The method of three-parameter Weibull distribution estimation

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ABSTRACT. In this paper we develop Maximum Likelihood (ML) and Improved Analytical (IA) numerical algorithms to estimate parameters of the Weibull distribution, namely, location, scale and shape parameters, using order statistics of a noncensored sample. Since ML method leads to multiextremal numerical problem we establish conditions to localize extremes of the ML function, which enables us to avoid problems related with ML estimation failure and to create a simple estimation procedure by solving one-dimensional equation. IA estimation also has been developed by solving the equation in one variable. The estimates proposed are studied by computer modeling and compared with the theoretical ones with respect to sample size and number of order statistics used for estimation. Recommendations for implementation of the estimates are also discussed.

1. Introduction

Weibull distribution (Weibull, 1951) has many applications in engineering and plays an important role in reliability and maintainability analysis. The Weibull distribution is one of the extreme-value distributions which is applied also in optimality testing of Markov type optimization algorithms (Haan (1981), Zilinskas & Zhigljavsky (1991), Bartkute & Sakalauskas (2004)). Because of useful applications, its parameters need to be evaluated precisely, and efficiently. However estimating the parameters of a three-parameter Weibull distribution has historically been complicated since classical estimation procedures such as ML estimation have become almost too fraught to implement. Some questions of estimation of the location, scale and shape parameters of this distribution for both censored and noncensored samples were considered by several authors (Rockette *et al.* (1974), Lemon (1975),

Hirose (1991), etc.). However, iterative computational methods for the estimation are needed in most cases (Hirose (1991), Bartolucci (1999)). In the paper Bartkute & Sakalauskas (2007) it was proposed an approach for three-parameter Weibull estimation solving univariate equations. In this paper, we develop in details two algorithms (ML and IA) for estimating Weibull parameters, namely, location, scale and shape parameters, using order statistics of a noncensored sample and making some simplifications which enable us to construct reliable and computationally efficient procedures for estimation.

2. Maximum likelihood method

The three-parameter Weibull distribution (Weibull, 1951) has the cumulative distribution function (cdf):

$$W(x, \alpha, c, A) = 1 - \mathrm{e}^{-c \cdot (x-A)^{\alpha}}, \quad \alpha > 0, \quad x \geqslant A, \quad c > 0, \tag{1}$$

where c, A and α denote the scale, location and shape parameters, respectively.

In this paper we compare analytical and ML methods for the estimation of these parameters by order statistics of a noncensored sample. The standard ML method for estimating the parameters of the Weibull model can have problems since the regularity conditions are not sometimes met, i.e., the ML estimate does not exist (Blischke (1974), Zanakis & Kyparisis (1986), Murthy et $al.$ (2004)). The probability of existence of ML estimator is studied further in more details. Besides, numerical implementation of the method requires complicated optimization software. To overcome the "non-regularity" and computational problems just mentioned, we will apply a modification suggested by Hall (1982).

Let m be the number of order statistics $\eta_{(1)} \leqslant \eta_{(2)} \leqslant \ldots \leqslant \eta_{(m)}$ from the sample of size N from the Weibull distribution. Then likelihood function of these order statistics can be expressed as follows (Hall (1982), Balakrishnan & Cohen (1990)):

$$\Phi_{\eta_{(1)}, \eta_{(2)}, \ldots, \eta_{(m)}}(x_1, x_2, \ldots, x_m) = \frac{N!}{(N-m)!} \cdot \left(1 - W(x_m)\right)^{N-m} \cdot \prod_{j=1}^{m} w(x_j),$$

where

$$w(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x}$$

is the density function.

Now we may write down likelihood function for the Weibull distribution which depends on three parameters c, A, and α. Generally ML equations are nonlinear in these three parameters and they can be solved only using nonlinear optimization techniques.

Let us propose a modification to simplify the estimation. From the Taylor expansion of cdf (1) we have:

$$W(x, \alpha, c, A) = c(x - A)^\alpha + o\big((x - A)^\alpha\big). \tag{2}$$

Note, order statistics are concentrated in the neighbourhood of the minimum point when sample size increases faster than number of order statistics. Thus the shape of the distribution in the neighborhood of the minimum point characterizes best the behavior of order statistics. Due to this reason it is enough to study only the first term of the distribution (2) (Hall (1982)). Hence consider instead of the asymptotic expansion (2) the main term in it for $x \in [A, A + \delta]$:

$$c(x - A)^\alpha. \tag{3}$$

Now the likelihood function is of the form:

$$L(\eta_{(0)}, \ldots, \eta_{(m)}; A, c; \alpha) = \frac{N!}{(N - m)!} \cdot (c \cdot \alpha)^m \big(1 - c \cdot (\eta_{(m)} - A)^\alpha\big)^{N-m}$$
$$\cdot \prod_{j=1}^{m} (\eta_{(j)} - A)^{\alpha-1}. \tag{4}$$

Derivatives of the log-likelihood function $\ln L = \ln\big(L\big(\eta_{(1)}, \ldots, \eta_{(m)}; A, c; \alpha\big)\big)$ are:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{m}{\alpha} - (N - m) \cdot \frac{c(\eta_{(m)} - A)^\alpha \cdot \ln(\eta_{(m)} - A)}{1 - c(\eta_{(m)} - A)^\alpha} + \sum_{j=1}^{m} \ln(\eta_{(j)} - A);$$

$$\frac{\partial \ln L}{\partial c} = \frac{m}{c} - (N - m) \cdot \frac{(\eta_{(m)} - A)^\alpha}{1 - c(\eta_{(m)} - A)^\alpha};$$

$$\frac{\partial \ln L}{\partial A} = (N - m) \cdot \frac{c \cdot \alpha(\eta_{(m)} - A)^{\alpha-1}}{1 - c(\eta_{(m)} - A)^\alpha} - (\alpha - 1) \cdot \sum_{j=1}^{m} \frac{1}{\eta_{(j)} - A}.$$

Setting the partial derivatives equal to 0, we get estimates $\hat{\alpha}$, $\hat{c}$:

$$\hat{\alpha} = \frac{m}{\sum_{j=1}^{m-1} \ln(1 + \beta_j(\hat{A}))}, \tag{5}$$

$$\hat{c} = \frac{m}{N(\eta_{(m)} - \hat{A})^\alpha}, \tag{6}$$

where $\beta_j(\hat{A}) = \frac{\eta_{(m)} - \eta_{(j)}}{\eta_{(j)} - \hat{A}}$ and $\hat{A}$, $\hat{A} < \eta_{(1)}$, is the solution of the equation:

$$\frac{1}{\sum_{j=1}^{m-1} \ln(1 + \beta_j(\hat{A}))} - \frac{1}{\sum_{j=1}^{m-1} \beta_j(\hat{A})} = \frac{1}{m}. \tag{7}$$

Denote

$$y = \frac{\eta_{(m)} - \eta_{(1)}}{\eta_{(1)} - \hat{A}}, \quad z_j = \frac{\eta_{(m)} - \eta_{(j)}}{\eta_{(m)} - \eta_{(1)} + y \cdot (\eta_{(j)} - \eta_{(1)})}, \quad j = 1, \ldots, m - 1. \tag{8}$$

Define the function $F(y)$ as

$$F(y) = \frac{1}{\sum_{j=1}^{m-1} \ln(1 + y \cdot z_j)} - \frac{1}{\sum_{j=1}^{m-1} y \cdot z_j} - \frac{1}{m}, \tag{9}$$

Note, due to absolute continuity of the distribution (1), the assumption $\eta_{(i)} \neq \eta_{(j)}$, $i \neq j$, $i, j = 1, \ldots, m$, holds with probability 1.

It is easy to show that the derivative of the ML function with respect to A, expressed through y by (5), (6), and function (9) are of opposite sign and equal to zero at the same points. Solution $y*$ of equation $F(y) = 0$ can be finite or not. If the solution exists, the ML estimator is obtained for $y*$ such that the function $F(y)$ changes sign at $F(y*)$.

In order to examine the existence of the solution we need to explore behavior of the function (9) and its first order derivative in neighborhood of zero and infinity. Differentiation of (9) gives us:

$$F'(y) = \frac{1 + (\eta_{(m)} - \eta_{(1)}) \cdot \sum_{j=2}^{m-1} \frac{(z_j)^2}{\eta_{(m)} - \eta_{(j)}}}{y^2 \cdot \left(1 + \sum_{j=2}^{m-1} z_j\right)^2}$$
$$- \frac{\frac{1}{1+y} \cdot \left(1 + \sum_{j=2}^{m-1} z_j\right)}{\left(\ln(1 + y) + \sum_{j=2}^{m-1} \ln(1 + y \cdot z_j)\right)^2}. \tag{10}$$

Straightforwardly

$$\lim_{y \to \infty} F(y) = -\frac{1}{m} \tag{11}$$

and

$$\lim_{y \to \infty} F'(y) = 0. \tag{12}$$

Taking into account corresponding limits we obtain

$$\lim_{y \to 0} F(y) = \frac{1}{2} \cdot \frac{\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^2}{\left(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})\right)^2} - \frac{1}{m} \tag{13}$$

and

$$\lim_{y \to 0} F'(y) = \frac{\frac{2}{3} \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)}) \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^3}{\left(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})\right)^3 \cdot (\eta_{(m)} - \eta_{(1)})}$$
$$- \frac{\frac{3}{4} \cdot \left(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^2\right)^2}{\left(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})\right)^3 \cdot (\eta_{(m)} - \eta_{(1)})}. \tag{14}$$

Denote the maximum point $y_{\max} = \arg\max_{0 \leqslant y < \infty} F(y)$.

Proposition 1. *Let $\eta_{(1)} < \eta_{(2)} < \ldots < \eta_{(m)}$. If $F(y_{\max}) > 0$, $0 < y_{\max} < \infty$, and*

$$\frac{1}{2} \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^2 \cdot \Big(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)}) \Big)^{-2} \leqslant \frac{1}{m}, \tag{15}$$

then equation $F(y) = 0$ has a solution, $y > 0$, and the function $F(y)$ changes its sign at $F(y*)$. If $F(y_{\max}) > 0$, $0 < y_{\max} < \infty$, and*

$$\frac{8}{9} \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)}) \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^3 - \Big(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^2 \Big)^2 \leqslant 0, \tag{16}$$

then the function $F(y)$ has the minimum point $0 < y* < \infty$.

Proposition 1 follows from formulas (11)–(14).

We established, in fact, that function $F(y)$ has one maximum point, $0 < y_{\max} < \infty$. This together with Proposition 1 helps us to localize the interval of the solution of the equation $F(y) = 0$. Extremal points as well as the solution of equation (7) can be found by dichotomy or Newton methods using expressions (9) of function $F(y)$ and its derivative (10) and Proposition 1. If we use Newton method with initial value $y_0 = 0$, and apply (9), (13), (14), then we obtain

$$y_1 = \frac{4}{3} (\eta_{(m)} - \eta_{(1)}) \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^3$$

$$\cdot \frac{\frac{1}{2} \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^2 \cdot \big(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)}) \big)^{-2} - \frac{1}{m}}{\big(\sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^2 \big)^2 - \frac{8}{9} \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)}) \cdot \sum_{j=1}^{m-1} (\eta_{(m)} - \eta_{(j)})^3} \tag{17}$$

which is used as an initial value for numerical solving of (7).

After substituting $y*$ into (8), parameters $\hat{\alpha}$, $\hat{c}$ and $\hat{A}$ can be expressed as

$$\hat{A} = \eta_{(1)} - \frac{\eta_{(m)} - \eta_{(1)}}{y*}, \tag{18}$$

$$\hat{\alpha} = \frac{m}{\sum_{j=1}^{m-1} \ln \big(1 + \frac{y*(\eta_{(m)} - \eta_{(j)})}{(\eta_{(m)} - \eta_{(1)}) + y*(\eta_{(j)} - \eta_{(1)})} \big)}, \tag{19}$$

$$\hat{c} = \frac{m}{N \cdot (\eta_{(m)} - \eta_{(1)})^{\hat{\alpha}} \cdot \big(1 + \frac{1}{y*} \big)^{\hat{\alpha}}}. \tag{20}$$

Equation (7) has no solution if maximum of $F(y)$ is negative and $0 \leqslant y* < \infty$. In the last case we propose to take $\hat{A} = \eta_{(1)}$, $y* = \infty$ and $\alpha = 0$.

If condition (15) does not hold then equation (7) has a solution at the point in which the likelihood function obtains minimum. Then likelihood

function has no maximum in the interval $0 < y < \infty$. In this case we propose to take $\hat{A} = -\infty$, $y* = 0$ ir $\alpha = \infty$.

Computer modeling results for the ML approach are given in Section 4.

3. Analytical estimation

A simple analytical estimate of the shape parameter was proposed by Haan (1981):

$$\hat{\alpha} = \ln\left(\frac{m}{q}\right) \cdot \left(\ln\left(\frac{\eta_{(m)} - \eta_{(1)}}{\eta_{(q)} - \eta_{(1)}}\right)\right)^{-1}, \tag{21}$$

where $N \to \infty$, $\frac{m^2}{N} \to 0$, $m \to \infty$, $\frac{q}{m} \to \tau$, $0 < \tau < 1$. Haan (1981) and Zhigljavsky (1985) recommend to take $\tau = 0.2$.

Computer simulation shows that the estimator (21) is biased and this bias increases when α increases. Better results are obtained when $\eta_{(1)}$ is changed by a linear estimate of the location parameter (Balakrishnan & Cohen (1990)):

$$A_{m,N} = \eta_{(1)} - c(\alpha, m) \cdot (\eta_{(m)} - \eta_1), \tag{22}$$

where $c(\alpha, m) = \frac{m! \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)}{\Gamma\left(1 + m + \frac{1}{\alpha}\right) - m! \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)}$.

After simplification we obtain

$$c(\alpha, m) = \left(\prod_{j=1}^{m}\left(1 + \frac{1}{j \cdot \alpha}\right) - 1\right)^{-1}.$$

We can assume again $\eta_{(i)} \neq \eta_{(j)}$, $i \neq j$, $i, j = 1, \ldots, m$, with probability 1 due to absolute continuity of measure (1).

Thus better estimate is given by the solution α^* of the equation

$$\begin{aligned}
\alpha^* &= \ln\left(\frac{m}{q}\right) \cdot \left(\frac{\ln\left(\frac{m}{q}\right)}{\hat{\alpha}} + \ln\left(\frac{1 + c(\alpha^*, m)}{1 + c(\alpha^*, q)}\right)\right)^{-1} \\
&= \ln\left(\frac{m}{q}\right) \cdot \left(\ln\left(\frac{\eta_{(m)} - \eta_{(1)}}{\eta_{(q)} - \eta_{(1)}}\right) + \ln\left(\frac{1 + c(\alpha^*, m)}{1 + c(\alpha^*, q)}\right)\right)^{-1},
\end{aligned}$$

where $\hat{\alpha}$ is taken from (21).

To study this equation we denote

$$f(z, \alpha) = \frac{\ln\left(\frac{m}{q}\right)}{\ln(z) + \omega\left(\frac{1}{\alpha}\right)}, \tag{23}$$

where

$$z = \frac{\eta_{(m)} - \eta_{(1)}}{\eta_{(q)} - \eta_{(1)}},$$

$$\omega(x) = \ln\left(1 - \left(\prod_{j=1}^{q}\left(1 + \frac{x}{j}\right)\right)^{-1}\right) - \ln\left(1 - \left(\prod_{j=1}^{m}\left(1 + \frac{x}{j}\right)\right)^{-1}\right),$$

$$x = \frac{1}{\alpha}.$$

Thus, the improved estimate must satisfy the equality:

$$\alpha^* = f(z, \alpha^*). \qquad (24)$$

The solution of (24) is obtained by a simple iteration method:

$$\alpha_{t+1} = f(z, \alpha_t), \qquad (25)$$

where α_0 is the initial value.

Lemma 1. *Equation* (24) *has solution if* $z > \left(\sum_{j=1}^{m}\frac{1}{j}\right) \cdot \left(\sum_{j=1}^{q}\frac{1}{j}\right)^{-1}$.

The proof is given in Appendix.

Lemma 2. *The derivative of* $f(z, \alpha)$ *in* (23) *is bounded at* $\alpha = \alpha^*$,

$$f'(z, \alpha^*) \leqslant \frac{1}{2} \cdot \frac{\sum_{j=q+1}^{m}\frac{1}{j} - \sum_{j=1}^{m}\frac{1}{j^2} \cdot \left(\sum_{j=1}^{m}\frac{1}{j}\right)^{-1}}{\ln(m) - \ln(q)}$$

$$+ \frac{\sum_{j=1}^{q}\frac{1}{j^2} \cdot \left(\sum_{j=1}^{q}\frac{1}{j}\right)^{-1}}{\ln(m) - \ln(q)} \leqslant \frac{1}{2}. \qquad (26)$$

The proof is given in Appendix.

Theorem 1. *Assume* $z > \sum_{j=1}^{m} j^{-1} \cdot \left(\sum_{j=1}^{q} j^{-1}\right)^{-1}$. *Then for any* $\frac{1}{2} < \nu < 1$ *there exists* $\varepsilon > 0$ *such that the sequence* (25) *converges to the solution of* (24) *with linear rate:* $|\alpha_t - \alpha^*| = O(v^t)$ *if* $|\alpha_0 - \alpha *| \leqslant \varepsilon$.

The proof is given in Appendix.

Initial value for the equation (24) is recommended to take of the form:

$$\alpha_0 = \ln\left(\frac{m}{q}\right) \cdot \left(\ln(z) + \ln\left(\sum_{j=1}^{q} j^{-1}\right) - \ln\left(\sum_{j=1}^{m} j^{-1}\right)\right)^{-1}. \qquad (27)$$

4. Computer simulation

We investigate the methods developed by simulation. Samples from Weibull distribution ($c = 1$, $A = 0$) have been simulated and parameters have been estimated for each sample. Number of repetitions $M = 100$.

Fig. 1 presents dependence of the frequency of nonexistence of ML estimate from number of iterations and number of order statistics ($k = 10, 20, 30$).
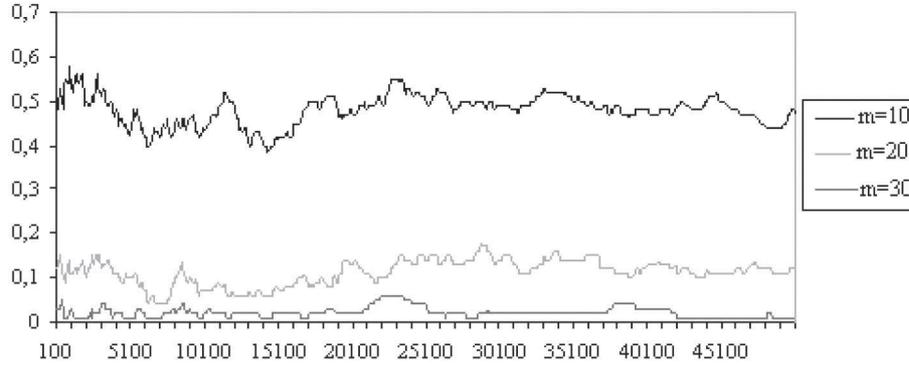
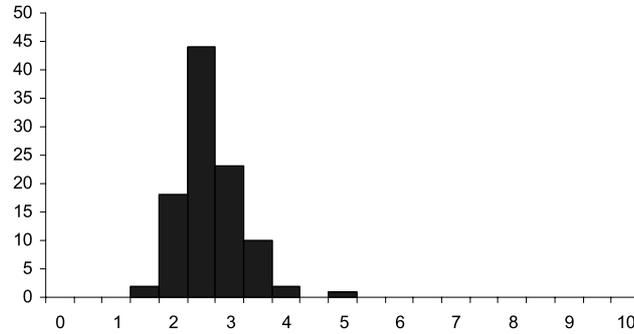FIGURE 1. Frequency of nonexistence of ML estimate ($\alpha = 5.0$).

FIGURE 2. Histogram of ML estimators of α ($\alpha = 2.5$, 100 trials).

Thus, the frequency of nonexistence of ML estimate decreases to zero when k increases. In Figs. 2, 3 and 4 histograms of the ML estimator and IA estimator are depicted. From these figures we can see that the variance of estimators decreases when number of order statistics increases. Averages of estimates of α, A, c are also presented in the Table 1 for different sample size N and different numbers of order statistics m and q.

It follows from simulation that ML and IA estimators enable us to evaluate three parameters of Weibull distribution with tolerable accuracy, when number of order statistics increases. If number of order statistics is taken small ($m < 40, \alpha \leqslant 5$), ML estimator may not exist. In order to obtain reliable estimates of Weibull distribution when shape parameter α increases, sample size and number of order statistics should increase too. Say, when $\alpha = 2.5$, it is enough to take only $m = 100$, $q = 20$. When $\alpha = 5.0$, more order statistics are needed for the three-parameter Weibull distribution estimation, e.g. $m = 500, q = 100$. The average variance of ML estimates is less than variance of analytical estimates that is in correspondence with theoretical properties of ML estimators.

TABLE 1. Monte-Carlo averages of estimates of α (100 trials) (MLE – maximum likelihood estimator, AE – analytical estimator, IAE – improved analytical estimator)

| $m = 100$, $q = 20$, $\alpha = 2.5$, $c = 1$, $A = 0$				
	α	Variance of α	A	c
$N = 1000$				
MLE	2.4254	0.2776	0.0144	1.032
AE	2.0302	0.0904	0.0195	1.2072
IAE	2.6107	0.4560	$-$ 0.0090	1.0346
$N = 10000$				
MLE	2.4342	0.3747	0.0038	1.1917
AE	2.0462	0.1507	0.0063	1.6540
IAE	2.7188	1.1151	-0.0077	1.1673
$N = 20000$				
MLE	2.3774	0.4143	0.0046	1.4267
AE	1.9687	0.1063	0.0072	1.9706
IAE	2.4981	0.5099	$-$ 0.0007	1.3352
$N = 50000$				
MLE	2.3979	0.5743	0.0030	1.3964
AE	1.9538	0.0961	0.0052	2.2493
IAE	2.4581	0.4129	0.0001	1.4110
$m = 500$, $q = 100$, $\alpha = 5.0$, $c = 1$, $A = 0$				
	α	Variance of α	A	c
$N = 1000$				
MLE	4.9313	0.7662	0.0129	0.9881
AE	3.4307	0.1428	0.1157	0.9241
IAE	5.1715	1.7762	-0.0272	1.0286
$N = 10000$				
MLE	5.0345	0.9592	-0.0035	1.0141
AE	3.4632	0.1715	0.0676	1.1513
IAE	5.2958	1.8477	-0.0284	1.0373
$N = 20000$				
MLE	5.0011	1.2829	0.00006	1.0190
AE	3.4105	0.1493	0.0667	1.2233
IAE	5.1052	1.5999	-0.0089	1.0278
$N = 50000$				
MLE	5.0931	1.0688	-0.0036	1.0156
AE	3.4121	0.1515	0.0564	1.3350
IAE	5.1278	1.9152	-0.0074	1.0335
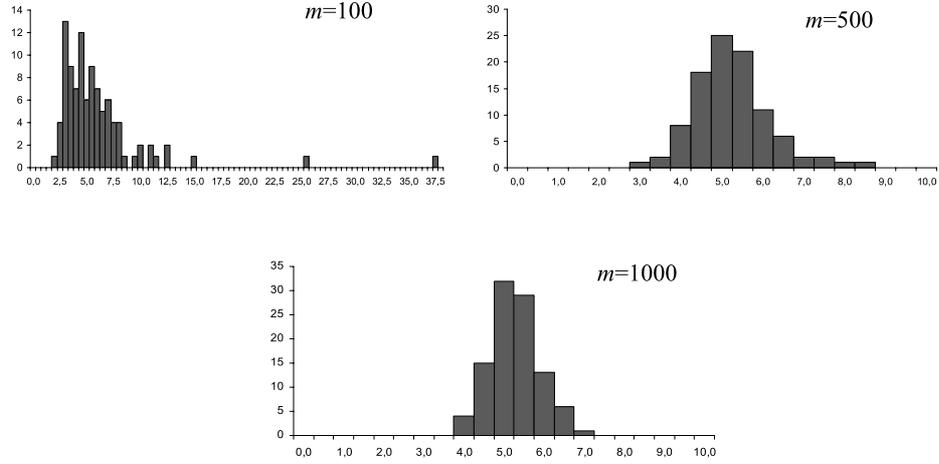
FIGURE 3. Histogram of ML estimators of parameter α, depending on number of order statistics ($\alpha = 5$, 100 trials).
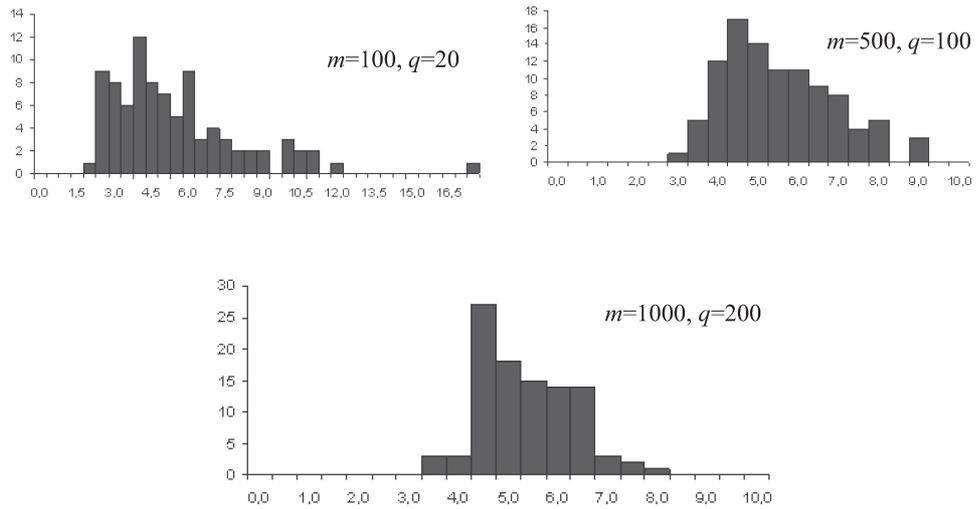

FIGURE 4. Histogram of IA estimators of parameter α, depending on number of order statistics ($\alpha = 5$, 100 trials).

5. Conclusions

In this paper we present and compare the ML method and the IA estimation method to estimate parameters of the Weibull distribution, namely, location, scale and shape parameters, using order statistics of a noncensored sample. In general, the ML method fails for three-parameter Weibull distribution. We establish conditions to localize extremes of ML function

that enables us to avoid problems related with ML estimation failure and to create a simple estimation procedure by solving one-dimensional equation. IA estimation also has been developed by solving equation in one variable. Computer simulation confirmed that ML and IA methods allow us to estimate Weibull distribution parameters for practical purposes with acceptable accuracy.

We recommend ML and IA methods for three-parameter Weibull distribution estimation. The practical recommendation is to increase numbers of order statistics m and q to obtain reliable estimates. When estimated α increases, the number of order statistics should increase too.

Appendix

Proof of Lemma 1. The derivative of the function $\omega(x)$ is

$$
\begin{aligned}
\omega'(x) = \sum_{j=1}^{q} \left(\frac{1}{x+j} \right) \cdot \left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1 \right)^{-1} \\
- \sum_{j=1}^{m} \left(\frac{1}{x+j} \right) \cdot \left(\prod_{j=1}^{m} \left(1 + \frac{x}{j} \right) - 1 \right)^{-1}.
\end{aligned}
\tag{1A}
$$

Now prove that $\sum_{j=1}^{q} \left(\frac{1}{x+j} \right) \cdot \left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1 \right)^{-1}$ increases when q increases.

Upon differentiating we have:

$$
S_q(x) = \left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) \right)' = \sum_{j=1}^{q} \left(\frac{1}{x+j} \right) \cdot \prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) \geqslant 0,
$$

$$
S_q'(x) = \prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) \left(\left(\sum_{j=1}^{q} \frac{1}{x+j} \right)^2 - \sum_{j=1}^{q} \left(\frac{1}{x+j} \right)^2 \right) \geqslant 0. \tag{2A}
$$

Further,

$$
\begin{aligned}
&\frac{\sum_{j=1}^{q+1} \left(\frac{1}{x+j} \right)}{\prod_{j=1}^{q+1} \left(1 + \frac{x}{j} \right) - 1} \\
&= \frac{\left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1 \right) \cdot \frac{1}{x+q+1} - \frac{x}{q+1} \cdot \sum_{j=1}^{q} \left(\frac{1}{x+j} \right) \cdot \prod_{j=1}^{q} \left(1 + \frac{x}{j} \right)}{\left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) \cdot \left(1 + \frac{x}{q+1} \right) - 1 \right) \left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1 \right)} \\
&\quad + \frac{\sum_{j=1}^{q} \left(\frac{1}{x+j} \right)}{\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1} \\
&\leqslant \frac{\sum_{j=1}^{q} \left(\frac{1}{x+j} \right)}{\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1}
\end{aligned}
\tag{3A}
$$

by virtue of the Lagrange formula and

$$\left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1 \right) = x \cdot S_q(\xi) \leqslant x \cdot S_q(x)$$

$$= x \cdot \sum_{j=1}^{q} \left(\frac{1}{x+j} \right) \cdot \prod_{j=1}^{q} \left(1 + \frac{x}{j} \right), \quad 0 \leqslant \xi \leqslant x.$$

Thus $\omega'(x) \geqslant 0$ because $q < m$. It follows that function $\omega(x)$ increases monotonically from the limit

$$\lim_{x \to 0} \omega(x) = \ln \left(\sum_{j=1}^{q} \frac{1}{j} \right) - \ln \left(\sum_{j=1}^{m} \frac{1}{j} \right), \tag{4A}$$

obtained by L'Hôpital's Rule, to $\lim_{x \to \infty} \omega(x) = 0$. The function $f(z, \alpha)$ monotonically increases from $\lim_{\alpha \to 0} f(z, \alpha) = \frac{\ln \left(\frac{m}{q} \right)}{\ln z}$ to $\lim_{\alpha \to \infty} f(z, \alpha) = \ln \left(\frac{m}{q} \right) - \left(\ln z + \ln \left(\sum_{j=1}^{q} \frac{1}{j} \right) - \ln \left(\sum_{j=1}^{m} \frac{1}{j} \right) \right)$, when α increases.

It follows from this that the curve $f(z, \alpha)$ has an intersection with line up α, if $z > \sum_{j=1}^{m} \frac{1}{j} \cdot \left(\sum_{j=1}^{q} \frac{1}{j} \right)^{-1}$. □

Proof of Lemma 2. We have

$$f'(z, \alpha) = \ln \left(\frac{m}{q} \right) \cdot \frac{1}{\alpha^2} \cdot \left(\ln(z) + \omega \left(\frac{1}{\alpha} \right) \right)^{-2} \cdot \omega' \left(\frac{1}{\alpha} \right).$$

Let us make sure that

$$\lim_{x \to 0} \omega'(x) = \lim_{x \to 0} \left(\sum_{j=1}^{q} \left(\frac{1}{x+j} \right) \cdot \left(\prod_{j=1}^{q} \left(1 + \frac{x}{j} \right) - 1 \right)^{-1} \right.$$

$$\left. - \sum_{j=1}^{m} \left(\frac{1}{x+j} \right) \cdot \left(\prod_{j=1}^{m} \left(1 + \frac{x}{j} \right) - 1 \right)^{-1} \right) \tag{5A}$$

$$= \frac{1}{2} \cdot \left(\sum_{j=1+q}^{m} \frac{1}{j} + \sum_{j=1}^{q} \frac{1}{j^2} \cdot \left(\sum_{j=1}^{q} \frac{1}{j} \right)^{-1} - \sum_{j=1}^{m} \frac{1}{j^2} \cdot \left(\sum_{j=1}^{m} \frac{1}{j} \right)^{-1} \right).$$

Now prove that $\omega'(x)$ is a monotonically decreasing function. This follows from the inequality:

$$\omega''(x) = W(m, x) - W(q, x) \leqslant 0,$$

where

$$W(i,x) = \frac{\sum_{j=1}^{i} \frac{1}{(x+j)^2} \cdot \left(\prod_{j=1}^{i} \left(1 + \frac{x}{j}\right) - 1 \right)}{\left(\prod_{j=1}^{i} \left(1 + \frac{x}{j}\right) - 1 \right)^2}$$
$$+ \frac{\prod_{j=1}^{i} \left(1 + \frac{x}{j}\right) \cdot \left(\sum_{j=1}^{i} \frac{1}{x+j} \right)^2}{\left(\prod_{j=1}^{i} \left(1 + \frac{x}{j}\right) - 1 \right)^2}, \quad i = 1,2,3,\ldots.$$

However it is enough to prove that $W(i,x)$ is decreasing in i, namely $W(i+1,x) < W(i,x)$. This follows from the inequality

$$\sum_{j=1}^{i} \frac{1}{x+j} \leqslant x \cdot \left(\left(\sum_{j=1}^{i} \frac{1}{x+j} \right)^2 \cdot \left(\prod_{j=1}^{i} \left(1 + \frac{x}{j}\right) - 1 \right)^{-1} \right. \tag{6A}$$
$$\left. + \frac{1}{2} \cdot \left(\sum_{j=1}^{i} \frac{1}{(x+j)^2} + \left(\sum_{j=1}^{i} \frac{1}{x+j} \right)^2 \right) \right)$$

after simple but rather cumbersome manipulation.

Since $\omega'(x)$ decreases monotonically from

$$\lim_{x \to 0} \omega'(x) = \frac{1}{2} \cdot \left(\sum_{j=q+1}^{m} \frac{1}{j} + \sum_{j=1}^{m} \frac{1}{j^2} \cdot \left(\sum_{j=1}^{m} \frac{1}{j} \right)^{-1} \right.$$
$$\left. - \sum_{j=1}^{q} \frac{1}{j^2} \cdot \left(\sum_{j=1}^{q} \frac{1}{j} \right)^{-1} \right) \tag{7A}$$
$$\leqslant \frac{1}{2} \cdot \ln \left(\frac{m}{q} \right)$$

to $\lim_{x \to \infty} \omega'(x) = 0$, it follows that

$$f'(z,\alpha) \leqslant \frac{1}{2} \cdot \ln^2 \left(\frac{m}{q} \right) \cdot \frac{1}{\alpha^2} \cdot \left(\ln(z) + \omega\left(\frac{1}{\alpha}\right) \right)^{-2}. \tag{8A}$$

This and equation (24) imply the Lemma. □

Proof of Theorem 1. The solution of equation (24) $\alpha*$ exists due to assumptions and Lemma 1. By virtue of Lemma 2 we have that for any $\frac{1}{2} < \nu < 1$ there exists $\varepsilon > 0$ such, that $|f'(\alpha)| \leqslant \nu$, if $|\alpha - \alpha *| \leqslant \varepsilon$. Thus, according to Lagrange theorem and (24), (25) we have:

$$|\alpha_t - \alpha^*| \leqslant \left| f'\big(z, \alpha^* + \tau(\alpha_{t-1} - \alpha^*)\big) \right| \cdot |\alpha_{t-1} - \alpha^*|$$
$$\leqslant q \cdot |\alpha_{t-1} - \alpha^*| \leqslant q^t \cdot |\alpha_0 - \alpha^*|,$$

because $|f'(z, \alpha^* + \tau(\alpha_{t-1} - \alpha^*))| \leqslant q$, $t = 1, 2, \ldots$. □

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