. Then $\mathbf{V}$ will be a linear combination of the matrices in this basis and the first part of the statement is established. Likewise, if $\mathbf{M}_1, \ldots, \mathbf{M}_l$ commute, they will belong to a CJA with principal basis $\{\mathbf{R}_1, \ldots, \mathbf{R}_v\}$, with $v > d$. Now, $\boldsymbol{\mu}$ will span $\mathrm{R}(\mathbf{X})$, with $\mathbf{X} = [\mathbf{X}_1 \quad \cdots \quad \mathbf{X}_m]$. Since $\mathrm{R}(\mathbf{X}) = \mathrm{R}(\mathbf{M})$, with $\mathbf{M} = \mathbf{X}\mathbf{X}' = \sum_{i=1}^{m} \mathbf{M}_i$, the orthogonal projection matrix $\mathbf{T}$ on $\mathrm{R}(\mathbf{M})$ will also belong to this CJA so that it will commute with $\mathbf{V}$. □

To distinguish it from the canonical form, we call this last form of the model the functional form, since its terms are usually connected with factor effects and factor interactions (see [3]). Moreover, if matrices $\mathbf{M}_1, \dots, \mathbf{M}_l$ commute, according to Proposition 2, $M = \{\mathbf{M}_1, \dots, \mathbf{M}_l\}$ will be a basis for $\mathscr{A}(M)$ if and only if these matrices are linearly independent and $M$ has eigenindex $l$. Thus the model we are considering will be (see [4]) strictly associated to $\mathscr{A}(M)$. It is also clear, according to Proposition 5, that if a model is associated to $\mathscr{A}(M)$, matrices $\mathbf{M}_1, \dots, \mathbf{M}_l$ will commute and the model will have COBS.

We now establish

**Proposition 6.** *If the $\mathbf{M}_1, \dots, \mathbf{M}_l$ commute and, moreover, if*

$$\left(\sum_{i=1}^{m} \mathbf{M}_i\right)\left(\sum_{j=m+1}^{l} \mathbf{M}_j\right) = \mathbf{0}_{n \times n}$$

*the model will have COBS with $k_1 = 1$.*

*Proof.* The $\mathrm{R}(\mathbf{M}_i)$, $i = 1, \dots, m$, are subspaces of $\mathrm{R}(\sum_{i=1}^{m} \mathbf{M}_i)$ since

$$\mathrm{R}\left(\sum_{i=1}^{m} \mathbf{M}_i\right) = \mathrm{R}([\mathbf{X}_1 \quad \cdots \quad \mathbf{X}_m]) \,.$$

Likewise, the $\mathrm{R}(\mathbf{M}_i)$, $i = m+1, \dots, l$, are subspaces of $\mathrm{R}\left(\sum_{i=m+1}^{l} \mathbf{M}_i\right)$ since $\mathrm{R}\left(\sum_{m+i=1}^{l} \mathbf{M}_i\right) = \mathrm{R}([\mathbf{X}_{m+1} \quad \cdots \quad \mathbf{X}_l])$. Then $\Omega = \mathrm{R}(\sum_{i=1}^{m} \mathbf{M}_i)$ will be orthogonal to the $\mathrm{R}(\mathbf{M}_i)$, $i = m+1, \dots, l$, and so $\mathbf{T}\mathbf{M}_i = \mathbf{0}_{n \times n}$, $i = m+1, \dots, l$.

$M_2 = \{\mathbf{M}_{m+1}, \dots, \mathbf{M}_l\}$ generates a CJA $\mathscr{A}(M_2)$ with principal basis $\{\mathbf{R}_1, \dots, \mathbf{R}_v\}$, and $\mathbf{M}_i = \sum_{j=1}^{v} b_{i,j}\mathbf{R}_j$, $i = m+1, \dots, l$. The $b_{i,j}$, $i = m+1, \dots, l$, cannot be all zeros since, otherwise, $\mathbf{R}_j$ would be discarded from the principal basis of $\mathscr{A}(M_2)$, $j = 1, \dots, v$. Thus $\mathrm{R}(\mathbf{R}_j)$ is a subspace of at least one of the $\mathrm{R}(\mathbf{M}_i)$, $i = m+1, \dots, l$, and so of $\mathrm{R}\left(\sum_{i=m+1}^{l} \mathbf{M}_i\right)$, thus making $\Omega$ orthogonal to the $\mathrm{R}(\mathbf{R}_j)$, $j = 1, \dots, v$. Therefore $\mathbf{T}\mathbf{R}_j = \mathbf{0}_{n \times n}$, $j = 1, \dots, v$. Adding $\mathbf{R}_{v+1} = \mathbf{I}_n - \sum_{j=1}^{v} \mathbf{R}_j$ to the matrices in the principal basis of $\mathscr{A}(M_2)$ we get the principal basis of a CJA that contains $\mathbf{V}$ since this matrix is a linear combination of the $\mathbf{M}_i$, $i = m+1, \dots, l$, and of $\mathbf{I}_n$. Thus we will have $\mathbf{V} = \sum_{j=1}^{v+1} \alpha_j \mathbf{R}_j$ and since $\mathbf{T}\mathbf{R}_{v+1}$ is non-null whilst $\mathbf{T}\mathbf{R}_j = \mathbf{0}_{\mathscr{N}(\times,n)}$, $j = 1, \dots,$, $k_1 = 1$. □

Consider binary operations in models with COBS. Assume the models are written in their functional form. Given a pair

$$\mathbf{y}_d = \sum_{j_d=1}^{m_d} \mathbf{X}_{j_d}\boldsymbol{\beta}_{j_d} + \sum_{j_d=m_d+1}^{l_d} \mathbf{X}_{j_d}\boldsymbol{\tau}_{j_d} + \mathbf{e}_d;\ d = 1, 2,$$

of such models, when we cross them we obtain a model $\mathbf{y}_1 \boxtimes \mathbf{y}_2$ expressed by

$$\mathbf{y}_1 \boxtimes \mathbf{y}_2 = \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \mathbf{X}_{j_1,j_2} \boldsymbol{\beta}_{j_1,j_2} + \sum_{j_1=m_1+1}^{l_1} \sum_{j_2=m_2+1}^{l_2} \mathbf{X}_{j_1,j_2} \boldsymbol{\tau}_{j_1,j_2} + \mathbf{e},$$

where, representing by $\otimes$ the Kronecker product,

$$\mathbf{X}_{j_1,j_2} = \mathbf{X}_{1,j_1} \otimes \mathbf{X}_{2,j_2};\ j_1 = 1, \ldots, l_1;\ j_2 = 1, \ldots, l_2.$$

The vectors $\boldsymbol{\beta}_{j_1,j_2}$ are fixed for $j_1 = 1, \ldots, m_1$ and $j_2 = 1, \ldots, m_2$, otherwise we assume that $\boldsymbol{\beta}_{j_1,j_2} \sim \mathscr{N}\Big(\mathbf{0}, \sigma^2_{j_1,j_2} \mathbf{I}_n\Big)$, and $\mathbf{e} \sim \mathscr{N}\big(\mathbf{0}, \sigma^2_e \mathbf{I}_n\big)$. All these vectors are assumed to be independent. We now establish

**Proposition 7.** *Crossing models with COBS give models with COBS.*

*Proof.* According to Proposition 5 we have only to show that matrices $\mathbf{M}_{j_1,j_2} = \mathbf{X}_{j_1,j_2} \mathbf{X}'_{j_1,j_2}$, $j_1, \ldots, l_1$, $j_2 = 1, \ldots, l_2$, commute. Now,

$$\begin{aligned}
\mathbf{M}_{j_1,j_2} \mathbf{M}_{j_1',j_2'} &= (\mathbf{M}_{j_1} \otimes \mathbf{M}_{j_2})(\mathbf{M}_{j_1'} \otimes \mathbf{M}_{j_2'}) \\
&= (\mathbf{M}_{j_1} \mathbf{M}_{j_1'}) \otimes (\mathbf{M}_{j_2} \mathbf{M}_{j_2'}) \\
&= (\mathbf{M}_{j_1'} \mathbf{M}_{j_1}) \otimes (\mathbf{M}_{j_2'} \mathbf{M}_{j_2}) \\
&= (\mathbf{M}_{j_1'} \otimes \mathbf{M}_{j_2'})(\mathbf{M}_{j_1} \otimes \mathbf{M}_{j_2}),
\end{aligned}$$

$j_1, j_1' = 1, \ldots, l_1$, $j_2, j_2' = 1, \ldots, l_2$, so the statement is established. □

Let $\mathbf{X}_{2,1} = \mathbf{1}_{n_2}$ and $\mathbf{E}_2$ be the unity of the CJA to which the first model is strictly associated. Then nesting the treatments of the second model into the treatments of the first one originates a model $\mathbf{y}_1 \boxdot \mathbf{y}_2$ expressed by

$$\mathbf{y}_1 \boxdot \mathbf{y}_2 = \sum_{j=1}^{l_1} (\mathbf{X}_{1,j} \otimes \mathbf{1}_{n_2}) \boldsymbol{\beta}_j + \sum_{j=2}^{l_2} (\mathbf{E}_1 \otimes \mathbf{X}_{2,j}) \boldsymbol{\beta}_{l_1+j} + \mathbf{e}.$$

Since random effect factors do not nest fixed effect factors, if the first model has random factors, the second one must be a random model. Likewise, if the second model has fixed effect factors, the first one must be a fixed effects model. Reasoning as to establish Proposition 7, we get

**Proposition 8.** *Nesting models with COBS give models with COBS.*

## 3. Inference

**3.1. Sufficient statistics.** Let us assume that $\mathbf{y} \sim \mathscr{N}(\boldsymbol{\mu}, \mathbf{V})$ with $\boldsymbol{\mu} \in \Omega = \mathrm{R}(\mathbf{T})$ and $\mathbf{V} = \sum_{i=1}^{w} \sigma_i^2 \mathbf{Q}_i$. Then, from the proof of Proposition 4,

$$\boldsymbol{\mu} = \sum_{h=1}^{k_1} \mathbf{A}_h \boldsymbol{\zeta}_h,$$

$$\mathbf{V} = \sum_{h=1}^{k_1+k_2} \gamma_i \mathbf{A}_h \mathbf{A}_h'$$

and, assuming $\mathbf{V}$ is non-singular, (see [4])

$$\mathbf{V}^{-1} = \sum_{h=1}^{k_1+k_2} \gamma_i^{-1} \mathbf{A}_h \mathbf{A}_h',$$

$$\det(\mathbf{V}) = \prod_{h=1}^{k_1+k_2} \gamma_i^{g_i}.$$

Thus

$$\begin{aligned}(\mathbf{y} - \boldsymbol{\mu})'\mathbf{V}^{-1}(\mathbf{y} - \boldsymbol{\mu}) &= \sum_{h=1}^{k_1+k_2} \gamma_i^{-1} (\mathbf{y} - \boldsymbol{\mu})' \mathbf{A}_h \mathbf{A}_h' (\mathbf{y} - \boldsymbol{\mu}) \\ &= \sum_{h=1}^{k_1} \frac{\|\boldsymbol{\eta}_h - \boldsymbol{\zeta}_h\|^2}{\gamma_h} + \sum_{h=k_1+1}^{k_1+k_2} \frac{S_h}{\gamma_h},\end{aligned}$$

where $S_h = \|\boldsymbol{\eta}_h\|^2 = \|\mathbf{A}_h'\mathbf{y}\|^2$, $h = k_1 + 1, \ldots, k_1 + k_2$. So, the observation vector has the density

$$n(\mathbf{y}) = \frac{\exp\left(-\frac{1}{2}\sum_{h=1}^{k_1} \frac{\|\boldsymbol{\eta}_h - \boldsymbol{\zeta}_h\|^2}{\gamma_h} - \frac{1}{2}\sum_{h=k_1+1}^{k_1+k_2} \frac{S_h}{\gamma_h}\right)}{\sqrt{(2\pi)^n \prod_{h=1}^{k_1+k_2} \gamma_i^{g_i}}}.$$

The next proposition summarizes these results.

**Proposition 9.** *The $\boldsymbol{\eta}_h$, $h = 1, \ldots, k_1$, and the $S_h$, $h = k_1+1, \ldots, k_1+k_2$, constitute a complete and sufficient set of statistics.*

*Proof.* Given the expression of the density, sufficiency follows from the factorization theorem whilst completeness results from the fact that the normal distribution belongs to the exponential family of densities and (see [6], p. 142) that the parameter space contains a product of non-degenerate intervals. □

By construction $S_h/\gamma_h$ has the central chi-square distribution with $g_h$ degrees of freedom, from where $S_h \sim \gamma_h \chi^2_{(g_h)}$, $h = k_1+1, \ldots, k_1+k_2$, whilst $\boldsymbol{\eta}_\mathbf{h} \sim \mathscr{N}(\boldsymbol{\zeta}, \gamma_h \mathbf{I}_{g_h})$, $h = 1, \ldots, k_1$. Thus, according to the Blackwell-Lehmann-Scheffé theorem (see [6]),

$$\tilde{\gamma}_h = \frac{S_h}{g_h};\ h = k_1+1, \ldots, k_1+k_2$$
$$\boldsymbol{\eta}_\mathbf{h};\ h = 1, \ldots, k_1$$

will be Uniformly Best Linear Unbiased Estimators – UMVUE.

**3.2. Variance components.** Since $S_h \sim \gamma_h \chi^2_{(g_h)}$ we can obtain two-sided and one-sided confidence intervals for $\gamma_h$, $h = k_1+1, \ldots, k_1+k_2$. Let $x_{g,p}$ be the $p$-quantile of a central chi-square distribution with $g$ degrees of freedom. We have for $\gamma_h$ the $1-q$ level confidence intervals

$$\left[\frac{S_h}{x_{g_h,1-\frac{q}{2}}}, \frac{S_h}{x_{g_h,\frac{q}{2}}}\right],$$
$$\left[0, \frac{S_h}{x_{g_h,q}}\right],$$
$$\left[\frac{S_h}{x_{g_h,1-q}}, +\infty\right[,$$

which enable us to test, through duality,

$$\mathrm{H}_0(h) : \gamma_h = c,$$
$$\mathrm{H}_0(h) : \gamma_h > c,$$
$$\mathrm{H}_0(h) : \gamma_h < c,$$

respectively. These tests will have level $q$ and the null hypothesis is rejected when $\tilde{\gamma}_h$ is not contained in the corresponding interval.

Moreover, as

$$\lambda_{h_1 h_2} = \frac{\gamma_{h_1}}{\gamma_{h_2}};\ h_1, h_2 = k_1+1, \ldots, k_1+k_2;\ h_1 \neq h_2,$$

it is easy to see that the statistic

$$F_{h_1 h_2} = \frac{g_2 S_{h_1}}{g_1 S_{h_2}};\ h_1, h_2 = k_1+1, \ldots, k_1+k_2;\ h_1 \neq h_2,$$

will be the product by $\lambda_{h_1,h_2}$ of a central $\mathscr{F}$ distribution with $g_1$ and $g_2$ degrees of freedom, $h_1, h_2 = k_1+1, \ldots, k_1+k_2$, $h_1 \neq h_2$. Thus, with $f_{r,s,p}$ the $p$-quantile of a central $\mathscr{F}$ distribution with $r$ and $s$ degrees of freedom, we have the $(1-q)$-level confidence intervals for $\lambda_{h_1,h_2}$

$$\left[\frac{F_{h_1h_2}}{f_{g_{h_1},g_{h_2},1-\frac{q}{2}}}, \frac{F_{h_1h_2}}{f_{g_{h_1},g_{h_2},\frac{q}{2}}}\right],$$

$$\left[0, \frac{F_{h_1h_2}}{f_{g_{h_1},g_{h_2},q}}\right],$$

$$\left[\frac{F_{h_1h_2}}{f_{g_{h_1},g_{h_2},1-q}}, +\infty\right[.$$

These intervals may be used to obtain, through duality, $\mathscr{F}$ tests for

$$\begin{aligned}
\mathrm{H}_0(h_1,h_2) &: \lambda_{h_1h_2} = c,\\
\mathrm{H}_0(h_1,h_2) &: \lambda_{h_1h_2} > c,\\
\mathrm{H}_0(h_1,h_2) &: \lambda_{h_1h_2} < c,
\end{aligned}$$

respectively. These tests will have level $q$.

**3.3. Estimable vectors.** In models with COBS the LSE estimators of estimable vectors are BLUE. Further, if $\Omega = \mathrm{R}(\mathbf{X})$, and the column vectors of $\mathbf{X}$ are linearly independent,

$$\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$$

and

$$\boldsymbol{\beta} = \mathbf{X}^+\boldsymbol{\mu},$$

whilst the estimable vectors are

$$\boldsymbol{\phi} = \mathbf{G}\boldsymbol{\beta} = \mathbf{G}\mathbf{X}^+\boldsymbol{\mu}.$$

When normality holds, the estimator for the mean vector can be expressed as

$$\tilde{\boldsymbol{\mu}} = \sum_{h=1}^{k_1} \mathbf{A}_h\boldsymbol{\eta}_h$$

and is, according to the Blackwell-Lehmann-Scheffé Theorem, the UMVUE of $\boldsymbol{\mu}$, while

$$\tilde{\boldsymbol{\phi}} = \mathbf{G}\mathbf{X}^+\tilde{\boldsymbol{\mu}}$$

will be the UMVUE of $\boldsymbol{\phi}$.

When $k_1 = 1$ we have

$$\begin{aligned}
\tilde{\boldsymbol{\mu}} &= \mathbf{A}_1\boldsymbol{\eta}_1 \sim \mathscr{N}(\boldsymbol{\mu}, \gamma_1\mathbf{P}_1)\\
\tilde{\boldsymbol{\phi}} &= \mathbf{G}\mathbf{X}^+\tilde{\boldsymbol{\mu}} \sim \mathscr{N}\left(\boldsymbol{\phi}, \gamma_1\mathbf{G}\mathbf{X}^+\mathbf{P}_1(\mathbf{X}^+)'\mathbf{G}'\right).
\end{aligned}$$

To shorten the writing, let

$$\mathbf{W} = \mathbf{G}\mathbf{X}^{+}\mathbf{P}_1(\mathbf{X}^{+})'\mathbf{G}'.$$

Since $k_1 = 1$, there is only one matrix $\mathbf{Q}_i$, $i = 1, \ldots, w$, such that $\mathbf{Q}_i\mathbf{T} = \mathbf{0}_{n\times n}$, say $\mathbf{Q}_{i_0}$. Let us assume that $\mathbf{P}_{h_0} = \mathbf{Q}_{i_0}\mathbf{U}$. Thus $S_{h_0} \sim \gamma_1\chi^2_{(g_0)}$, with $g_0 = \mathrm{rank}(\mathbf{P}_{h_0})$, independent from $\boldsymbol{\eta}_1$ and from $\boldsymbol{\phi}$. Hence

$$(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})'\mathbf{W}^{+}(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}) \sim \gamma_1\chi^2_{(c)},$$

with $c = \mathrm{rank}(\mathbf{W})$. Thus

$$F = \frac{g_0(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})'\mathbf{W}^{+}(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})}{cS_{h_0}}$$

will have central $\mathscr{F}$ distribution with $c$ and $g_0$ degrees of freedom. We then obtain the $(1-q)$-level confidence ellipsoid

$$\left\{\mathbf{x} : (\mathbf{x} - \tilde{\boldsymbol{\phi}})'\mathbf{W}^{+}(\mathbf{x} - \tilde{\boldsymbol{\phi}}) < cf_{c,g_0,1-p}\frac{S_0}{g_0}\right\}$$

for $\boldsymbol{\phi}$. Through duality we get a $q$-level $\mathscr{F}$ test for

$$\mathrm{H}_0(\mathbf{c}) : \boldsymbol{\phi} = \mathbf{c}$$

with $c$ and $g_0$ degrees of freedom.

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