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On representations of stochastic processes by Radon measures on $D(0,1)$

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Abstract. We provide a necessary and sufficient condition for a stochastic process $X = \{X_t : t \in [0,1]\}$, taking values in a real Banach space B, for the existence of a probability Radon measure on the space $D((0,1),B)$ such that the process $\{e_t : t \in [0,1]\}$ consisting of evaluation functionals is distributed as X. The condition may be easy verified for Levy processes.

Throughout the paper $X = \{X_t : t \in [0,1]\}$ is a stochastic process on a probability space (Ω, Σ, P) and B is a real Banach space with a norm $\|\cdot\|_B$. We denote by $D((0,1),B)$ the Banach space of all functions $f : [0,1] \to B$ that are right continuous at each point of $[0,1)$ with left-hand limit at each point of $(0,1]$ equipped with the norm $\|f\| = \sup\{\|f(t)\|_B : t \in [0,1]\}$. In the paper we find a necessary and sufficient condition ($\circledast$) for a stochastic process X, taking values in a Banach space B, for the existence of a probability Radon measure on $D((0,1),B)$ such that the process $\{e_t : t \in [0,1]\}$ consisting of evaluation functionals is distributed as X (i.e., has the same finite dimensional probability laws). The process $\{e_t : t \in [0,1]\}$ is called the process of evaluation functionals. By the Phillips-Grothendieck theorem this is the same as to represent X by a probability Radon measure on the space $D((0,1),B)$ equipped with the weak topology (see [10]). Our condition ($\circledast$) is quite technical but it may be easily verified for Levy processes.

For a given subset Q of $[0,1]$ we denote by $D_Q((0,1),B)$ the subspace of $D((0,1),B)$ consisting of all functions continuous at every point of the set $[0,1] \setminus Q$. If Q and R are subsets of $[0,1)$ such that $Q \cap R \subset \{0\}$, then $D_Q((0,1),B) \cap D_R((0,1),B) = C([0,1],B)$, the Banach space of all B-valued continuous functions on $[0,1]$. The evaluation functional $e_t : D((0,1),B) \to B$ at a point $t \in [0,1]$ is given by $e_t(f) = f(t)$. For every $t \in [0,1)$ we define the function $\pi_t : [0,1] \to \mathbb{R}$ by $\pi_t = \chi_{[t,1]}$ and $\pi_1 = \chi_{\{1\}}$. Then for every

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$0 \leqslant t_1 < t_2 < \cdots < t_n \leqslant 1$ and $y_1, \ldots, y_n \in B$ we have

$$\Big\|\sum_{j=1}^{n} y_j \pi_{t_j}\Big\|_{D((0,1),B)} = \Big\|\Big(\sum_{j=1}^{n} y_j\Big)\pi_{t_n} + \sum_{k=2}^{n}\Big(\sum_{j=1}^{k-1} y_j\Big)(\pi_{t_k} - \pi_{t_{k-1}})\Big\|_B$$
$$= \max_{1\leqslant k\leqslant n}\Big\|\sum_{j=1}^{k} y_j\Big\|_B.$$

It is easy to see that the closed linear hull of functions $\{\pi_t : t \in Q\}$ coincides with $D_Q((0,1),\mathbb{R})$ for every dense subset Q of $[0,1]$. Consequently, the closed linear hull of functions $\{y\pi_t : t \in Q, y \in B\}$ coincides with $D_Q((0,1),B)$ for every dense subset Q of $[0,1]$.

A function $f : \Omega \to B$ is strongly measurable if there exists a sequence (f_n) of Σ-simple B-valued functions such that $f = \lim_{n\to\infty} f_n$ P-almost everywhere (we briefly write P-a.e.). If B is finite dimensional, the notion coincides with the notion of random variable. In the present paper we consider only stochastic processes consisting of strongly measurable functions. The space of all B-valued strongly measurable random variables on (Ω,Σ,P), equipped with the topology of convergence in probability, is denoted by $L(B)$. It is a complete metric space with the metric given by $d(x,z) = \int_\Omega \frac{\|x-z\|}{1+\|x-z\|}\,dP$ for every $x, z \in L(B)$. We say that X is continuous in probability if the map $t \to X_t$ from $[0,1]$ into $L(B)$ is continuous. If the left- and right-hand limits of X in $L(B)$ exist, we denote them by $X_{t-} = \lim_{s\to t-} X_s$ and $X_{t+} = \lim_{s\to t+} X_s$, respectively. We say that $T = (T_n)$ is a partition of $[0,1]$ if

1) $T_n = \{0 = t_{0,n} < t_{1,n} < \cdots < t_{N_n,n} = 1\}$ is a finite subset of $[0,1]$ for every $n \in \mathbb{N}$,
2) $T_n \subset T_{n+1}$ for every $n \in \mathbb{N}$ and
3) $\lim_{n\to\infty} \max_{1\leqslant j\leqslant N_n} t_{j,n} - t_{j-1,n} = 0$.

For a partition $T = (T_n)$ we put $\tilde{T} = \bigcup_{n=1}^{\infty} T_n$. For every integer $k \geqslant 2$ the partition $(\{\frac{j}{k^n} : j = 0,\ldots,k^n\})$ of $[0,1]$ is denoted by T^k.

For a given stochastic process $X = \{X_t : t \in [0,1]\}$, a partition T, $\varepsilon > 0$ and $m > n$ we denote

$$A_{X,T,n,m,\varepsilon} = \big\{\omega \in \Omega : \max_{0\leqslant j<N_n} \max_{k_j<l<k_{j+1}} \|(X_{t_{l,m}} - X_{t_{j,n}})(\omega)\|_B \geqslant \varepsilon\big\}$$

where $t_{j,n} = t_{k_j,m}$ for $j = 0,1,\ldots,N_n$. We say that a stochastic process $\{X_t : t \in [0,1]\}$ with values in B on a probability space (Ω,Σ,P) has the property ($\circledast$) with respect to a partition T of $[0,1]$ if

a) X_t is a strongly measurable function for every $t \in [0,1]$,
b) $\lim_{h\to 0^+} X_{t+h} = X_t$ in probability for every $t \in [0,1) \setminus \tilde{T}$,

c) for every $\varepsilon > 0$

$$\lim_{N\to\infty} P\Big(\bigcup_{n=N}^{\infty}\bigcup_{m=n+1}^{\infty} A_{X,T,n,m,\varepsilon}\Big) = 0.$$

The space $D((0,1),B)$, equipped with the weak topology, is denoted by $(D((0,1),B), weak)$.

Theorem 1. *If $X = \{X_t : t \in [0,1]\}$ is a stochastic process with values in B with the property $(\circledast)$ with respect to a partition T of $[0,1]$, then there exists a probability Radon measure on $D_{\tilde{T}}((0,1),B)$ such that the process of evaluation functionals $\{e_t : t \in [0,1]\}$ is distributed as X.*

For every probability Radon measure on $(D((0,1),B), weak)$ there exists a countable dense subset Q of $[0,1]$ such that the stochastic process $\{e_t : t \in [0,1]\}$ has the property $(\circledast)$ with respect to any partition T of $[0,1]$ such that $Q \subset \tilde{T}$.

Proof. For every finite subset $R = \{t_0, t_1, \dots, t_n\}$ of $[0,1]$ with $0 = t_0 < t_1 < \cdots < t_n = 1$ we define the map $f_R : \Omega \to D((0,1),B)$ by

$$f_R = \sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})\pi_{t_j}.$$

It is clear that for every finite subset R the function f_R is strongly measurable (it is a linear combination of strongly measurable functions) and takes its values in $D_R((0,1),B)$. Moreover, $e_0(f_R) = 0$ and for every $1 \leqslant k \leqslant n$

$$e_{t_k}(f_R) = \sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})\pi_{t_j}(t_k) = \sum_{j=1}^{k}(X_{t_j} - X_{t_{j-1}}) = X_{t_k} - X_0.$$

Let $T = (T_n) = (\{0 = t_{0,n} < t_{1,n} < \cdots < t_{N_n,n} = 1\})$ be the partition. For $m > n$ we put $t_{j,n} = t_{k_j,m}$ for $j = 0,1,\dots,N_n$. Since $\sum_{i=k_j+1}^{k_{j+1}}(X_{t_{i,m}} - X_{t_{i-1,m}}) = X_{t_{j+1,n}} - X_{t_{j,n}}$ for every $j = 1,\dots,N_n$, we have

$$\|f_{T_m}(\omega) - f_{T_n}(\omega)\| = \max_{0\leqslant j<N_n}\ \max_{k_j<l<k_{j+1}}\Big\|X_{t_{l,m}}(\omega) - X_{t_{j,n}}(\omega)\Big\|_B$$

for every $\omega \in \Omega$. It is clear that

$$A_{X,T,n,m,\varepsilon} = \{\omega \in \Omega : \|f_{T_m}(\omega) - f_{T_n}(\omega)\|_B \geqslant \varepsilon\}.$$

By the property $(\circledast)$ the set $D_T = \{\omega \in \Omega : f_{T_n}(\omega) \text{ does not converges}\} = \bigcup_{k=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{m=n+1}^{\infty} A_{X,T,n,m,\frac{1}{k}}$ has measure zero in (Ω,Σ,P). We define $f_T(\omega) = \lim_{n\to\infty} f_{T_n}(\omega)$ for every $\omega \in \Omega \setminus D_T$. Since (f_{T_n}) is a sequence of strongly measurable functions taking values in $D_{\tilde{T}}((0,1),B)$, the function f_T is strongly measurable (by the Pettis measurability theorem, see [10, Thm. 3-1-3]) and takes its values in $D_{\tilde{T}}((0,1),B)$. It is clear that $e_t(f_T(\omega)) = X_t(\omega) - X_0(\omega)$ for every $t \in \tilde{T}$ and for every $\omega \in \Omega \setminus D_T$. For every $t \in [0,1) \setminus \tilde{T}$ there exists a decreasing sequence $(t_n) \subset \tilde{T}$ such that $t = \lim_{n\to\infty} t_n$. Since $f_T(\omega)$ is a right continuous function on $[0,1] \setminus \tilde{T}$, $e_t(f_T(\omega)) = \lim_{n\to\infty} e_{t_n}(f_T(\omega)) = \lim_{n\to\infty} X_{t_n}(\omega) - X_0(\omega)$ for every $\omega \in \Omega \setminus D_T$. Since (X_{t_n}) converges to X_t in probability, $e_t(f_T) = X_t - X_0$ P-a.e.. Let $F_T = f_T + X_0\pi_0$. It is clear that F_T is strongly measurable and $e_t(F_T) = X_t - X_0 + X_0 = X_t$ P-a.e. for every $t \in [0,1]$. By the Pettis measurability theorem there exist a closed separable subspace B_0 of B and a subset E of Ω with $P(E) = 0$ such that $F_T(\omega) \in D_{\tilde{T}}((0,1), B_0)$ for every $\omega \in \Omega \setminus (D_T \cup E)$. Let P_1 be the Borel measure on the space $D_{\tilde{T}}((0,1), B)$ given by the formula $P_1(A) = P(F_T^{-1}(A \cap D_{\tilde{T}}((0,1), B_0))$ for every Borel subset A of $D_{\tilde{T}}((0,1), B)$. Since $D_{\tilde{T}}((0,1), B_0)$ is a separable Banach space the measure P_1 is Radon and the process $\{e_t : t \in [0,1]\}$ on the probability space $(D_{\tilde{T}}((0,1), B)$, the Borel σ-algebra of $D_{\tilde{T}}((0,1), B), P_1)$ is distributed as X.

Due to the Phillips-Grothendieck theorem (see [10, Thm. 2-3-4]) for every probability Radon measure P on $(D((0,1), B), weak)$ there exists a separable subspace D_0 of $D((0,1), B)$ such that $P(A) = P(A \cap D_0)$. Therefore, there exists a countable dense subset Q and a separable closed subspace B_0 of B such that $D_0 \subset D_Q((0,1), B_0)$. Hence for every $t \in [0,1]$ the function e_t takes P-almost all its values in B_0. By the Pettis measurability theorem e_t is strongly measurable for every $t \in [0,1]$. Since $D_Q((0,1), B_0)$ consists of right continuous functions on $[0,1) \setminus Q$, the process $E = \{e_t : t \in [0,1]\}$ satisfies the condition b) of the property ($\circledast$). Let $T = (\{0 = t_{0,n} < t_{1,n} < \cdots < t_{N_n,n} = 1\})$ be a partition of $[0,1]$ such that $Q \subset \tilde{T}$. Let $R_n : D((0,1), B) \to D_Q((0,1), B)$ be the projection given by $R_n(f) = f(1)\pi_1 + \sum_{i=0}^{N_n-1} f(t_{i,n})(\pi_{t_{i+1,n}} - \pi_{t_{i,n}})$. It is clear that $\|R_n\| \leqslant 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} R_n(y\pi_t) = y\pi_t$ for every $y \in B_0$ and $t \in \tilde{T}$. Therefore $\lim_n R_n(f) = f$ for every $f \in D_Q((0,1), B_0)$. Consequently, $(R_n(f))$ is a Cauchy sequence in $D((0,1), B)$ for every $f \in D_Q((0,1), B_0)$. Hence

$$P\Big(\bigcup_{j=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{r=n+1}^{\infty} \{f \in D((0,1), B) : \|R_r(f) - R_n(f)\|_B \geqslant \tfrac{1}{j}\}\Big) = 0.$$

Therefore

$$\lim_{N\to\infty} P\Big(\bigcup_{n=N}^{\infty} \bigcup_{r=n+1}^{\infty} A_{E,T,n,r,\varepsilon}\Big) =$$

$$\lim_{N\to\infty} P\Big(\bigcup_{n=N}^{\infty} \bigcup_{r=n+1}^{\infty} \{f \in D((0,1), B) : \max_{0 \leqslant j < N_n} \max_{k_j < l < k_{j+1}} \|f(t_{l,r}) - f(t_{j,n})\|_B \geqslant \varepsilon\}\Big)$$

$$= \lim_{N\to\infty} P\Big(\bigcup_{n=N}^{\infty} \bigcup_{r=n+1}^{\infty} \{f \in D((0,1), B) : \|R_r(f) - R_n(f)\|_B \geqslant \varepsilon\}\Big) = 0$$

for each $\varepsilon > 0$, where $t_{j,n} = t_{k_j,r}$ for $j = 0, 1, \ldots, N_n$, $r > n$. Thus we show that the process $\{e_t : t \in [0,1]\}$ has the property ($\circledast$) with respect to any partition T of $[0,1]$ such that $Q \subset \tilde{T}$. □

Corollary 2. *Let $X = \{X_t : t \in [0,1]\}$ be a stochastic process with values in B. There exists a probability Radon measure on $C([0,1], B)$ such that the process $\{e_t : t \in [0,1]\}$ is distributed as X, if*

a) *X has the property ($\circledast$) with respect to partitions T and R of $[0,1]$ such that $T \cap R = \{0,1\}$ and $\lim_{h\to 0^+} X_{1-h} = X_1$ in probability,*

or

b) *X has the property ($\circledast$) with respect to a partition T of $[0,1]$ and $X_t = \lim_{h\to 0^+} X_{t-h}$ in probability for every $t \in \tilde{T} \setminus \{0\}$.*

Moreover, for every probability Radon measure on $C([0,1], B)$ the stochastic process $\{e_t : t \in [0,1]\}$ is continuous in probability and has the property ($\circledast$) with respect to any partition T of $[0,1]$.

Proof. a) For the partitions T and R we get strongly measurable functions F_T and F_R (see the proof above) taking P-almost all its values in $D_{\tilde{T}\cup\tilde{R}}((0,1), B)$ such that $e_t(F_T) = X_t = e_t(F_R)$ P-a.e. for every $t \in [0,1]$. Evaluation operators $\{e_t : t \in \tilde{T} \cup \tilde{R}\}$ separate points of $D_{\tilde{T}\cup\tilde{R}}((0,1), B)$. Therefore $F_T = F_R$ P-a.e.. Hence F_T takes P-almost all its values in $D_{\tilde{R}}((0,1), B) \cap D_{\tilde{T}}((0,1), B)$. Since $\tilde{P} \cap \tilde{R} = \{0,1\}$ and $\lim_{h\to 0^+} F_T(\omega)(1-h)$ exists for P-almost all $\omega \in \Omega$ and $\lim_{h\to 0^+} X_{1-h} = X_1$ in probability, the function F_T takes P-almost all its values in $C([0,1], B)$.

b) We show in the proof of Theorem 1 that

$$P\Big(\bigcup_{t\in[0,1]\setminus\tilde{T}} \{\omega \in \Omega : F_T(\omega) \text{ is discontinuous at } t\}\Big) = 0.$$

Let $t \in \tilde{T}$. Since $\lim_{h\to 0^+} F_T(\omega)(t-h)$ exists for P-almost all $\omega \in \Omega$ and $\lim_{h\to 0^+} X_{t-h} = X_t$ in probability, $P\big(\{\omega \in \Omega : \limsup_{h\to 0^+} |X_{t-h}(\omega) - X_t(\omega)| \neq 0\}\big) = 0$. Therefore

$$P\Big(\bigcup_{t\in[0,1]} \{\omega \in \Omega : F_T(\omega) \text{ is discontinuous at } t\}\Big) = 0.$$

The second part is a straightforward consequence of the fact that $C([0,1], B)$ is a subspace of the $D_Q((0,1), B)$ for every dense subset Q of $[0,1]$. □

Corollary 3. *Let $X = \{X_t : t \in [0,1]\}$ be a stochastic process consisting of strongly measurable functions taking values in B. If X is continuous in probability, then the following assertions are equivalent:*

a) *there exists a stochastic process $Y = \{Y_t : t \in [0,1]\}$ on a probability space $(\Omega_1, \Sigma_1, P_1)$ consisting of strongly measurable functions taking values in B distributed as X such that the function $t \to Y_t(\omega)$ from*

$[0,1]$ into B is continuous for P_1-almost every $\omega \in \Omega_1$ (= Y has continuous realizations P_1-a.e.),

b) *there exists a probability Radon measure on $C([0,1],B)$ such that the process $\{e_t : t \in [0,1]\}$ is distributed as X,*

c) *there exists a probability Radon measure on $(D((0,1),B), weak)$ such that the process $\{e_t : t \in [0,1]\}$ is distributed as X,*

d) *there exists a partition T of $[0,1]$ such that X has the property ($\circledast$) with respect to T,*

e) *X has the property ($\circledast$) with respect to any partition T of $[0,1]$.*

Proof. The implication b)⇒c) is obvious. The implication c)⇒d) is a part of Theorem 1. The implication d)⇒e) follows from Corollary 2 b) and c). The implication d)⇒b) follows from Corollary 2 a). The implication b)⇒a) is obvious. We only need to show that a)⇒b).

Let Q be a countable dense subset of $[0,1]$. By the Pettis measurability theorem (see [10, Thm. 3-1-3]) for every $t \in Q$ there exists a separable subset B_t of B such that $P(Y_t^{-1}(B_t)) = 1$. Let C be a closed linear hull of the set $\bigcup_{t\in Q} B_t$ in B. It is clear that C is a separable subspace of B. For P-almost all $\omega \in \bigcap_{t\in Q} Y_t^{-1}(B_t)$ the function $t \to Y_t(\omega)$ from $[0,1]$ into B is continuous and $\{Y_t(\omega) : t \in Q\} \subset C$. It shows that this function takes its values in C. Hence the map $\Psi : \Omega \to C([0,1],C)$ given by $\Psi(\omega)(t) = Y_t(\omega)$ is strongly measurable. Therefore the measure P_1 given by $P_1(A) = P(\Psi^{-1}(A \cap C([0,1],C))$ for every Borel subset A of $C([0,1],B)$ is a probability Radon measure on $C([0,1],B)$. □

A strongly measurable function $f : \Omega \to B$ is Bochner integrable if $\int_\Omega \|f\|\, dP < \infty$. The reader may find more information about Bochner integrable functions and martingales in Banach spaces in [3].

Corollary 4. *Let $X = \{X_t : t \in [0,1]\}$ be a continuous in probability martingale consisting of Bochner integrable functions with values in B. If*

$$\lim_{n\to\infty} \int_\Omega \max_{1\leqslant j\leqslant k^n} \|X_{\frac{j+1}{k^n}} - X_{\frac{j}{k^n}}\|\, dP = 0,$$

for some integer $k \geqslant 2$, then X has the property ($\circledast$) with respect to any partition T of $[0,1]$.

Proof. Let $M_{n,m,l} : \Omega \to B^{k^n}$ be given by $M_{n,m,l} = (X_{\frac{j}{k^n}+\frac{l}{k^m}} - X_{\frac{j}{k^n}})_{j=0}^{k^n-1}$. We equip B^{k^n} with the norm $\|(y_1,\ldots,y_{k^n})\| = \max_{1\leqslant j\leqslant k^n} \|y_j\|_B$. Then $\{\|M_{n,m,l}\| : l = 1,\ldots,k^{m-n}\}$ is a submartingale. By the Doob inequality (see [9, p. 493]) for every $\varepsilon > 0$ and $m > n$ we get

$$P(A_{X,T^k,n,m,\varepsilon}) \leqslant \frac{1}{\varepsilon} \int_\Omega \|M_{n,m,k^{m-n}}\| dP = \frac{1}{\varepsilon} \int_\Omega \max_{1\leqslant j\leqslant k^n} \|X_{\frac{j+1}{k^n}} - X_{\frac{j}{k^n}}\|\, dP.$$

This shows that $P(\bigcup_{m=n+1}^{\infty} A_{X,T^k,n,m,\varepsilon}) \leqslant \frac{1}{\varepsilon}\int_\Omega \max_{1\leqslant j\leqslant k^n} \|X_{\frac{j+1}{k^n}} - X_{\frac{j}{k^n}}\| \, dP$. Let (l_n) be a sequence of integers such that $\sum_{n=1}^{\infty}\int_\Omega \max_{1\leqslant j\leqslant k^{l_n}} \|X_{\frac{j+1}{k^{l_n}}} - X_{\frac{j}{k^{l_n}}}\| \, dP < \infty$. Then X has the property ($\circledast$) with respect to the partition $T = (\{\frac{j}{k^{l_n}} : j = 0, \ldots, k^{l_n}\})$ of $[0,1]$. An appeal to Corollary 3 completes the proof. □

The proof of our next result is based on the following modification of the vector valued Ottaviani inequality.

Proposition 5. *If $\{X_{j,k} : 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant N_j\}$ are independent strongly measurable random variables with values in B, then for every $\varepsilon > 0$*

$$P(\{\max_{1\leqslant j\leqslant n}\max_{1\leqslant l\leqslant N_j}\Big\|\sum_{k=1}^{l} X_{j,k}\Big\| \geqslant 2\varepsilon\})(1 - \max_{1\leqslant j\leqslant n}\max_{1\leqslant l< N_j}\{P(\Big\|\sum_{k=l+1}^{N_j} X_{j,k}\Big\| \geqslant \varepsilon)\})$$

$$\leqslant 2P(\max_{1\leqslant j\leqslant n}\Big\|\sum_{k=1}^{N_j} X_{j,k}\Big\| \geqslant \varepsilon).$$

Proof. For $n = 1$ the inequality follows from the Ottaviani inequality (see [11, Thm. 11.3.1]). Let $A_j = \{\max_{1\leqslant l\leqslant N_j} \|\sum_{k=1}^{l} X_{j,k}\| \geqslant 2\varepsilon\}$ and $B_j = \{\|\sum_{k=1}^{N_j} X_{j,k}\| \geqslant \varepsilon\}$. Let $f : [0,1]^2 \to [0,1]$ be given by $f(x,y) = x + y - xy$. The function f is increasing in each variable separately and strictly increasing in each variable separately on $[0,1)^2$. Suppose that $c^{-1} = 1 - \max_{1\leqslant j\leqslant n}\max_{1\leqslant l\leqslant N_j}\{P(\big\|\sum_{k=l+1}^{N_j} X_{j,k}\big\| \geqslant \varepsilon)\} \neq 0$. Otherwise there is nothing to prove. We need to show that $P(A_1 \cup \cdots \cup A_n) \leqslant 2cP(B_1 \cup \cdots \cup B_n)$. By the Ottaviani inequality $P(A_j) \leqslant cP(B_j)$ for every $j = 1, \ldots, n$. Moreover, functions $\chi_{A_1}, \ldots, \chi_{A_n}$ as well as $\chi_{B_1}, \ldots, \chi_{B_n}$ are stochastically independent.

If $c = 1$, then

$$P(A_1 \cup A_2) = f(P(A_1), P(A_2)) \leqslant f(P(B_1), P(B_2)) = P(B_1 \cup B_2).$$

It is clear that for $n > 2$ functions $\chi_{A_1\cup A_2}, \chi_{A_3}, \ldots, \chi_{A_n}$ as well as $\chi_{B_1\cup B_2}, \chi_{B_3}, \ldots, \chi_{B_n}$ are stochastically independent. We repeat $n-1$ times the procedure above to get the inequality $P(A_1 \cup \cdots \cup A_n) \leqslant P(B_1 \cup \cdots \cup B_n)$.

Assume now that $c > 1$. Without loss of generality we may assume that there exists $1 \leqslant k \leqslant n$ such that $P(B_j) < P(A_j)$ for every $j \leqslant k$ and $P(A_j) \leqslant P(B_j)$ for every $j > k$. Suppose that $k \geqslant 2$. Let $0 \leqslant d_j = \frac{cP(B_j)-P(A_j)}{(c-1)P(A_j)} < 1$ and $e_j = \frac{P(B_j)-P(A_j)}{P(A_j)} < 0$ for $j = 1, 2$. Applying the fact $0 \leqslant P(A_j)d_j < 1$

for $j = 1, 2$ we get

$$\begin{aligned} &cP(B_1 \cup B_2) - P(A_1 \cup A_2) \\ &= c(P(B_1) + P(B_2) - P(B_1)P(B_2)) - P(A_1) - P(A_2) + P(A_1)P(A_2) \\ &= (c-1)f(P(A_1)d_1, P(A_2)d_2) + \frac{c}{c-1}P(A_1)P(A_2)e_1e_2 \geqslant 0. \end{aligned}$$

It is clear that for $k > 2$ functions $\chi_{A_1 \cup A_2}, \chi_{A_3}, \dots, \chi_{A_k}$ as well as $\chi_{B_1 \cup B_2}, \chi_{B_3}, \dots, \chi_{B_k}$ are stochastically independent. Moreover,

$$P(A_1 \cup A_2) = f(P(A_1), P(A_2)) > f(P(B_1), P(B_2)) = P(B_1 \cup B_2).$$

We repeat $k-1$ times the procedure above to get the inequality $P(A_1 \cup \dots \cup A_k) \leqslant cP(B_1 \cup \dots \cup B_k)$. It follows from the considerations above that $P(A_{k+1} \cup \dots \cup A_n) \leqslant P(B_{k+1} \cup \dots \cup B_n)$. After gathering together the above inequalities we get

$$\begin{aligned} &P(A_1 \cup \dots \cup A_n) \leqslant 2\max\{P(A_1 \cup \dots \cup A_k), P(A_{k+1} \cup \dots \cup A_n)\} \\ &\leqslant 2\max\{cP(B_1 \cup \dots \cup B_k), P(B_{k+1} \cup \dots \cup B_n)\} \leqslant 2cP(B_1 \cup \dots \cup B_n). \end{aligned}$$

□

Let $\tau_{t+} = \chi_{[0,t]}$ and $\tau_{t-} = \chi_{[0,t)}$ for $t \in (0,1]$ and $\tau_0 = \chi_{\{0\}}$. We denote by $\mathbb{L}$ the set $\{\tau_{t\pm} : t \in (0,1]\} \cup \{\tau_0\}$ equipped with the pointwise convergence topology. The space $\mathbb{L}$ is a Hausdorff, compact, sequentially compact, first countable, nonmetrizable space. It is a modification of *the two arrows space.* The reader may find more information about topological properties of this space in [4, p. 270] and [6]. If the left- and right-hand limits of X in $L(B)$ exist and $X_t = X_{t+}$ P-a.e. for every $t \in [0,1]$, then the map $\tau_s \to X_s$ from $\mathbb{L}$ to $L(B)$ is continuous. Consequently, the set $\{t \in [0,1] : d(X_t, X_{t-}) \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$. Therefore $\{t \in [0,1] : d(X_t, X_{t-}) > 0\}$ is countable. For a given stochastic process $X = \{X_t : t \in [0,1]\}$, a partition T, $\varepsilon > 0$ and n we put

$$E_{X,T,n,\varepsilon} = \big\{\omega \in \Omega : \max_{0 \leqslant j < N_n} \|(X_{t_{j+1,n}} - X_{t_{j,n}})(\omega)\|_B \geqslant \varepsilon\big\} \qquad \text{and}$$

$$E^-_{X,P,n,\varepsilon} = \big\{\omega \in \Omega : \max_{0 \leqslant j < N_n} \|(X_{t_{j+1,n}-} - X_{t_{j,n}})(\omega)\|_B \geqslant \varepsilon\big\}.$$

A process $X = \{X_t : t \in [0,1]\}$ has independent increaments if for every $0 < t_1 < \dots < t_n \leqslant 1$ the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

Corollary 6. *Let $X = \{X_t : t \in [0,1]\}$ be a stochastic process consisting of strongly measurable functions with values in B with independent increaments.*

a) *If X is continuous in probability, then X has has the property* ($\circledast$) *with respect to a partition T of $[0,1]$ if and only if*

$$\lim_{n\to\infty} P(E_{X,T^2,n,\varepsilon}) = 0$$

for every $\varepsilon > 0$.

b) *Suppose that* X *has left- and right-hand limits in* $L(B)$ *for every* $t \in [0,1]$ *and* $X_t = X_{t+}$ P*-a.e. for every* $t \in [0,1)$. *If* T *is a partition of* $[0,1]$ *such that* $\{t : d(X_t, X_{t-}) > 0\} \subset \tilde{T}$, *then* X *has the property* ($\circledast$) *with respect to* T *if and only if*

$$\lim_{n\to\infty} P(E^{-}_{X,T,n,\varepsilon}) = 0$$

for every $\varepsilon > 0$.

Proof. a) Suppose that X has the property ($\circledast$) with respect to a partition T of $[0,1]$. According to Corollary 3 the process X has the property ($\circledast$) with respect to T^2. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Applying the facts that for every $j = 1, \dots, 2^n$, there exists an increasing sequence $(s_{j,k}) \subset \tilde{T}^2$ such that $\lim_{k\to\infty} s_{j,k} = \frac{j}{2^n}$ and $\lim_{k\to\infty} X_{s_{j,k}} = X_{\frac{j}{2^n}}$ P-a.e. we get

$$P(E_{X,T^2,n,\varepsilon}) \leqslant P(\bigcup_{m=n+1}^{\infty} A_{X,T^2,n,m,\frac{\varepsilon}{2}}).$$

Therefore $\lim_{n\to\infty} P(E_{X,T^2,n,\varepsilon}) = 0$ for every $\varepsilon > 0$.

Assume now that $\lim_{n\to\infty} P(E_{X,T^2,n,\delta}) = 0$ for every $\delta > 0$. Let (l_k) be an increasing sequence of integers such that $\sum_{k=1}^{\infty} P(E_{X,T^2,l_k,\frac{1}{k}}) < \infty$. Let $T = (\{\frac{j}{2^{l_k}} : j = 0, \dots, 2^{l_k}\})$. Let $\varepsilon > 0$. Since the topology of convergence in probability is metric, there exists $\delta > 0$ such that $P(\|X_t - X_s\| \geqslant \frac{\varepsilon}{2})) \leqslant \frac{1}{2}$ for every $|t-s| < \delta$. Applying Proposition 5 for $\{Y_{j,k} = X_{\frac{j}{2^{l_n}}+\frac{k}{2^{l_m}}} - X_{\frac{j}{2^{l_n}}+\frac{k-1}{2^{l_m}}} : 0 \leqslant j \leqslant 2^{l_n} - 1, 1 \leqslant k \leqslant 2^{l_m - l_n}\}$ we get $P(A_{X,T,n,m,\varepsilon}) \leqslant 4P(E_{X,T^2,l_n,\frac{\varepsilon}{2}})$ for every n such that $2^{-l_n} < \delta$. Since $A_{X,T,n,m,\varepsilon} \subset A_{X,T,n,r,\varepsilon}$ for every $r \geqslant m$, $P(\bigcup_{m=n+1}^{\infty} A_{X,T,n,m,\varepsilon}) \leqslant 4P(E_{X,T^2,l_n,\frac{\varepsilon}{2}})$ for every n such that $2^{-l_n} < \delta$. Since $E_{X,T^2,l_n,\frac{\varepsilon}{2}} \subset E_{X,T^2,l_n,\frac{1}{n}}$ for almost all n, X has the property ($\circledast$) with respect to T.

The proof of part b) is similar. □

A stochastic process $X = \{X_t : t \in [0,1]\}$ is said to be a Levy process if it has the following properties:

1) $X_0 = 0$,
2) X has independent increaments,
3) for every $0 \leqslant s < t \leqslant 1$ the random variable $X_t - X_s$ has the same distribution as X_{t-s}.

For a real Levy process $X = \{X_t : t \in [0,1]\}$ the random variable X_1 has an infinite divisible distribution. The reader may find more information about this family of distributions in [5].

A stochastic process $N(c,Y) = \{X_t : t \in [0,1]\}$ is said to be a compound Poisson process if it can be represented for $t \geqslant 0$ by

$$X_t = \sum_{k=0}^{N_t} Y_k$$

where $\{N_t : t \in [0,1]\}$ is a Poisson process with the mean rate $c > 0$, $Y_0 = 0$ and (Y_n) is a sequence of independent random variables identically distributed as the random variable Y. Moreover, the process $\{N_t : t \in [0,1]\}$ and the sequence (Y_n) are assumed to be independent. If $Y = 1$, then $N(c,1) = \{N_t : t \in [0,1]\}$ is Poisson process with the mean rate c. For any random variable Y and $c > 0$ the process $N(c,Y)$ is a Levy process. The characteristic function of $N(c,Y)_t$ is given by

$$\varphi_{N(c,Y)_t}(u) = e^{ct(\varphi_Y(u)-1)}.$$

The reader may find more information about compound Poisson processes in [7].

Theorem 7. *A Levy process $X = \{X_t : t \in [0,1]\}$ taking values in $\mathbb{R}^n$ has the property $(\circledast)$ with respect to a partition T of $[0,1]$ if and only if it is Gaussian.*

Proof. Suppose first that $n = 1$. By the Levy-Khintchine representation theorem the characteristic function of X_1 has the following form

$$\varphi_{X_1}(u) = e^{iua + \int_{\mathbb{R}} (e^{iux} - 1 - \frac{iux}{1+x^2}) \frac{1+x^2}{x^2}\, d\mu}$$

for some constant $a \in \mathbb{R}$ and a positive Borel measure μ on $\mathbb{R}$ where the function under the integral sign is equal to $-\frac{u^2}{2}$ at the point $x = 0$ (see [5, Thm. 5.5.1], [8, Thm. 1.16]). If $\mu = 0$, then the above theorem is obvious. We assume that $\mu(\mathbb{R}) > 0$. Let Y_1 be a random variable with the characteristic function

$$\varphi_{Y_1}(u) = e^{\int_{\mathbb{R}} (e^{iux} - 1)\, d\mu}.$$

It is clear that there exists a random variable V such that $Y_1 = N(\mu(\mathbb{R}), V)_1$. According to the Kolmogorov representation theorem (see [5, Thm. 5.5.3], [1, Thm. 28.1]) there exists a random variable Z_1 with the characteristic function

$$\varphi_{Z_1}(u) = e^{iua + \int_{\mathbb{R}} \frac{e^{iux} - 1 - iux}{x^2}\, d\mu}.$$

Then Y_1 and Z_1 are infinite divisible. Let $Y = \{Y_t : t \in [0,1]\}$ and $Z = \{Z_t : t \in [0,1]\}$ be the Levy processes generated by Y_1 and Z_1, respectively. We assume that Y and Z are independent. Then the process $\bar{X} = \{X_t + Y_t : t \in [0,1]\}$ is distributed as X. It is easy to check that $\{Y_t : t \in [0,1]\}$ is the Poisson compound process $N(\mu(\mathbb{R}), V)$. Suppose that $V \neq 0$. Let (V_n) be a sequence of independent random variables distributed as V on (Ω, Σ, P). Let $A_{0,\eta} = \Omega$ and $A_{k,\eta} = \{\omega \in \Omega : |\sum_{j=1}^k V_j| < \eta\}$. Let $\varepsilon > 0$ be such that $P(A_{1,3\varepsilon}) < 1$. It is easy to check that $\lim_{k\to\infty} 2^k(e^{\frac{\mu(\mathbb{R})}{2^k}} - 1 + \frac{\mu(\mathbb{R})}{2^k}(1 - P(A_{1,3\varepsilon})) = \mu(\mathbb{R})(P(A_{1,3\varepsilon}))$. Hence

$$\begin{aligned}
\limsup_{k\to\infty} P(|Y_{\frac{1}{2^k}}| < 3\varepsilon)^{2^k} &= \limsup_{k\to\infty}\Big(e^{-\frac{\mu(\mathbb{R})}{2^k}}\sum_{j=0}^{\infty}\frac{\mu(\mathbb{R})^j P(A_{j,3\varepsilon})}{2^{jk}j!}\Big)^{2^k} \\
&\leqslant e^{-\mu(\mathbb{R})}\lim_{k\to\infty}\Big(1+\frac{\mu(\mathbb{R})}{2^k}P(A_{1,3\varepsilon})+\sum_{j=2}^{\infty}\frac{\mu(\mathbb{R})^j}{2^{jk}j!}\Big)^{2^k} \\
&= e^{-\mu(\mathbb{R})}\lim_{k\to\infty}\Big(e^{\frac{\mu(\mathbb{R})}{2^k}}-\frac{\mu(\mathbb{R})}{2^k}(1-P(A_{1,3\varepsilon})\Big)^{2^k} \\
&= e^{-\mu(\mathbb{R})(1-P(A_{1,3\varepsilon}))} < 1.
\end{aligned}$$

For $t \in [0,1]$ such that $e^{-\mu(\mathbb{R})t} > \frac{1}{2}$ we have

$$\begin{aligned}
P(|Y_t + Z_t| < \varepsilon) &\leqslant P(\bigcup_{k\in\mathbb{Z}}\{|Y_t - k\varepsilon| < 2\varepsilon, k\varepsilon \leqslant Z_t < (k+1)\varepsilon\} \\
&= \sum_{k\in\mathbb{Z}} P(\{|Y_t - k\varepsilon| < 2\varepsilon\})P(\{k\varepsilon \leqslant Z_t < (k+1)\varepsilon\}) \\
&\leqslant \sup_{k\in\mathbb{Z}} P(\{|Y_t - k\varepsilon| < 2\varepsilon\})\sum_{k\in\mathbb{Z}} P(\{k\varepsilon \leqslant Z_t < (k+1)\varepsilon\}) \\
&\leqslant P(\{|Y_t| < 3\varepsilon\}).
\end{aligned}$$

Hence

$$\begin{aligned}
\limsup_{k\to\infty}\big(1-P(E_{X,T^2,k,\varepsilon})\big) &= \limsup_{k\to\infty} P(|Y_{\frac{1}{2^k}} + Z_{\frac{1}{2^k}}| < \varepsilon)^{2^k} \\
&\leqslant \limsup_{k\to\infty} P(\{|Y_{\frac{1}{2^k}}| < 3\varepsilon\})^{2^k} \leqslant e^{-\mu(\mathbb{R})(1-P(A_{1,3\varepsilon}))} < 1.
\end{aligned}$$

The process X is continuous in probability and does not have the property $(\circledast)$ with respect to T^2 if $V \neq 0$. According to Corollary 6 the process X does not have the property $(\circledast)$ with respect to any partition T of $[0,1]$ if $V \neq 0$. If $V = 0$, then $\mu = \mu(\mathbb{R})\delta_0$, where δ_0 is the Dirac measure concentrated at 0. Then the random variable X_1 has the following characteristic function

$$\varphi_{X_1}(u) = e^{iua+\frac{cu^2}{2}}.$$

Suppose now that $n > 1$. Let $p_j : \mathbb{R}^n \to \mathbb{R}$ be the projection given by $p_j((x_1,\dots,x_n)) = x_j$ for each $j = 1,\dots,n$. For every $a_1,\dots,a_n \in \mathbb{R}$ the process $\{\sum_{j=1}^n a_j p_j \circ X_t : t \in [0,1]\}$ is a Levy process. It is clear that if X has the property $(\circledast)$ with respect a partition to T of $[0,1]$, then also the process $\{\sum_{j=1}^n a_j p_j \circ X_t : t \in [0,1]\}$ has the property $(\circledast)$ with respect to T. According to the first part of the proof if X has the property $(\circledast)$ with respect to any partition T of $[0,1]$ then for every $t \in [0,1]$ and for every $a_1, \dots, a_n \in \mathbb{R}$ the random variable $\sum_{j=1}^n a_j p_j \circ X_t$ has a Gaussian distribution. Therefore X is Gaussian (see [9, p. 301]).

It is a well-known fact that if X is a Gaussian Levy process, then there exists a stochastic process $Y = \{Y_t : t \in [0,1]\}$ on a probability space $(\Omega_1, \Sigma_1, P_1)$ distributed as X with continuous realizations P_1-a.e. (see [2]). But the fact we also get by applying Corollary 6. Since X is Gaussian, there exist $c \in \mathbb{R}^n$ and a selfadjoint linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ such that the joint characteristic function of X_t is given by

$$\varphi_{X_t}(u) = e^{it\langle c,u\rangle - \frac{t\langle A(u),A(u)\rangle}{2}}$$

for every $u \in \mathbb{R}^n$. Moreover, for every $t \in [0,1]$ the random variable X_t has the same distribution as the random variable $\omega \to tc+\sqrt{t}A(U_1(\omega), \dots, U_n(\omega))$ where $U_1, \dots, U_n$ are independent normal $\mathcal{N}(0,1)$ random variables. Suppose that $A \neq 0$. Otherwise there is nothing to prove. Let $F_\eta = \{x \in \mathbb{R}^n : \|x\| < \eta\}$. Let $\varepsilon > 0$. Then

$$\begin{aligned}
1 &\geqslant \lim_{k\to\infty} 1 - P(E_{X,T^2,k,\varepsilon}) = \lim_{k\to\infty} P(\|X_{\frac{1}{2^k}}\| < \varepsilon)^{2^k} \\
&= \lim_{k\to\infty} \left(P(\{\omega \in \Omega : c2^{-k} + 2^{-\frac{k}{2}} A(U_1(\omega), \dots, U_n(\omega)) \in F_\varepsilon\})\right)^{2^k} \\
&= \lim_{k\to\infty} \left((2\pi)^{-\frac{n}{2}} \int_{A^{-1}(2^{\frac{k}{2}} F_\varepsilon - 2^{-\frac{k}{2}} c)} e^{-\frac{\langle x,x\rangle}{2}} d\lambda_n(x)\right)^{2^k} \\
&\geqslant \lim_{k\to\infty} \left((2\pi)^{-\frac{n}{2}} \int_{A^{-1}(F_{2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|})} e^{-\frac{\langle x,x\rangle}{2}} d\lambda_n(x)\right)^{2^k} \\
&\geqslant \lim_{k\to\infty} \left((2\pi)^{-\frac{n}{2}} \int_{F_{\|A\|^{-1}(2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|)}} e^{-\frac{\langle x,x\rangle}{2}} d\lambda_n(x)\right)^{2^k} \\
&\geqslant \lim_{k\to\infty} \left(\frac{1}{2\pi} \int_{-\frac{2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|}{\sqrt{n}\|A\|}}^{\frac{2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|}{\sqrt{n}\|A\|}} \int_{-\frac{2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|}{\sqrt{n}\|A\|}}^{\frac{2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|}{\sqrt{n}\|A\|}} e^{-\frac{x^2+y^2}{2}} dxdy\right)^{n2^{k-1}} \\
&\geqslant \lim_{k\to\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\frac{2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|}{\sqrt{n}\|A\|}} e^{-\frac{r^2}{2}} r\, dr dt\right)^{n2^{k-1}} \\
&= \lim_{k\to\infty} \left(1 - e^{-\frac{(2^{\frac{k}{2}}\varepsilon - 2^{-\frac{k}{2}}\|c\|)^2}{2n\|A\|^2}}\right)^{n2^{k-1}} = 1,
\end{aligned}$$

where λ_n is the Lebesgue measure on $\mathbb{R}^n$ □

Corollary 8. *If a Levy process $X = \{X_t : t \in [0,1]\}$ with values in a Banach space B has continuous realizations P-a.e., then for any continuous linear operator $A : B \to \mathbb{R}^n$ the process $\{A \circ X_t : t \in [0,1]\}$ is Gaussian.*

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