\section{Introduction}

As first defined by Ruckle \cite{5}, \(\phi\)-topologies in sequence spaces have many valuable properties. Two that stand out the most are: (a) the completion of a sequence space which has a \(\phi\)-topology is also a sequence space and (b) any matrix transformation, defined by a row-finite matrix, between sequence spaces \(E\) and \(F\) that have the strongest \(\phi\)-topologies of all, is a continuous mapping.

The \(\beta\)-topologies, as defined in the article \cite{1}, are significant because the summability field \(c_A\) of any summability matrix \(A = (a_{nk})\) is a separable FK-space such that the FK-topology is a \(\beta\)-topology.

This note aims to analyse the link between \(\phi\)- and \(\beta\)-topologies in FK-spaces. It is quite apparent that every \(\phi\)-topology is a \(\beta\)-topology, the opposite is not always true. In the second section we detail \(\phi\) and \(\beta\)-topologies in separable FK-spaces and confirm the existence of a \(\beta\)-topology that is not a \(\phi\)-topology.

In the third section we demonstrate that if an FK-space has AB, then every \(\beta\)-topology on it is also a \(\phi\)-topology. Furthermore, we show that in such a case the FK-topology is generated by a sequence of monotone seminorms. An analogous statement for BK-spaces is well-known (cf. \cite{4}, Lemma 2.1).
The terminology from the theory of locally convex spaces and summability is standard, we refer to Wilansky [6] and [7].

Let $X$ be a vector space and let $M$ be a total subspace of $X^*$ (the algebraic dual space of $X$). Then $X$ and $M$ by means of the bilinear form

$$\langle \ , \ \rangle : \ X \times M \to \mathbb{K}, \ (x, f) \mapsto \langle x, f \rangle := f(x),$$

form a dual pair $\langle X, M \rangle$. A locally convex topology $\tau$ on $X$ is said to be $\langle X, M \rangle$-polar, if it is generated by a family of seminorms $\{p_\gamma \}_{\gamma \in \Gamma}$ having the property

$$\forall \gamma \in \Gamma \ \exists S_\gamma \subset M \ \forall x \in X \ p_\gamma(x) = \sup_{f \in S_\gamma} |f(x)|. \quad (1)$$

We call a seminorm $M$-polar if it satisfies (1). Note that $\tau$ is an $\langle X, M \rangle$-polar topology on $X$ if and only if there exists a $\tau$-neighbourhood basis $\mathcal{B}$ of zero such that

$$\forall U \in \mathcal{B} \ M \cap U^0 \text{ is } \sigma(X', X)\text{-dense in } U^0,$$

(cf. [1], Proposition 4.2), where $X'$ is the topological dual of $(X, \tau)$ and $U^0 := \{f \in X' \mid \forall x \in U \ |f(x)| \leq 1\}$ is the the polar set of $U$.

2. $\varphi$- and $\beta$-topologies

Let $\omega$ be the space of all complex or real sequences $x = (x_k)$ and $\varphi$ the space of all finitely non-zero sequences. A linear subspace of $\omega$ is called a sequence space. The space $\varphi$ is clearly the linear span of $\{e^k \mid k \in \mathbb{N}\}$, where

$$e^k := (\delta_{ki}) = (0, \ldots, 0, 1, 0, \ldots) \ (k \in \mathbb{N}).$$

For a sequence space $E$ its $\beta$-dual $E^\beta$ is defined by

$$E^\beta := \left\{ y \in \omega \mid \sum_k x_k y_k \text{ converges for each } x \in E \right\}.$$

A K-space is a sequence space endowed with a locally convex topology such that the coordinate functionals $\pi_k$ defined by $\pi_k(x) := x_k \ (k \in \mathbb{N})$ are continuous. A Fréchet (Banach, LB-)K-space is called an FK (BK, LBK)-space.

Let $E$ be a sequence space containing $\varphi$. By means of the natural pairing

$$\langle x, u \rangle := \sum_k x_k u_k \ (x \in E, \ u \in E^\beta),$$

the sequence space $E$ is in duality with both $E^\beta$ and $\varphi$. Obviously, the inclusion $\varphi \subset E'$ holds for each K-space $E$. If $(E, \tau_E)$ is an FK-space, then $E^\beta \subset E'$.

A $\varphi$-topology on a sequence space $E$ is defined as an $\langle E, \varphi \rangle$-polar topology. It can be verified that a K-topology $\tau$ on $E$ is a $\varphi$-topology if and only if
there exists a \( \tau \)-neighbourhood basis of zero consisting of \( \tau_\omega \o E \)-closed subsets, where \( \tau_\omega \) is the FK-topology of the sequence space \( \omega \).

An \( \langle E, E^\beta \rangle \)-polar topology on a sequence space \( E \) is called a \( \beta \)-topology (cf. [1], Definition 4.1).

Let \( A = (a_{nk}) \) be an infinite matrix. Then its summability field \( c_A \), defined by

\[
c_A := \left\{ x \in \omega \mid \exists \lim_n \sum_k a_{nk}x_k =: \lim A x \right\},
\]

is a separable FK-space, where the FK-topology is generated by the seminorms

\[
p_0(x) := \sup_n \left| \sum_k a_{nk}x_k \right|,
\]

\[
p_n(x) := \sup_m \left| \sum_{k=1}^m a_{nk}x_k \right| \quad (n \in \mathbb{N})
\]

and

\[
r_k(x) := |x_k| \quad (k \in \mathbb{N})
\]

for each \( x \in c_A \). The seminorms \( p_0, p_n \) and \( r_k \) are \( E^\beta \)-polar, thus the FK-topology of \( c_A \) is a \( \beta \)-topology. (Note that in this instance we accept that \( \varphi \subset c_A \), i.e., the columns \( (a_{nk})_{n \in \mathbb{N}} \) \( (k \in \mathbb{N}) \) are all converging sequences.) If \( A = (a_{nk}) \) is row-finite, that is \( (a_{nk})_{k \in \mathbb{N}} \in \varphi \) \( (n \in \mathbb{N}) \), then its summability field \( c_A \) is an FK-space such that the FK-topology is a \( \varphi \)-topology.

**Proposition 1.** Let \( (E, \tau_E) \) be a separable FK-space. If \( \tau_E \) is a \( \beta \)-topology, then for every \( f \in E' \) there exists a matrix \( A = (a_{nk}) \) such that \( E \subset c_A \) and

\[
f(x) = \lim A x \quad (x \in E).
\]

Moreover, if \( \tau_E \) is a \( \varphi \)-topology, then the matrix \( A = (a_{nk}) \) can be chosen row-finite.

**Proof.** Let \( (E, \tau_E) \) be a separable FK-space such that \( \tau_E \) is a \( \beta \)-topology. If \( f \in E' \), then there exists a \( \tau_E \)-neighbourhood \( U \) of zero such that \( |f(x)| \leq 1 \quad (x \in U) \). Because \( \tau_E \) is a \( \beta \)-topology, we may choose \( U \) such that \( U = U^0 \cap E^\beta \langle E', E \rangle \). Since \( (E, \tau_E) \) is separable, \( (U^0, \sigma(E', E)|U^0) \) is a metrizable topological space (cf. [6], 9.5.3). Hence, for the functional \( f \in \underbrace{U^0 \cap E^\beta \langle E', E \rangle} \) there exists a sequence \( (a^{(n)}) \), consisting of elements from \( U^0 \cap E^\beta \), such that \( a^{(n)} \to f \) in \( (E', \sigma(E', E)) \). Since \( a^{(n)} = (a_{nk})_{k \in \mathbb{N}} \in E^\beta \) for every \( n \in \mathbb{N} \),

\[
f(x) = \lim_n \langle x, a^{(n)} \rangle = \lim_n \sum_k a_{nk}x_k = \lim A x
\]
for each \( x \in E \). Moreover, \( E \subset c_A \).

If \( \tau_E \) is a \( \varphi \)-topology, then we may choose \( U \) such that \( U^0 = U^0 \cap \sigma(E',E) \). Therefore the matrix \( A = (a_{nk}) \) can be chosen so that \( a^{(n)} \in \varphi \) (\( n \in \mathbb{N} \)), that is, the matrix \( A \) is row-finite. \( \square \)

As already mentioned, every FK-topology that is a \( \varphi \)-topology is also a \( \beta \)-topology. The following example demonstrates that the opposite is not always true.

**Example.** Erdős and Piranian [3] (see also [1], Remarks 4.3 and Example 4.4) examined a matrix \( B = (b_{nk}) \) defined by

\[
b_{nk} := \begin{cases} 2^{-p} & \text{if } k = 2^n(2p - 1) \ (p, n \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}
\]

The matrix \( B \) is regular, that is, \( c_B \supset c \) (where \( c \) is the space of all converging sequences) and \( \lim_B x = \lim_n x_n \) for each \( x \in c \). Erdős and Piranian showed that if \( C \) is any regular matrix such that \( c_B \subset c_C \), then \( C \) is not row-finite. We know that \( c_B \) is a separable FK-space such that the FK-topology is a \( \beta \)-topology. Since \( \lim_B \in (c_B)' \), by Proposition 1 there exists a matrix \( A \) such that \( c_B \subset c_A \) and

\[
\lim_B x = \lim_A x \quad \text{for all } x \in c_B.
\]

Therefore \( A \) is a regular matrix, which implies that it is not row-finite. By Proposition 1, the FK-topology on \( c_B \) is not a \( \varphi \)-topology.

### 3. Monotone seminorms

Let \( (E, \| \|) \) be a BK-space. The norm \( \| \| \) is called **monotone** (see [7], 7.1.1) if

\[
\| x \| = \sup_n \| x^{[n]} \|,
\]

where \( x^{[n]} \) is the \( n^{th} \) section of \( x \) defined by

\[
x^{[n]} := (x_1, x_2, \ldots, x_n, 0, \ldots) = \sum_{k=1}^{n} x_k e_k \ (n \in \mathbb{N}).
\]

If the norm \( \| \| \) is monotone, then \( \| x^{[m]} \| \leq \| x^{[n]} \| \) for each \( m \geq n \). Any BK-space \( (E, \| \|) \) that has a monotone norm also has AB, i.e., for each \( x \in E \) the subset \( \{ x^{[n]} : n \in \mathbb{N} \} \) is bounded in \( (E, \| \|) \). The converse is not always true, that is, there exist AB-BK-spaces that do not have a monotone norm. An example is the sum \( bs + c_0 \) (cf. [7], 10.3.7), where \( c_0 \) is the space of all sequences which converge to zero and

\[
bs := \left\{ x \in \omega \mid \| x \|_{bs} := \sup_m \left| \sum_{k=1}^{m} x_k \right| < \infty \right\}.
\]
It is well-known, that a BK-space \((E, \| \|)\) has a monotone norm if and only if \((E, \| \|)\) is an AB-BK-space that has a \(\varphi\)-topology (cf. [4], Lemma 2.1). Our aim is to demonstrate that an analogous statement holds for AB-FK-spaces and that in such a scenario every \(\beta\)-topology is also a \(\varphi\)-topology.

Let \(E\) be a sequence space. Then we call a seminorm \(q : E \to \mathbb{R}\) monotone if

\[
q(x) = \sup_m q(x^{[m]}) \quad \text{for each } x \in E.
\]

If \((E, \tau_E)\) is an FK-space, then there exists a sequence \(\{p_n \mid n \in \mathbb{N}\}\) of seminorms generating the FK-topology \(\tau_E\) such that

\[
p_n(x) \leq p_{n+1}(x) \quad \text{for each } x \in E \text{ and } n \in \mathbb{N}.
\]

For the subsets \(U_n := \{x \in E \mid p_n(x) \leq 1\}\) we get that \(U_1 \supseteq U_2 \supseteq \ldots\). Let

\[
G_n := \left\{ f \in E' \mid \left\| f \right\|_{G_n} := \sup_{x \in U_n} |f(x)| \right\} < \infty\}
\]

and

\[
F_n := \left\{ u \in E^\beta \mid \left\| u \right\|_{F_n} := \sup_{x \in U_n} \left| \sum_{k=1}^m u_k x_k \right| < \infty\right\}
\]

for each \(n \in \mathbb{N}\). Then \((G_n, \left\| G_n \right\|)\) is a Banach space and \((F_n, \left\| F_n \right\|)\) is a BK-space for every \(n \in \mathbb{N}\). We denote by \(B_{F_n}\) the unit ball of \(F_n\). Note that

\[
\sup_{u \in B_{F_n}} \left| \sum_{k=1}^\infty u_k x_k \right| = \sup_{u \in B_{F_n}} \left| \sum_{k=1}^m u_k x_k \right| \quad \text{for each } x \in E.
\]

If the seminorm \(p_n\) is monotone, then

\[
\sup_{f \in U_n^{0,\varphi}} \left| \sum_{k=1}^m x_k f(e^k) \right| = \sup_{f \in U_n^{0,\varphi}} \left| \sum_{k=1}^\infty x_k f(e^k) \right| = \sup_{f \in U_n^{0,\varphi} \cap E^\beta} \left| \sum_{k=1}^\infty x_k f(e^k) \right|.
\]

**Proposition 2.** Let \((E, \tau_E)\) be an AB-FK-space. Then the following statements are equivalent.

(a) \(\tau_E\) is generated by a sequence of monotone seminorms.

(b) \(\tau_E\) is a \(\varphi\)-topology.

(c) \(\tau_E\) is a \(\beta\)-topology.

**Proof.** (a)⇒(b) Let \((E, \tau_E)\) be an AB-FK-space where \(\tau_E\) is generated by a sequence \(\{p_n \mid n \in \mathbb{N}\}\) of monotone seminorms. We may assume that (2) holds. For every monotone seminorm \(p_n\) we get (cf. (4))

\[
p_n(x) = \sup_m p_n(x^{[m]}) = \sup_{f \in U_n^{0,\varphi}} \left| \sum_{k=1}^m x_k f(e^k) \right| \quad \text{for each } x \in E.
\]
Since \( (E, \tau_E) \) is an AB-FK-space, \( E^\beta \cap G_n \) is closed in the Banach space \( G_n \) for every \( n \in \mathbb{N} \) (cf. [2], Theorem 3.4). Therefore \( (E^\beta \cap G_n, \| \cdot \|_{G_n}) \) is a BK-space for every \( n \in \mathbb{N} \). Consequently, the sequences \( (F_n)_{n \in \mathbb{N}} \) and \( (E^\beta \cap G_n)_{n \in \mathbb{N}} \), both consisting of BK-spaces, determine the same sequence space \( E^\beta \), that is,

\[
E^\beta = \bigcup_{n \in \mathbb{N}} F_n = \bigcup_{n \in \mathbb{N}} (E^\beta \cap G_n).
\]

Hence, the inductive sequences \( (F_n)_{n \in \mathbb{N}} \) and \( (E^\beta \cap G_n)_{n \in \mathbb{N}} \) are equivalent. Then for each \( n \in \mathbb{N} \) there exist \( i \in \mathbb{N} \) and \( \rho_{n,i} > 0 \), such that

\[
U_n = (U^0_n \cap E^\beta)^0 \supset \frac{1}{\rho_{n,i}} (B_{F_i})^0.
\]  

(5)

Conversely, from the inclusion \( F_n \subset E^\beta \cap G_n \) we get that there exists \( \mu_n > 0 \) such that

\[
(B_{F_n})^0 \supset \frac{1}{\mu_n} (U^0_n \cap E^\beta)^0 = \frac{1}{\mu_n} U_n \quad (n \in \mathbb{N}).
\]  

(6)

Let us define

\[
q_i(x) := \sup_{u \in B_{F_i}} \left| \sum_{k=1}^{\infty} u_k x_k \right| = \sup_{u \in B_{F_i}, m \in \mathbb{N}} \left| \sum_{k=1}^{m} u_k x_k \right| \quad (x \in E)
\]

(cf. (3)). Then

\[
\sup_m q_i(x^m) = \sup_{m \in \mathbb{N}} \left| \sum_{k=1}^{n} (u^m)_k x_k \right| = \sup_{u \in B_{F_i}, l \in \mathbb{N}} \left| \sum_{k=1}^{l} u_k x_k \right| = q_i(x)
\]

for each \( x \in E \) and \( i \in \mathbb{N} \). So \( \{q_i \mid i \in \mathbb{N}\} \) is a sequence of monotone seminorms. Since

\[
\{x \in E \mid q_i(x) \leq 1\} = \left\{x \in E \mid \sup_{u \in B_{F_i}} \left| \sum_{k=1}^{\infty} u_k x_k \right| \leq 1\right\} = (B_{F_i})^0,
\]

from (5) and (6) we see that the locally convex topology \( \tau' \) generated by \( \{q_i \mid i \in \mathbb{N}\} \) coincides with \( \tau_E \). Thus we have proved that the \( \beta \)-topology \( \tau_E \) can be determined by a sequence of monotone seminorms.

\( \square \)
References


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