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φ- and β-topologies in sequence spaces

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Abstract. The aim of this note is to examine the relationship between φ-topologies and β-topologies in FK-spaces. Every φ-topology on an FK-space is a β-topology, the converse statement is not always true. Still, in AB-BK-spaces the statement holds, i.e., every β-topology is a φ-topology. We establish that an analogous statement is true for AB-FK-spaces.

1. Introduction

As first defined by Ruckle [5], φ-topologies in sequence spaces have many valuable properties. Two that stand out the most are: (a) the completion of a sequence space which has a φ-topology is also a sequence space and (b) any matrix transformation, defined by a row-finite matrix, between sequence spaces E and F that have the strongest φ-topologies of all, is a continuous mapping.

The β-topologies, as defined in the article [1], are significant because the summability field c_A of any summability matrix $A = (a_{nk})$ is a separable FK-space such that the FK-topology is a β-topology.

This note aims to analyse the link between φ- and β-topologies in FK-spaces. It is quite apparent that every φ-topology is a β-topology, the opposite is not always true. In the second section we detail φ- and β-topologies in separable FK-spaces and confirm the existence of a β-topology that is not a φ-topology.

In the third section we demonstrate that if an FK-space has AB, then every β-topology on it is also a φ-topology. Furthermore, we show that in such a case the FK-topology is generated by a sequence of monotone seminorms. An analogous statement for BK-spaces is well-known (cf. [4], Lemma 2.1).

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The terminology from the theory of locally convex spaces and summability is standard, we refer to Wilansky [6] and [7].

Let X be a vector space and let M be a total subspace of X^* (the algebraic dual space of X). Then X and M by means of the bilinear form

$$\langle\ ,\ \rangle :\ X \times M \to \mathbb{K},\ (x,f) \mapsto \langle x,f\rangle := f(x),$$

form a dual pair $\langle X, M\rangle$. A locally convex topology τ on X is said to be $\langle X, M\rangle$-*polar*, if it is generated by a family of seminorms $\{p_\gamma\}_{\gamma\in\Gamma}$ having the property

$$\forall\gamma \in \Gamma\ \ \exists S_\gamma \subset M\ \ \forall x \in X \quad p_\gamma(x) = \sup_{f\in S_\gamma} |f(x)|\,. \tag{1}$$

We call a seminorm M-*polar* if it satisfies (1). Note that τ is an $\langle X, M\rangle$-polar topology on X if and only if there exists a τ-neighbourhood basis $\mathfrak{B}$ of zero such that

$$\forall U \in \mathfrak{B} \quad M \cap U^0 \text{ is } \sigma(X', X)\text{-dense in } U^0,$$

(cf. [1], Proposition 4.2), where X' is the topological dual of (X, τ) and $U^0 := \{f \in X' \mid \forall x \in U \quad |f(x)| \le 1\}$ is the the polar set of U.

2. φ- and β-topologies

Let ω be the space of all complex or real sequences $x = (x_k)$ and φ the space of all finitely non-zero sequences. A linear subspace of ω is called a *sequence space*. The space φ is clearly the linear span of $\{e^k \mid k \in \mathbb{N}\}$, where

$$e^k := (\delta_{ki}) = (0, \dots, 0, 1, 0, \dots)\ (k \in \mathbb{N}).$$

For a sequence space E its β-dual E^β is defined by

$$E^\beta := \left\{ y \in \omega \mid \sum_k x_k y_k \text{ converges for each } x \in E \right\}.$$

A K-*space* is a sequence space endowed with a locally convex topology such that the coordinate functionals π_k defined by $\pi_k(x) := x_k$ $(k \in \mathbb{N})$ are continuous. A Fréchet (Banach, LB-)K-space is called an FK(BK, LBK)-*space*.

Let E be a sequence space containing φ. By means of the natural pairing

$$\langle x, u\rangle := \sum_k x_k u_k\ (x \in E,\ u \in E^\beta),$$

the sequence space E is in duality with both E^β and φ. Obviously, the inclusion $\varphi \subset E'$ holds for each K-space E. If (E, τ_E) is an FK-space, then $E^\beta \subset E'$.

A φ-*topology* on a sequence space E is defined as an $\langle E, \varphi\rangle$-polar topology. It can be verified that a K-topology τ on E is a φ-topology if and only if there exists a τ-neighbourhood basis of zero consisting of $\tau_\omega|_E$-closed subsets, where τ_ω is the FK-topology of the sequence space ω.

An $\langle E, E^\beta\rangle$-polar topology on a sequence space E is called a *β-topology* (cf. [1], Definition 4.1).

Let $A = (a_{nk})$ be an infinite matrix. Then its summability field c_A, defined by

$$c_A := \left\{ x \in \omega \mid \exists \lim_n \sum_k a_{nk} x_k =: \lim_A x \right\},$$

is a separable FK-space, where the FK-topology is generated by the seminorms

$$p_0(x) := \sup_n \left| \sum_k a_{nk} x_k \right|,$$
$$p_n(x) := \sup_m \left| \sum_{k=1}^m a_{nk} x_k \right| \quad (n \in \mathbb{N})$$

and

$$r_k(x) := |x_k| \ \ (k \in \mathbb{N})$$

for each $x \in c_A$. The seminorms p_0, p_n and r_k are E^β-polar, thus the FK-topology of c_A is a β-topology. (Note that in this instance we accept that $\varphi \subset c_A$, i.e., the columns $(a_{nk})_{n\in\mathbb{N}}$ $(k \in \mathbb{N})$ are all converging sequences.) If $A = (a_{nk})$ is row-finite, that is $(a_{nk})_{k\in\mathbb{N}} \in \varphi$ $(n \in \mathbb{N})$, then its summability field c_A is an FK-space such that the FK-topology is a φ-topology.

Proposition 1. *Let (E, τ_E) be a separable FK-space. If τ_E is a β-topology, then for every $f \in E'$ there exists a matrix $A = (a_{nk})$ such that $E \subset c_A$ and*

$$f(x) = \lim_A x \ \ (x \in E).$$

Moreover, if τ_E is a φ-topology, then the matrix $A = (a_{nk})$ can be chosen row-finite.

Proof. Let (E, τ_E) be a separable FK-space such that τ_E is a β-topology. If $f \in E'$, then there exists a τ_E-neighbourhood U of zero such that $|f(x)| \leq 1$ $(x \in U)$. Because τ_E is a β-topology, we may choose U such that $U^0 = \overline{U^0 \cap E^\beta}^{\sigma(E',E)}$. Since (E, τ_E) is separable, $(U^0, \sigma(E', E)|_{U^0})$ is a metrizable topological space (cf. [6], 9.5.3). Hence, for the functional $f \in \overline{U^0 \cap E^\beta}^{\sigma(E',E)}$ there exists a sequence $(a^{(n)})$, consisting of elements from $U^0 \cap E^\beta$, such that $a^{(n)} \to f$ in $(E', \sigma(E', E))$. Since $a^{(n)} = (a_{nk})_{k\in\mathbb{N}} \in E^\beta$ for every $n \in \mathbb{N}$,

$$f(x) = \lim_n \langle x, a^{(n)}\rangle = \lim_n \sum_k a_{nk} x_k = \lim_A x$$

for each $x \in E$. Moreover, $E \subset c_A$.

If τ_E is a φ-topology, then we may choose U such that $U^0 = \overline{U^0 \cap \varphi}^{\sigma(E',E)}$. Therefore the matrix $A = (a_{nk})$ can be chosen so that $a^{(n)} \in \varphi$ $(n \in \mathbb{N})$, that is, the matrix A is row-finite. □

As already mentioned, every FK-topology that is a φ-topology is also a β-topology. The following example demonstrates that the opposite is not always true.

Example. Erdös and Piranian [3] (see also [1], Remarks 4.3 and Example 4.4) examined a matrix $B = (b_{nk})$ defined by

$$b_{nk} := \begin{cases} 2^{-p} & \text{if } k = 2^{n-1}(2p-1) \quad (p, n \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

The matrix B is regular, that is, $c_B \supset c$ (where c is the space of all converging sequences) and $\lim_B x = \lim_k x_k$ for each $x \in c$. Erdös and Piranian showed that if C is any regular matrix such that $c_B \subset c_C$, then C is not row-finite. We know that c_B is a separable FK-space such that the FK-topology is a β-topology. Since $\lim_B \in (c_B)'$, by Proposition 1 there exists a matrix A such that $c_B \subset c_A$ and

$$\lim_B x = \lim_A x \text{ for all } x \in c_B.$$

Therefore A is a regular matrix, which implies that it is not row-finite. By Proposition 1, the FK-topology on c_B is not a φ-topology.

3. Monotone seminorms

Let $(E, \| \ \|)$ be a BK-space. The norm $\| \ \|$ is called *monotone* (see [7], 7.1.1) if

$$\|x\| = \sup_n \|x^{[n]}\|,$$

where $x^{[n]}$ is the n^{th} section of x defined by

$$x^{[n]} := (x_1, x_2, \dots, x_n, 0, \dots) = \sum_{k=1}^{n} x_k e^k \ (n \in \mathbb{N}).$$

If the norm $\| \ \|$ is monotone, then $\|x^{[n]}\| \leq \|x^{[m]}\|$ for each $m \geq n$. Any BK-space $(E, \| \ \|)$ that has a monotone norm also has AB, i.e., for each $x \in E$ the subset $\{x^{[n]} \mid n \in \mathbb{N}\}$ is bounded in $(E, \| \ \|)$. The converse is not always true, that is, there exist AB-BK-spaces that do not have a monotone norm. An example is the sum $bs + c_0$ (cf. [7], 10.3.7), where c_0 is the space of all sequences which converge to zero and

$$bs := \left\{ x \in \omega \mid \|x\|_{bs} := \sup_m \left| \sum_{k=1}^{m} x_k \right| < \infty \right\}.$$

It is well-known, that a BK-space $(E, \| \ \|)$ has a monotone norm if and only if $(E, \| \ \|)$ is an AB-BK-space that has a φ-topology (cf. [4], Lemma 2.1). Our aim is to demonstrate that an analogous statement holds for AB-FK-spaces and that in such a scenario every β-topology is also a φ-topology.

Let E be a sequence space. Then we call a seminorm $q : E \to \mathbb{R}$ *monotone* if

$$q(x) = \sup_m q(x^{[m]}) \text{ for each } x \in E.$$

If (E, τ_E) is an FK-space, then there exists a sequence $\{p_n \mid n \in \mathbb{N}\}$ of seminorms generating the FK-topology τ_E such that

$$p_n(x) \leq p_{n+1}(x) \text{ for each } x \in E \text{ and } n \in \mathbb{N}. \tag{2}$$

For the subsets $U_n := \{x \in E \mid p_n(x) \leq 1\}$ we get that $U_1 \supset U_2 \supset \dots$. Let

$$G_n := \left\{ f \in E' \mid \|f\|_{G_n} := \sup_{x \in U_n} |f(x)| < \infty \right\}$$

and

$$F_n := \left\{ u \in E^\beta \mid \|u\|_{F_n} := \sup_{x \in U_n,\ m \in \mathbb{N}} \left| \sum_{k=1}^m u_k x_k \right| < \infty \right\}$$

for each $n \in \mathbb{N}$. Then $(G_n, \| \ \|_{G_n})$ is a Banach space and $(F_n, \| \ \|_{F_n})$ is a BK-space for every $n \in \mathbb{N}$. We denote by B_{F_n} the unit ball of F_n. Note that

$$\sup_{u \in B_{F_n}} \left| \sum_{k=1}^\infty u_k x_k \right| = \sup_{u \in B_{F_n},\ m \in \mathbb{N}} \left| \sum_{k=1}^m u_k x_k \right| \text{ for each } x \in E. \tag{3}$$

If the seminorm p_n is monotone, then

$$\sup_{f \in U_n^0,\ m \in \mathbb{N}} \left| \sum_{k=1}^m x_k f(e^k) \right| = \sup_{f \in U_n^0 \cap \varphi} \left| \sum_{k=1}^\infty x_k f(e^k) \right| = \sup_{f \in U_n^0 \cap E^\beta} \left| \sum_{k=1}^\infty x_k f(e^k) \right|. \tag{4}$$

Proposition 2. *Let (E, τ_E) be an AB-FK-space. Then the following statements are equivalent.*

(a) *τ_E is generated by a sequence of monotone seminorms.*
(b) *τ_E is a φ-topology.*
(c) *τ_E is a β-topology.*

Proof. (a)$\Rightarrow$(b) Let (E, τ_E) be an AB-FK-space where τ_E is generated by a sequence $\{p_n \mid n \in \mathbb{N}\}$ of monotone seminorms. We may assume that (2) holds. For every monotone seminorm p_n we get (cf. (4))

$$p_n(x) = \sup_m p_n(x^{[m]}) = \sup_{f \in U_n^0,\ m \in \mathbb{N}} \left| \sum_{k=1}^m x_k f(e^k) \right| = \sup_{f \in U_n^0 \cap \varphi} \left| \sum_{k=1}^\infty x_k f(e^k) \right|$$

for each $x \in E$. Hence, p_n is φ-polar.

The implication (b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(a) Let (E, τ_E) be an AB-FK-space, where the β-topology τ_E is generated by a sequence $\{p_n \mid n \in \mathbb{N}\}$ of seminorms with the property (2). Because τ_E is a β-topology, we may assume that $(U_n^0 \cap E^\beta)^0 = (U_n^0)^0 = U_n$, and thus

$$p_n(x) = \sup_{u \in U_n^0 \cap E^\beta} \left| \sum_{k=1}^\infty x_k u_k \right| \ (x \in E,\ n \in \mathbb{N}).$$

Since (E, τ_E) is an AB-FK-space, $E^\beta \cap G_n$ is closed in the Banach space G_n for every $n \in \mathbb{N}$ (cf. [2], Theorem 3.4). Therefore $(E^\beta \cap G_n, \| \ \|_{G_n})$ is a BK-space for every $n \in \mathbb{N}$. Consequently, the sequences $(F_n)_{n \in \mathbb{N}}$ and $(E^\beta \cap G_n)_{n \in \mathbb{N}}$, both consisting of BK-spaces, determine the same sequence space E^β, that is,

$$E^\beta = \bigcup_{n \in \mathbb{N}} F_n = \bigcup_{n \in \mathbb{N}} (E^\beta \cap G_n).$$

Hence, the inductive sequences $(F_n)_{n \in \mathbb{N}}$ and $(E^\beta \cap G_n)_{n \in \mathbb{N}}$ are equivalent. Then for each $n \in \mathbb{N}$ there exist $i \in \mathbb{N}$ and $\rho_{n,i} > 0$, such that

$$U_n = (U_n^0 \cap E^\beta)^0 \supset \frac{1}{\rho_{n,i}} (B_{F_i})^0. \tag{5}$$

Conversely, from the inclusion $F_n \subset E^\beta \cap G_n$ we get that there exists $\mu_n > 0$ such that

$$(B_{F_n})^0 \supset \frac{1}{\mu_n} (U_n^0 \cap E^\beta)^0 = \frac{1}{\mu_n} U_n \ (n \in \mathbb{N}). \tag{6}$$

Let us define

$$q_i(x) := \sup_{u \in B_{F_i}} \left| \sum_{k=1}^\infty u_k x_k \right| = \sup_{u \in B_{F_i},\ m \in \mathbb{N}} \left| \sum_{k=1}^m u_k x_k \right| \ (x \in E)$$

(cf. (3)). Then

$$\sup_m q_i(x^{[m]}) = \sup_m \sup_{u \in B_{F_i},\ n \in \mathbb{N}} \left| \sum_{k=1}^n (u^{[m]})_k x_k \right| = \sup_{u \in B_{F_i},\ l \in \mathbb{N}} \left| \sum_{k=1}^l u_k x_k \right| = q_i(x)$$

for each $x \in E$ and $i \in \mathbb{N}$. So $\{q_i \mid i \in \mathbb{N}\}$ is a sequence of monotone seminorms. Since

$$\{x \in E \mid q_i(x) \leq 1\} = \left\{ x \in E \mid \sup_{u \in B_{F_i}} \left| \sum_{k=1}^\infty u_k x_k \right| \leq 1 \right\} = (B_{F_i})^0,$$

from (5) and (6) we see that the locally convex topology τ' generated by $\{q_i \mid i \in \mathbb{N}\}$ coincides with τ_E. Thus we have proved that the β-topology τ_E can be determined by a sequence of monotone seminorms. $\square$

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