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# On the osculator Lorentz spheres of timelike parallel $p_i$-equidistant ruled surfaces in the Minkowski 3-space $R_1^3$

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Abstract. In this paper, we present radii and curvature axes of osculator Lorentz spheres of the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve in the Minkowski 3-space $R_1^3$ and give the arc lengths of indicatrix curves of timelike base curves of these surfaces.

## 1. Introduction

I. E. Valeontis [3] defined parallel $p$-equidistant ruled surfaces in $E^3$ and gave some results related to the striction curves of these surfaces.

M. Masal and N. Kuruoğlu [2] studied arc lengths, curvature radii, curvature axes, spherical involute and areas of real closed spherical indicatrix curves of base curves of parallel $p$-equidistant ruled surfaces in $E^3$. And also, M. Masal and N. Kuruoğlu [1] defined timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve in the Minkowski 3-space and have studied dralls, the shape operators, Gaussian curvatures, mean curvatures, shape tensor, $q^{th}$ fundamental forms of these surfaces.

This paper is organized as follows. In Section 3 we have found radii and curvature axes of osculator Lorentz spheres of the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve in the Minkowski 3-space. And later in Section 4 we have given arc lengths of indicatrix curves of these surfaces.

## 2. Preliminaries

Let $\alpha : I \to R_1^3$ , $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a differentiable unit speed timelike curve in the Minkowski 3-space, where $I$ is an open interval

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in R containing the origin. Let $V_1$ be the tangent vector field of $\alpha$ , $D$ be the Levi–Civita connection on $R_1^3$ and $D_{V_1}V_1$ be a spacelike vector. If $V_1$ moves along $\alpha$, then a timelike ruled surface $M$ which is given by the parametrization

$$\varphi(t,v)=\alpha(t)+vV_1(t)$$

is obtained. Let $\{V_1,V_2,V_3\}$ be an orthonormal frame field along $\alpha$ in $R_1^3$, where $V_2$ and $V_3$ are spacelike vectors. If $k_1$ and $k_2$ are the natural curvature and torsion of $\alpha(t)$ , respectively, then for $\alpha$ the Frenet formulas are given by (see [4])

$$V_1^{'}=k_1V_2,\ V_2^{'}=k_1V_1-k_2V_3,\ V_3^{'}=k_2V_2. \tag{2.1}$$

Using $V_1=\alpha^{'}$ and $V_2=\dfrac{\alpha^{''}}{\|\alpha''\|}$ , we have $k_1=\left\|\alpha^{''}\right\|>0$, where " $'$ " means derivative with respect to time $t$ (see [1]).

**Definition 2.1** ([1])**.** The planes corresponding to subspaces $Sp\{V_1,V_2\}$, $Sp\{V_2,V_3\}$ and $Sp\{V_3,V_1\}$ along striction curves of timelike ruled surface $M$ are called *asymptotic plane, polar plane and central plane*, respectively.

Let us suppose that $\alpha^*=\alpha^*(t^*)$ is another differentiable timelike curve with arc-length and $\{V_1^*,V_2^*,V_3^*\}$ is the Frenet frame of this curve in three dimensional Minkowski space $R_1^3$. Hence, we define timelike ruled surface $M^*$ parametrically as follows:

$$\varphi^*(t^*,v^*)=\alpha^*(t^*)+v^*V_1^*(t^*),\ (t^*,v^*)\in I\times \mathrm{R}.$$

**Definition 2.2** ([1])**.** Let $M$ and $M^*$ be two timelike ruled surfaces and let $p_1$, $p_2$ and $p_3$ be the distances between the polar planes, central planes and asymptotic planes, respectively. If the directions of $M$ and $M^*$ are parallel and the distances $p_i$, $1\leqslant i\leqslant 3$, of $M$ and $M^*$ are constant, then the pair of ruled surfaces $M$ and $M^*$ is called *timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve*. If specifically $p_i$=0, then this pair of ruled surfaces is named as *timelike parallel $p_i$-equivalent ruled surfaces with a timelike base curve,* where the base curves of ruled surfaces $M$ and $M^*$ are of class $C^2$ .

Therefore the pair of timelike parallel $p_i$-equidistant ruled surfaces are defined parametrically as

$$M:\varphi(t,v)=\alpha(t)+vV_1(t),\ \ (t,v)\in I\times \mathrm{R},$$
$$M^*:\varphi^*(t^*,v^*)=\alpha^*(t^*)+v^*V_1(t^*),\ \ (t^*,v^*)\in I\times \mathrm{R},$$

where $t$ and $t^*$ are the arc parameters of curves $\alpha$ and $\alpha^*$, respectively. Let the striction curve of $M$ be the base curve of $M$ and let $\alpha^*$ be a base curve of $M^*$. In this case we can write

$$\alpha^* = \alpha + p_1 V_1 + p_2 V_2 + p_3 V_3,$$

where $p_1(t)$, $p_2(t)$ and $p_3(t)$ are of class $C^2$ (see [1]).

**Theorem 2.1** (see [1], Theorem 3.2 and Corollary 3.1)**.** *Let $M$ and $M^*$ be timelike parallel $p_i$-equidistant ruled surfaces.*

i) *The Frenet vectors of timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$ at $\alpha(t)$ and $\alpha^*(t^*)$ points are equivalent for $\frac{dt^*}{dt} > 0$.*

ii) *There is a relation between the natural curvatures $k_1(t)$ and $k_1^*(t^*)$ of base curves and the torsions $k_2(t)$ and $k_2^*(t^*)$ of $M$ and $M^*$ as follows:*

$$k_i^* = k_i \frac{dt}{dt^*}, \quad 1 \le i \le 2\,.$$

## 3. On the osculator Lorentz spheres of timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve

In this section, we will investigate radii and curvature axes of osculator Lorentz spheres of timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$ with a timelike base curve.

We compute the locus of center of the osculator sphere $S_1^2$ which is the fourth order contact with the base curve $\alpha$ of $M$. Let us consider the function $f$ defined by

$$\begin{aligned} f : & I \to \mathrm{R} \\ & t \to f(t) = \langle \alpha(t) - a, \alpha(t) - a \rangle - R^2, \end{aligned}$$

where $a$ and $R$ are the center and radius of $S_1^2$, respectively. Since $S_1^2$ is the fourth order contact with the curve $\alpha$, we can write

$$f(t) = f'(t) = f''(t) = f'''(t) = 0.$$

From $f(t) = 0$ we have

$$\langle \alpha(t) - a, \alpha(t) - a \rangle = R^2, \tag{3.1}$$

from $f'(t) = 0$ and $V_1(t) = \alpha'(t)$ we get

$$\langle V_1(t), \alpha(t) - a \rangle = 0, \tag{3.2}$$

from $f''(t) = 0$ and equation (2.1) we have

$$\langle V_2(t), \alpha(t) - a \rangle = \frac{1}{k_1(t)}\,. \tag{3.3}$$

Furthermore, for the vector $\alpha(t) - a$, we can write

$$\alpha(t) - a = m_1(t)V_1(t) + m_2(t)V_2(t) + m_3(t)V_3(t), \;\; m_i(t) \in \mathrm{R}, \tag{3.4}$$

where $\{V_1, V_2, V_3\}$ is the orthonormal frame field of $M$. From here we have

$$\begin{aligned} &\langle \alpha(t) - a, V_1(t)\rangle = -m_1(t), \ \langle \alpha(t) - a, V_2(t)\rangle = m_2(t), \\ &\langle \alpha(t) - a, V_3(t)\rangle = m_3(t). \end{aligned} \tag{3.5}$$

From equations (3.2) and (3.3) we get

$$m_1(t) = 0, \ \ m_2(t) = \frac{1}{k_1(t)} . \tag{3.6}$$

From equations (3.1), (3.4) and (3.6) we obtain

$$R = \sqrt{m_2^2 + m_3^2} \tag{3.7}$$

or

$$m_3 = \pm\sqrt{R^2 - m_2^2} . \tag{3.8}$$

Using (3.4), for the center $a$ of $S_1^2$, we can write

$$a = \alpha(t) - \frac{1}{k_1} V_2 - \lambda V_{3,} \ \ \lambda = m_3(t) \in \mathrm{R} .$$

From $f^{'''}(t) = 0$ we have

$$k_1^{'} \langle V_2(t), \alpha(t) - a\rangle + k_1 \langle V_2^{'}(t), \alpha(t) - a\rangle + k_1 \langle V_2(t), V_1(t)\rangle = 0.$$

So, from (2.1), (3.5) and (3.6) we obtain

$$m_3 = \frac{-k_1^{'}}{k_1^2 k_2} = \frac{m_2^{'}}{k_2} . \tag{3.9}$$

Similarly, we compute the locus of center of osculator sphere $S_1^{*2}$ which is the fourth order contact with the timelike base curve $\alpha^*$ of $M^*$. Let us consider the function $f^*$ defined as

$$\begin{aligned} &f^* : I \to \mathrm{R} \\ &\qquad t^* \to f^*(t^*) = \langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^*\rangle - R^{*2}, \end{aligned}$$

where $a^*$ and $R^*$ are the center and the radius of $S_1^{*2}$. In addition, for the vector $\alpha^*(t^*) - a^*$, we can write

$$\alpha^*(t^*) - a^* = m_1^*(t^*) V_1^*(t^*) + m_2^*(t^*) V_2^*(t^*) + m_3^*(t^*) V_3^*(t^*), \ m_i^*(t^*) \in \mathrm{R},$$

where $\{V_1^*, V_2^*, V_3^*\}$ is the orthonormal frame field of $M^*$.

In a similar way $m_1^*(t^*), m_2^*(t^*), m_3^*(t^*), R^*$ and $a^*$ of $S_1^{*2}$ are found to be

$$m_1^*(t^*) = 0, \ m_2^*(t^*) = \frac{1}{k_1^*(t^*)}, \ m_3^*(t^*) = \frac{m_2^{*'}}{k_2^*(t^*)}, \ R^* = \sqrt{m_2^{*2} + m_3^{*2}} \tag{3.10}$$

and

$$a^* = \alpha^*(t^*) - \frac{1}{k_1^*} V_2^* - \lambda^* V_3^*, \ \ \lambda^* = m_3^*(t^*) \in \mathrm{R} .$$

Now, we can compute the relations between the radii of osculator Lorentz spheres and curvature axes of the base curves of $M$ and $M^*$. From Theorem 2.1 ii), equations (3.6) and (3.10), we have

$$m_1^*(t^*) = m_1(t) = 0,\ m_2^*(t^*) = \frac{dt^*}{dt} m_2(t). \tag{3.11}$$

If $\dfrac{dt}{dt^*}$ is constant, then from Theorem 2.1 ii) we obtain

$$k_1^{*\prime} = k_1{}' \left( \frac{dt}{dt^*} \right)^2. \tag{3.12}$$

Hence, using (3.10), (3.12), (3.9) and Theorem 2.1 ii), we find

$$m_3^* = \frac{dt^*}{dt} m_3. \tag{3.13}$$

Combining (3.11), (3.13) and Theorem 2.1 i), we get

$$\alpha^* - a^* = \frac{dt^*}{dt} (\alpha - a)\,.$$

Similarly, combining (3.7), (3.8), (3.11) and (3.13), we have

$$R^{*2} = \left( \frac{dt^*}{dt} \right)^2 R^2$$

or

$$R^* = \left| \frac{dt^*}{dt} \right| R\,.$$

So, we have proved the following theorem.

**Theorem 3.1.** *Let $M$ and $M^*$ be the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve.*

i) *If $q_\alpha$ and $q_{\alpha^*}$ are the curvature axes (the locus of center of osculator Lorentz spheres) of the base curves $\alpha$ and $\alpha^*$ of $M$ and $M^*$, then we have*

$$q_{\alpha^*} - \alpha^* = \frac{dt^*}{dt} (q_\alpha - \alpha)\,.$$

ii) *If $R$ and $R^*$ are the radiuses of osculator Lorentz spheres of base curves $\alpha$ and $\alpha^*$ of $M$ and $M^*$, then we have*

$$R^* = \left| \frac{dt^*}{dt} \right| R\,.$$

## 4. Arc lengths of indicatrix curves of the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve

In this section, we will investigate arc lengths of indicatrix curves of timelike base curves of the timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$ with timelike base curve.

Since $V_2$ and $V_3$ are spacelike vectors, the curves $(V_2)$ and $(V_3)$ generated by the spacelike vectors $V_2$ and $V_3$ on the pseudosphere $S_1^2$ are called the pseudo-spherical indicatrix curves. The curve $(V_1)$ generated by the vector $V_1$ on the pseudohyperbolic space $H_1^2$ is called indicatrix curve. Let $S_{V_i}$ and $S_{V_i^*}$ denote the arc lengths of indicatrix curves $(V_i)$ and $\left(V_i^*\right)$ generated by the vector fields $V_i$ and $V_i^*$, respectively. So we can write

$$S_{V_i} = \int \left\| V_i' \right\| dt \quad \text{and} \quad S_{V_i^*} = \int \left\| V_i^{*'} \right\| dt^* , \ \ 1 \leq i \leq 3 .$$

Using the Frenet formulas and Theorem 2.1 ii), we get

$$S_{V_{\mathrm{i}}^*} = \int k_1 \, dt = S_{V_i} , \ S_{V_2^*} = \int \sqrt{\left| k_2^2 - k_1^2 \right|} \, dt = S_{V_2} , \ S_{V_3^*} = \int |k_2| \, dt = S_{V_3} ,$$

where $\dfrac{dt}{dt^*} > 0$.

Similarly, for the arc lengths $S_\alpha$ and $S_{\alpha^*}$ of the indicatrix curves $(\alpha)$ and $\left(\alpha^*\right)$ generated by the timelike curves $\alpha$ and $\alpha^*$ on the pseudosphere $S_1^2$, respectively, we can write

$$S_\alpha = \int \left\| \alpha' \right\| dt = \int dt \quad \text{and} \quad S_{\alpha^*} = \int \left\| \alpha^{*'} \right\| dt^* = \int dt^* .$$

If $\dfrac{k_1}{k_1^*}$ is constant, then using Theorem 2.1 ii), we obtain

$$S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha .$$

Thus we have proved the following theorems.

**Theorem 4.1.** *If $S_{V_i}$ and $S_{V_i^*}$ , $1 \leq i \leq 3$, are the arc lengths of indicatrix curves of Frenet vectors $V_i$ and $V_i^*$ of timelike base curves $\alpha$ and $\alpha^*$ of the timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$, respectively, then we have*

$$S_{V_i^*} = S_{V_i} , \ \ 1 \leq i \leq 3 .$$

**Theorem 4.2.** *Let $S_\alpha$ and $S_{\alpha^*}$ be the arc lengths of indicatrix curves of timelike base curves $\alpha$ and $\alpha^*$ of the timelike parallel $p_i$-equidistant ruled* *surfaces* $M$ *and* $M^*$, *respectively. If* $\dfrac{k_1}{k_1^*}$ *is constant, then we have* $S_{\alpha^*} = \dfrac{k_1}{k_1^*} S_\alpha$.

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