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# Approximation of functions belonging to the class $\mathrm{L}^p(\omega)_\beta$ by linear operators

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Abstract. We prove results which correspond to the theorems of M. L. Mittal, B. E. Rhodes, V. N. Mishra [International Journal of Mathematics and Mathematical Sciences, Volume 2006 (2006), Article ID 53538, 10 pages] on the rate of norm and pointwise approximation of conjugate functions by the matrix summability means of their Fourier series.

## 1. Introduction

Let $L^p$ $(1 < p < \infty)$ be the class of all $2\pi$-periodic real-valued functions integrable in the Lebesgue sense with $p$-th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| = \|f(\cdot)\|_{L^p} = \left( \int_Q \mid f(t) \mid^p dt \right)^{1/p}. \tag{1}$$

Consider its trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x)$$

and the conjugate one

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (b_\nu(f)\cos\nu x - a_\nu(f)\sin\nu x)$$

with the partial sums $S_k f$ and $\widetilde{S}_k f$, respectively. We know that if $f \in L$, then

$$\widetilde{f}(x) := -\frac{1}{\pi}\int_0^\pi \psi_x(t)\,\frac{1}{2}\cot\frac{t}{2}dt = \lim_{\epsilon\to 0}\widetilde{f}(x,\epsilon),$$

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where

$$\widetilde{f}\left(x,\epsilon\right):=-\frac{1}{\pi}\int_{\epsilon}^{\pi}\psi_{x}\left(t\right)\frac{1}{2}\cot\frac{t}{2}dt$$

with

$$\psi_{x}\left(t\right):=f\left(x+t\right)-f\left(x-t\right),$$

exists for almost all $x$ [7, Th. (3.1)IV].

Now, we define two classes of sequences (see [2]).

A sequence $c:=(c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c\in RBVS$, if it has the property

$$\sum_{k=m}^{\infty}\left|c_n-c_{n+1}\right|\leq K(c)c_m \tag{2}$$

for all natural numbers $m$, where $K(c)$ is a constant depending only on $c$.

A sequence $c:=(c_n)$ of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly $c\in HBVS$, if it has the property

$$\sum_{k=0}^{m-1}\left|c_n-c_{n+1}\right|\leq K(c)c_m \tag{3}$$

for all natural numbers $m$, or only for all $m\leq N$ if the sequence $c$ has only finite number of nonzero terms and the last nonzero term is $c_N$.

We assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, there exists a constant $K$ such that

$$0\leq K(\alpha_n)\leq K$$

holds for all $n$, where $K(\alpha_n)$ denotes the sequence of constants appearing in the inequalities (2) or (3) for the sequence $\alpha_n:=(a_{n,k})_{k=0}^{n}$. Now we can give the conditions to be used later on. We assume that for all $n$ and $0\leq m\leq n$

$$\sum_{k=m}^{\infty}\left|a_{n,k}-a_{n,k+1}\right|\leq Ka_{n,m} \tag{4}$$

or

$$\sum_{k=0}^{m-1}\left|a_{n,k}-a_{n,k+1}\right|\leq Ka_{n,m} \tag{5}$$

holds if $\alpha_n:=(a_{n,k})_{k=0}^{n}$ belongs to $RBVS$ or $HBVS$, respectively.

Let $A:=(a_{n,k})$ be a lower triangular infinite matrix of real numbers such that

$$a_{n,k}\geq 0,\quad \sum_{k=0}^{n}a_{n,k}=1\quad (k,n=0,1,2,...),$$

and let the $A$-transformations of $(S_k f)$ and $\left(\widetilde{S}_k f\right)$ be given by

$$T_{n,A} f(x) := \sum_{k=0}^{n} a_{n,k} S_k f(x) \qquad (n = 0, 1, 2, ...)$$

and

$$\widetilde{T}_{n,A} f(x) := \sum_{k=0}^{n} a_{n,k} \widetilde{S}_k f(x) \qquad (n = 0, 1, 2, ...),$$

respectively. Let for $k = 1, 2, .., n + 1$

$$A_{n,k} = \sum_{i=0}^{k-1} a_{n,i} \quad \text{and} \quad \overline{A}_{n,k} = \sum_{i=n-k+1}^{n} a_{n,i} \quad \left(A_{n,0} = \overline{A}_{n,0} = 0\right).$$

As a measure of approximation by the above quantities we use the generalized moduli of continuity of $f$ in the space $L^p$ defined for $\beta \geq 0$ by the formulas

$$\widetilde{\omega}_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ \left|\sin \frac{t}{2}\right|^{\beta p} \int_0^\pi |\psi_x(t)|^p \, dx \right\}^{\frac{1}{p}},$$

$$\omega_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ \left|\sin \frac{t}{2}\right|^{\beta p} \int_0^\pi |\varphi_x(t)|^p \, dx \right\}^{\frac{1}{p}},$$

where

$$\varphi_x(t) := f(x + t) + f(x - t) - 2f(x).$$

It is clear that for $\beta > \alpha \geq 0$

$$\widetilde{\omega}_\beta f(\delta)_{L^p} \leq \widetilde{\omega}_\alpha f(\delta)_{L^p} \quad \text{and} \quad \omega_\beta f(\delta)_{L^p} \leq \omega_\alpha f(\delta)_{L^p},$$

and it is easily seen that $\widetilde{\omega}_0 f(\cdot)_{L^p} = \widetilde{\omega} f(\cdot)_{L^p}$, $\omega_0 f(\cdot)_{L^p} = \omega f(\cdot)_{L^p}$ are the classical moduli of continuity.

The deviation $\widetilde{T}_{n,A} f - \widetilde{f}$ was estimated in the norm of $L^p$ by S. Lal and H. Nigam [1]. Their result was generalized by M. L. Mittal, B. E. Rhoades and V. N. Mishra [3] in the following form:

*Let $A = (a_{n,k})$ be an infinite regular triangular matrix with nonnegative entries satisfying*

$$\sum_{k=0}^{r} (k + 1) |a_{n,n-k} - a_{n,n-k-1}| = O\left(\sum_{k=n-r}^{n} a_{n,k}\right), \quad 0 \leq r \leq n. \tag{6}$$

*Then the degree of approximation of function $\widetilde{f}$, conjugate to a $2\pi$-periodic function $f$ belonging to the class*

$$W\left(L^{p},\omega_{0}\right)=\left\{f\in L^{p}:\left(\int\limits_{0}^{2\pi}\left|\left[f\left(x+t\right)-f\left(x\right)\right]\sin^{\beta}x\right|^{p}dx\right)^{\frac{1}{p}}=O\left(\omega_{0}\left(t\right)\right)\right\},$$

*where $p>1$ and $\beta\geq 0$, is given by*

$$\left\|\widetilde{T}_{n,A}f-\widetilde{f}\,\right\|=O\left(\left(n+1\right)^{\beta+1/p}\omega_{0}\left(\frac{\pi}{n+1}\right)\right) \tag{7}$$

*provided that $\omega_0$ satisfies*

$$\left\{\int_{0}^{\pi/(n+1)}\left(\frac{t\left|\psi_{x}\left(t\right)\right|}{\omega_{0}\left(t\right)}\right)^{p}\sin^{\beta p}tdt\right\}^{1/p}=O\left(\left(n+1\right)^{-1}\right) \tag{8}$$

*and*

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}\left|\psi_{x}\left(t\right)\right|}{\omega_{0}\left(t\right)}\right)^{p}dt\right\}^{1/p}=O\left(\left(n+1\right)^{\gamma}\right)$$

*uniformly in $x$, and $\omega_0\left(t\right)/t$ is nonincreasing in $t$, in which $\gamma$ is an arbitrary positive number with $q(1-\gamma)-1>0$, where $p^{-1}+q^{-1}=1$, $1\leq p\leq\infty$.*

The assumptions of this theorem are not sufficient for the estimation (7). More precisely, condition (8) leads to the divergent integral of type $\left(\int_0^{\pi/n}t^{-(2+\beta)q}dt\right)^{1/q}$. Moreover, condition (6) gives the following estimate

$$\left\{\int_{\pi/n}^{\pi}\left(t^{\gamma-\beta}\overline{A}_{n,n+1}\right)^{q}dt\right\}^{1/q}=O\left(n^{\beta-\gamma-1/q}\right)$$

which is incorrect for e.g. $\beta=0$.

Taking $\beta=0$ one has the above-mentioned earlier result [1]:

*If $A=(a_{n,k})$ is an infinite regular triangular matrix such that the elements $a_{n,k}$ are nonnegative and nondecreasing with $k$, then the degree of approximation of a function $\widetilde{f}$, conjugate to a $2\pi$-periodic function $f$ belonging to $Lip\left(\omega_1,p\right)$, is given by*

$$\left\|\widetilde{T}_{n,A}f-\widetilde{f}\,\right\|=O\left(\left(n+1\right)^{1/p}\omega_{1}\left(\frac{\pi}{n+1}\right)\right) \tag{9}$$

*provided that $\omega_1$ satisfies*

$$\left\{\int_{0}^{1/n}\left(\frac{t\left|\psi_{x}\left(t\right)\right|}{\omega_{1}\left(t\right)}\right)^{p}dt\right\}^{1/p}=O\left(\left(n+1\right)^{-1}\right)$$

*and*

$$\left\{ \int_{1/n}^{\pi} \left( \frac{t^{-\gamma} \left| \psi_x \left( t \right) \right|}{\omega_1 \left( t \right)} \right)^p dt \right\}^{1/p} = O \left( \left( n+1 \right)^{\gamma} \right)$$

*uniformly in $x$, where $\gamma$ is an arbitrary number such that $q(1-\gamma)-1>0$, where $p^{-1}+q^{-1}=1$, $1 \leq p \leq \infty$.*

In [1], there are similar mistakes in the proof as in [2].

As we noted, the assumption (8) is, in general, not proper. In the results formulated below we give, instead of (8), another condition (12) which guarantees the estimate (9).

The estimates of the deviation $\widetilde{T}_{n,A}f - \widetilde{f}$ were also obtained by K. Qureshi [4, 5] in case $\beta = 0$ and for monotonic sequences $(a_{n,k})_{k=0}^{n}$.

In this note we shall consider the same deviation and additionally the deviations $\widetilde{T}_{n,A}f\left(\cdot\right) - \widetilde{f}\left(\cdot, \frac{\pi}{n+1}\right)$ and $T_{n,A}f - f$. In our theorems we formulate the general and precise conditions for the functions and moduli of continuity. Finally, we also give some results on the norm approximation.

We shall write $I_1 \ll I_2$ if there exists a positive constant $K$, sometimes depending on some parameters, such that $I_1 \leq K I_2$.

## 2. Statement of the results

Let us consider a function $\omega$ of modulus of continuity type on the interval $[0, 2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\omega\left(0\right) = 0$, $\omega\left(\delta_1 + \delta_2\right) \leq \omega\left(\delta_1\right) + \omega\left(\delta_2\right)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$. It is easy to conclude that the function $\delta^{-1}\omega\left(\delta\right)$ is a nondecreasing function of $\delta$. Let

$$L^p\left(\widetilde{\omega}\right)_\beta = \left\{ f \in L^p : \widetilde{\omega}_\beta f\left(\delta\right)_{L^p} \leq \widetilde{\omega}\left(\delta\right) \right\},$$

$$L^p\left(\omega\right)_\beta = \left\{ f \in L^p : \omega_\beta f\left(\delta\right)_{L^p} \leq \omega\left(\delta\right) \right\},$$

where $\omega$ and $\widetilde{\omega}$ are also the functions of modulus of continuity type. It is clear that for $\beta > \alpha \geq 0$

$$L^p\left(\widetilde{\omega}\right)_\alpha \subset L^p\left(\widetilde{\omega}\right)_\beta \quad \text{and } L^p\left(\omega\right)_\alpha \subset L^p\left(\omega\right)_\beta .$$

We can now formulate our main results using the following notation:

$$a_n = \begin{cases} a_{n,0} & \text{when} \quad (a_{n,k})_{k=0}^{n} \in RBVS, \\ a_{n,n} & \text{when} \quad (a_{n,k})_{k=0}^{n} \in HBVS. \end{cases}$$

At the beginning, we formulate the results on the degrees of pointwise summability of conjugate series.

**Theorem 1.** *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $\beta < 1 - \frac{1}{p}$, $(a_{n,k})_{k=0}^n \in HBVS$ *(or* $(a_{n,k})_{k=0}^n \in RBVS$*) and let* $\widetilde{\omega}$ *be such that*

$$\left\{ \int_0^{\pi/(n+1)} \left( \frac{t\,|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left( (n+1)^{-1} \right) \tag{10}$$

*and*

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left( \frac{t^{-\gamma}\,|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left( (n+1)^{\gamma} \right) \tag{11}$$

*hold with* $0 < \gamma < \beta + \frac{1}{p}$. *Then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}\left( x, \frac{\pi}{n+1} \right) \right| = O_x\left( (n+1)^{\beta + \frac{1}{p}} a_n(n+1)\, \widetilde{\omega}\left( \frac{\pi}{n+1} \right) \right)$$

*for considered* $x$.

**Theorem 2.** *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $\beta < 1 - \frac{1}{p}$, $(a_{n,k})_{k=0}^n \in HBVS$ *(or* $(a_{n,k})_{k=0}^n \in RBVS$*) and let* $\widetilde{\omega}$ *satisfy* (11) *with* $0 < \gamma < \beta + \frac{1}{p}$,

$$\left\{ \int_0^{\pi/(n+1)} \left( \frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left( (n+1)^{-1/p} \right) \tag{12}$$

*and*

$$\left\{ \int_0^{\pi/(n+1)} \left( \frac{\widetilde{\omega}(t)}{t \sin^{\beta} \frac{t}{2}} \right)^q dt \right\}^{1/q} = O\left( (n+1)^{\beta + 1/p}\, \widetilde{\omega}\left( \frac{\pi}{n+1} \right) \right), \tag{13}$$

*where* $q = p(p-1)^{-1}$. *Then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| = O_x\left( (n+1)^{\beta + \frac{1}{p}} a_n(n+1)\, \widetilde{\omega}\left( \frac{\pi}{n+1} \right) \right)$$

*for considered* $x$ *such that* $\widetilde{f}(x)$ *exists.*

Now we present the approximation properties of the operator $T_{n,A} f$.

**Theorem 3.** *Let* $f \in L^p(\omega)_\beta$ *with* $\beta < 1 - \frac{1}{p}$, $(a_{n,k})_{k=0}^n \in HBVS$ *(or* $(a_{n,k})_{k=0}^n \in RBVS$*) and let* $\omega$ *satisfy*

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left( \frac{t^{-\gamma}\,|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left( (n+1)^{\gamma} \right) \tag{14}$$

*with* $0 < \gamma < \beta + \frac{1}{p}$,

$$\left\{ \int_0^{\pi/(n+1)} \left( \frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left( (n+1)^{-1/p} \right) \tag{15}$$

*and*

$$\left\{ \int_0^{\pi/(n+1)} \left( \frac{\omega(t)}{t \sin^\beta \frac{t}{2}} \right)^q dt \right\}^{1/q} = O\left( (n+1)^{\beta+1/p} \omega\left( \frac{\pi}{n+1} \right) \right), \quad (16)$$

*where* $q = p(p-1)^{-1}$. *Then*

$$\left| T_{n,A} f(x) - f(x) \right| = O_x\left( (n+1)^{\beta+\frac{1}{p}} a_n (n+1) \omega\left( \frac{\pi}{n+1} \right) \right),$$

*for considered* $x$.

Finally, we formulate some remarks.

**Remark 1.** Considering the $L^p$norms of the deviations from our theorems instead of the pointwise one we can obtain the same estimations without any additional assumptions like (10), (11), (12) and (14), (15).

**Remark 2.** Under the additional assumptions $a_{n,n} = O\left(\frac{1}{n}\right)$, $\beta = 0$ and $\widetilde{\omega}(t) = O(t^\alpha)$ or $\omega(t) = O(t^\alpha)$ $(0 < \alpha \leq 1)$, the degrees of approximation in Theorems 1, 2 or 3, respectively, are $O\left(n^{\frac{1}{p}-\alpha}\right)$. We obtain in Theorems 1, 2 or 3 the same degrees of approximation under the assumption $a_{n,0} = O\left(\frac{1}{n}\right)$.

**Remark 3.** Due to the above remarks, in the special case when our sequences $(a_{n,k})_{k=0}^n$ are monotonic with respect to $k$ we have the corrected form of the result of S. Lal and H. K. Nigam [1].

## 3. Auxiliary results

We begin this section by some notation following A. Zygmund [7, Section 5 of Chapter II].

It is clear that

$$\widetilde{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \widetilde{D}_k(t) \, dt,$$

$$S_k f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t) \, dt$$

and

$$\widetilde{T}_{n,A} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k}(t) \, dt,$$

$$T_{n,A} f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{k=0}^{n} a_{n,k} D_k(t) \, dt,$$

where

$$\widetilde{D_k}(t) = \sum_{\nu=0}^{k} \sin \nu t = \frac{\cos \frac{t}{2} - \cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}},$$

$$D_k(t) = \frac{1}{2} + \sum_{\nu=1}^{k} \cos \nu t = \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Hence

$$\begin{aligned}\widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) &= -\frac{1}{\pi} \int_0^{\pi/(n+1)} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k}(t)\, dt \\ &\quad + \frac{1}{\pi} \int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k^\circ}(t)\, dt\end{aligned}$$

and

$$\widetilde{T}_{n,A} f(x) - \widetilde{f}(x) = \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k^\circ}(t)\, dt,$$

where

$$\widetilde{D_k^\circ}(t) = \frac{\cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}},$$

and

$$T_{n,A} f(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) \sum_{k=0}^{n} a_{n,k} D_k(t)\, dt.$$

Now, we formulate some estimates for the conjugate Dirichlet kernels.

**Lemma 1** (see [7])**.** *If* $0 < |t| \le \pi/2$, *then*

$$\left|\widetilde{D_k^\circ}(t)\right| \le \frac{\pi}{2|t|} \quad \text{and} \quad \left|\widetilde{D_k}(t)\right| \le \frac{\pi}{|t|},$$

*and for any real number* $t$ *we have*

$$\left|\widetilde{D_k}(t)\right| \le \frac{1}{2} k(k+1)|t| \quad \text{and} \quad \left|\widetilde{D_k}(t)\right| \le k+1.$$

More complicated estimates we give with proofs.

**Lemma 2.** *If* $(a_{n,k})_{k=0}^{n} \in HBVS$ *and* $\frac{1}{n} \le t \le \pi$, *then*

$$\left|\sum_{k=0}^{n} a_{n,k} \widetilde{D_k^\circ}(t)\right| = O\left(t^{-1} \overline{A}_{n,\tau}\right) = O\left(t^{-2} a_{n,n}\right),$$

*and if* $(a_{n,k})_{k=0}^{n} \in RBVS$ *for* $0 < t \le \pi$, *then*

$$\left|\sum_{k=0}^{n} a_{n,k} \widetilde{D_k^\circ}(t)\right| = O\left(t^{-1} A_{n,\tau}\right) = O\left(t^{-2} a_{n,0}\right),$$

*where* $\tau = \max\left(1, \left[t^{-1}\right]\right)$.

*Proof.* Let us consider the sum

$$\begin{aligned}\sum_{k=m}^{n} a_{n,k} \cos\frac{(2k+1)\,t}{2}\sin\frac{t}{2} \\ &= a_{n,m}\cos\frac{(2m+1)\,t}{2}\sin\frac{t}{2} \\ &\quad + \sum_{k=m+1}^{n-1}\sin\frac{t}{2}\sum_{\nu=m+1}^{k}\cos\frac{(2\nu+1)\,t}{2}\left(a_{n,k}-a_{n,k+1}\right) \\ &\quad + \sin\frac{t}{2}\sum_{\nu=m+1}^{n}\cos\frac{(2\nu+1)\,t}{2}a_{n,n} \\ &= a_{n,m}\cos\frac{(2m+1)\,t}{2}\sin\frac{t}{2} \\ &\quad + \sum_{k=m+1}^{n-1}\left(a_{n,k}-a_{n,k+1}\right)\sin\frac{(k-m-1)\,t}{2}\cos\frac{(k+m+1)\,t}{2} \\ &\quad + a_{n,n}\sin\frac{(n-m-1)\,t}{2}\cos\frac{(n+m+1)\,t}{2}.\end{aligned}$$

Hence, for $n \geq \tau = \left[\frac{\pi}{t}\right] \geq 0$,

$$\left|\sum_{k=0}^{n} a_{n,k}\cos\frac{(2k+1)\,t}{2}\sin\frac{t}{2}\right| \leq \sin\frac{t}{2}A_{n,\tau} + a_{n,\tau} + \sum_{k=\tau}^{n-1}\left|a_{n,k}-a_{n,k+1}\right| + a_{n,n}$$

or

$$\left|\sum_{k=0}^{n} a_{n,k}\cos\frac{(2k+1)\,t}{2}\sin\frac{t}{2}\right| \leq \sin\frac{t}{2}\overline{A}_{n,n-\tau} + a_{n,0} + \sum_{k=0}^{n-\tau}\left|a_{n,k}-a_{n,k+1}\right| + a_{n,n-\tau}.$$

Since $(a_{n,k})_{k=0}^{n} \in RBVS$ we have

$$a_{n,m} \leq \sum_{k=m}^{\infty}\left|a_{n,k}-a_{n,k+1}\right| \leq \sum_{k=r}^{\infty}\left|a_{n,k}-a_{n,k+1}\right| \ll a_{n,r} \quad (n \geq m \geq r \geq 0)$$

and therefore

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{n,k}\cos\frac{(2k+1)\,t}{2}\sin\frac{t}{2}\right| &\ll \sin\frac{t}{2}A_{n,\tau} + a_{n,\tau} \leq \sin\frac{t}{2}A_{n,\tau} + t a_{n,\tau}\sum_{k=0}^{\tau}1 \\ &\leq \sin\frac{t}{2}A_{n,\tau} + t\sum_{k=0}^{\tau}a_{n,k} \leq tA_{n,\tau} \leq a_{n,0}\ .\end{aligned}$$

Analogously, the relation $(a_{n,k})_{k=0}^{n} \in HBVS$ implies

$$|a_{n,m} - a_{n,r}| \leq \sum_{k=m}^{r-1} |a_{n,k} - a_{n,k+1}| \leq \sum_{k=0}^{r-1} |a_{n,k} - a_{n,k+1}| \ll a_{n,r} \quad (n \geq r \geq m \geq 0)$$

and

$$a_{n,m} \ll a_{n,r} \qquad (n \geq r \geq m \geq 0),$$

whence

$$\begin{aligned} \left| \sum_{k=0}^{n} a_{n,k} \cos \frac{(2k+1)\,t}{2} \sin \frac{t}{2} \right| &\ll \sin \frac{t}{2} \overline{A}_{n,n-\tau} + a_{n,n-\tau} \\ &\leq t\overline{A}_{n,n-\tau} + t a_{n,n-\tau} \sum_{k=n-\tau}^{n} 1 \\ &\leq t\overline{A}_{n,n-\tau} + t \sum_{k=n-\tau}^{n} a_{n,k} \\ &\ll t\overline{A}_{n,n-\tau} \ll a_{n,n}. \end{aligned}$$

□

Next, we present some known estimates for the Dirichlet kernel.

**Lemma 3** (see [7])**.** *If* $0 < |t| \leq \pi/2$, *then*

$$|D_k(t)| \leq \frac{\pi}{|t|}$$

*and for any real number* $t$ *we have*

$$|D_k(t)| \leq k+1 \quad .$$

We have a lemma similar to Lemma 2.

**Lemma 4** (cf. [2, 6])**.** *If* $(a_{n,k})_{k=0}^{n} \in HBVS$ *and* $\frac{1}{n} \leq t \leq \pi$, *then*

$$\left| \sum_{k=0}^{n} a_{n,k} D_k(t) \right| = O\left(t^{-1}\overline{A}_{n,\tau}\right) = O\left(t^{-2} a_{n,n}\right),$$

*and if* $(a_{n,k})_{k=0}^{n} \in RBVS$ *for* $0 < t \leq \pi$, *then*

$$\left| \sum_{k=0}^{n} a_{n,k} D_k(t) \right| = O\left(t^{-1} A_{n,\tau}\right) = O\left(t^{-2} a_{n,0}\right),$$

*where* $\tau = \max\left(1, \left[t^{-1}\right]\right)$.

*Proof.* Similarly as above, for $n \geq \tau = \left[\frac{\pi}{t}\right] \geq 0$,

$$\left|\sum_{k=0}^{n} a_{n,k} \sin\frac{(2k+1)\,t}{2} \sin\frac{t}{2}\right| \leq A_{n,\tau} + a_{n,\tau} + \sum_{k=\tau}^{n-1} |a_{n,k} - a_{n,k+1}| + a_{n,n}$$

or

$$\left|\sum_{k=0}^{n} a_{n,k} \sin\frac{(2k+1)\,t}{2} \sin\frac{t}{2}\right| \leq \overline{A}_{n,n-\tau} + a_{n,0} + \sum_{k=0}^{n-\tau} |a_{n,k} - a_{n,k+1}| + a_{n,n-\tau}\,.$$

Since $(a_{n,k})_{k=0}^{n} \in RBVS$ we have

$$a_{n,m} \leq \sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq \sum_{k=r}^{\infty} |a_{n,k} - a_{n,k+1}| \ll a_{n,r} \quad (n \geq m \geq r \geq 0)$$

and therefore

$$\left|\sum_{k=0}^{n} a_{n,k} \sin\frac{(2k+1)\,t}{2} \sin\frac{t}{2}\right| \ll tA_{n,\tau} + a_{n,\tau} \ll tA_{n,\tau} \ll a_{n,0}.$$

Analogously, the relation $(a_{n,k})_{k=0}^{n} \in HBVS$ implies

$$\begin{aligned} |a_{n,m} - a_{n,r}| &\leq \sum_{k=m}^{r-1} |a_{n,k} - a_{n,k+1}| \leq \sum_{k=0}^{r-1} |a_{n,k} - a_{n,k+1}| \\ &\ll a_{n,r} \quad (n \geq r \geq m \geq 0) \end{aligned}$$

and

$$a_{n,m} \ll a_{n,r} \qquad (n \geq r \geq m \geq 0)\,,$$

whence

$$\left|\sum_{k=0}^{n} a_{n,k} \sin\frac{(2k+1)\,t}{2} \sin\frac{t}{2}\right| \ll t\overline{A}_{n,n-\tau} + a_{n,n-\tau} \ll t\overline{A}_{n,n-\tau} \ll a_{n,n}.$$

□

## 4. Proofs of the results

**4.1. Proof of Theorem 1.** We start with the obvious relations

$$\begin{aligned} \widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) &= -\frac{1}{\pi} \int_0^{\pi/(n+1)} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k}(t)\, dt \\ &\quad + \frac{1}{\pi} \int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k^{\circ}}(t) \\ &=: \widetilde{I}_1 + \widetilde{I}_2^{\circ} \end{aligned}$$

and

$$\left|\widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \leq \left|\widetilde{I}_1\right| + \left|\widetilde{I}_2^{\circ}\right|.$$

By Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 1 and (10)

$$\begin{aligned}
\left|\widetilde{I}_1\right| &\leq (n+1)^2\int_0^{\pi/(n+1)} t\left|\psi_x(t)\right|dt \\
&\leq (n+1)^2\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{t\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{\omega}(t)}{\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\ll (n+1)^1\left\{\int_0^{\pi/(n+1)}\left[\frac{\widetilde{\omega}(t)}{t^\beta}\right]^q dt\right\}^{\frac{1}{q}} \ll (n+1)^{\beta+\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)
\end{aligned}$$

for $\beta<1-\frac{1}{p}$.

By Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 2 and (11)

$$\begin{aligned}
\left|\widetilde{I}_2^\circ\right| &\leq \frac{1}{\pi}\int_{\pi/(n+1)}^{\pi}\left|\psi_x(t)\right|\left|\sum_{k=0}^n a_{n,k}\widetilde{D_k^\circ}(t)\right|dt \ll a_n\int_{\pi/(n+1)}^{\pi}\frac{\left|\psi_x(t)\right|}{t^2}dt \\
&\leq a_n\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^{-\gamma}\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\ll a_n(n+1)^\gamma\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\ll a_n(n+1)^{\gamma+1}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[t^{\gamma-\beta-1}\right]^q dt\right\}^{\frac{1}{q}} \\
&\ll (n+1)^{\beta+\frac{1}{p}}a_n(n+1)\widetilde{\omega}\left(\frac{\pi}{n+1}\right)
\end{aligned}$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates we obtain the desired result.

**4.2. Proof of Theorem 2.** We start with the obvious relations

$$\begin{aligned}
\widetilde{T}_{n,A}f(x)-\widetilde{f}(x) &= \frac{1}{\pi}\int_0^{\pi/(n+1)}\psi_x(t)\sum_{k=0}^n a_{n,k}\widetilde{D_k^\circ}(t)\,dt \\
&\quad+\frac{1}{\pi}\int_{\pi/(n+1)}^{\pi}\psi_x(t)\sum_{k=0}^n a_{n,k}\widetilde{D_k^\circ}(t)\,dt \\
&=: \widetilde{I}_1^\circ+\widetilde{I}_2^\circ
\end{aligned}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right|\leq\left|\widetilde{I}_1^\circ\right|+\left|\widetilde{I}_2^\circ\right|.$$

By Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 1, (12) and (13)

$$\begin{aligned}\left|\widetilde{I}_1^{\circ}\right| &\leq \frac{1}{\pi}\int_0^{\pi/(n+1)}\frac{|\psi_x(t)|}{t}dt\\ &\leq \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{\omega}(t)}{t^1\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}\end{aligned}$$

$$\ll (n+1)^{-\frac{1}{p}}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\omega(t)}{t^{1+\beta}}\right]^q dt\right\}^{\frac{1}{q}} \ll (n+1)^\beta\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)$$

By the previous proof

$$\left|\widetilde{I}_2^{\circ}\right| \ll (n+1)^{\beta+\frac{1}{p}}\,a_n(n+1)\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates we obtain the desired result.

**4.3. Proof of Theorem 3.** Let

$$\begin{aligned}T_{n,A}f(x)-f(x) &= \frac{1}{\pi}\int_0^{\pi/(n+1)}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}D_k(t)\,dt\\ &\quad+\frac{1}{\pi}\int_{\pi/(n+1)}^{\pi}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}D_k(t)\,dt\\ &=: I_1^{\circ}+I_2^{\circ},\end{aligned}$$

then

$$\left|T_{n,A}f(x)-f(x)\right| \leq |I_1^{\circ}|+|I_2^{\circ}|.$$

By Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 3, (15) and (16),

$$\begin{aligned}|I_1^{\circ}| &\leq \frac{1}{\pi}\int_0^{\pi/(n+1)}\frac{|\varphi_x(t)|}{t}dt\\ &\leq \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{|\varphi_x(t)|}{\omega(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\omega(t)}{t^1\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}\end{aligned}$$

$$\ll (n+1)^{-\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\omega(t)}{t^{1+\beta}}\right]^q dt\right\}^{\frac{1}{q}} \ll (n+1)^\beta\,\omega\left(\frac{\pi}{n+1}\right)$$

By Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 4 and (14)

$$\begin{aligned}
|I_2^{\circ}| &\ll a_n \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left[ \frac{t^{-\gamma} |\varphi_x(t)|}{\omega(t)} \sin^{\beta} \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left[ \frac{\omega(t)}{t^{2-\gamma} \sin^{\beta} \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}} \\
&\ll a_n (n+1)^{\gamma+1} \omega\left(\frac{\pi}{n+1}\right) \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left[ t^{\gamma-\beta-1} \right]^q dt \right\}^{\frac{1}{q}} \\
&\ll (n+1)^{\beta+\frac{1}{p}} a_n (n+1) \omega\left(\frac{\pi}{n+1}\right)
\end{aligned}$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates we obtain the desired result.

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