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Regions of insensitivity for the variance of the estimator of a linear function of the mean parameters in the mixed linear model with type II constraints

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Abstract. The insensitivity regions for the variance of the estimator of a linear function of the mean parameters in a mixed linear model are already described in different sources. The aim of this paper is to outline one possible way of their utilization and to extend the concept of insensitivity to the estimators of a linear function of the mean parameters of the mixed linear model with type II constraints.

1. Introduction

Each measurement is characterized, among other parameters by its parameters of accuracy. If the actual values of the accuracy parameters are not known, their estimates need to be used when estimating the mean parameters, testing hypothesis about the mean parameters, or determining their confidence regions. The results of these procedures are (more or less) affected by using the estimated values of the accuracy parameters instead of the actual ones. We need to find some rules for the resolution whether the disparity between the result based on the actual values of the variance parameters and the result based on their estimated values is important. We propose to base such decision on the concept of insensitivity.

Foundations of sensitivity and insensitivity can be found in Chapter 6 of [7]. Regions of insensitivity for the variance of the estimator of a linear function of the mixed linear model parameters and for the confidence region of these parametrs can be found in [6]. Other important results regarding insensitivity can be found in [5], [8], [9], [1].

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Let us at first focus on the concept of the regions of insensitivity for the variance of the estimator of a linear function of the mean parameters in the mixed linear model without constraints.

2. Mixed linear model (without constraints)

Let $\mathbf{Y}$ be an n-dimensional random vector distribution of which belongs to a given class of distributions with parameters

$$\boldsymbol{\beta} \in \mathbb{R}^k$$

and

$$\boldsymbol{\theta} = (\theta_1, \theta_2 \dots, \theta_r) \in \boldsymbol{\Theta} \subset \mathbb{R}^r.$$

This class of distributions must satisfy two conditions:

$$\mathrm{E}\,(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta},$$

where $\mathbf{X}$ is a known matrix of $n \times k$ dimension, and

$$\mathrm{Var}\,(\mathbf{Y}) = \sum_{i=1}^{r} \theta_i \mathbf{V}_i = \boldsymbol{\Sigma}_{\boldsymbol{\theta}}, \tag{1}$$

where $\mathbf{V}_1, \dots, \mathbf{V}_r$ are known symmetrical matrices. Let us consider a regular model, i.e., $\mathbf{X}$ is of full column rank $r(\mathbf{X}) = k$ and $\boldsymbol{\Theta}$ is such an open subset of the Euclidean space $\mathbb{R}^r$ for which $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is a positive definite matrix for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. The variance parameters can be estimated via MINQUE (see [10]):

$$\widehat{\boldsymbol{\theta}} = \mathbf{S}^{-1}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+} \begin{pmatrix} \mathbf{Y}'\,(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+\,\mathbf{V}_1\,(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+\,\mathbf{Y} \\ \vdots \\ \mathbf{Y}'\,(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+\,\mathbf{V}_r\,(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+\,\mathbf{Y} \end{pmatrix}, \tag{2}$$

where $\mathbf{M_X}$ denotes the projection matrix on an orthogonal complement (in Euclidean sense) of the range space of matrix $\mathbf{X}$, $(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+$ denotes the Moore–Penrose generalized matrix inverse of the matrix $\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X}$ and $\mathbf{S}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X})^+}$ is a matrix with

$$\left\{\mathbf{S}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X})^+}\right\}_{i,j} = \mathrm{Tr}\left[\mathbf{V}_i(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X})^+\mathbf{V}_j(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X})^+\right]$$

on its (i,j)-th position, and $(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X})^+$ can be obtained in the form (cf. [6])

$$(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X})^+ = {\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}}^{-1} - {\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}}^{-1}\mathbf{X}(\mathbf{X}'{\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}}^{-1}\mathbf{X})^{-1}\mathbf{X}'{\boldsymbol{\Sigma}_{\boldsymbol{\theta_0}}}^{-1}.$$

We need a suitable choice of a starting value of the variance parameters $\boldsymbol{\theta}_0$, which enters (2). The $\boldsymbol{\theta}$ estimate $\widehat{\boldsymbol{\theta}}$ is a result of an iterative procedure based on (2). This estimate can be further used to estimate the first order parameters. The $\widehat{\boldsymbol{\theta}}$-LBLUE ($\widehat{\boldsymbol{\theta}}$-locally best linear unbiased estimator) of $\boldsymbol{\beta}$ is (cf. e.g. [6])

$$\widehat{\boldsymbol{\beta}}\left(\widehat{\boldsymbol{\theta}}\right) = \left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}{}^{-1}\mathbf{Y}. \tag{3}$$

Here $\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}$ denotes a matrix of type (1) based on $\widehat{\boldsymbol{\theta}}$ rather than $\boldsymbol{\theta}$.

Let us now discuss the problem outlined in Section 2 in more detail. What is the discrepancy between the $\boldsymbol{\beta}$-estimate obtained by using $\boldsymbol{\theta}^*$, the actual value of $\boldsymbol{\theta}$ (if it is known), and the estimate based on (3) with the input value $\widehat{\boldsymbol{\theta}}$ obtained by the iterative procedure (2) (if the actual value of $\boldsymbol{\theta}$ is unknown)? Can we get some criterion which enables to decide whether these two estimates are close (although we are not able to determine $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}^*)$ if not knowing $\boldsymbol{\theta}^*$)?

Another question arises. Do we have to estimate $\boldsymbol{\theta}$ to use the result as an input value of (2) even if we are not interested in the estimates of the variance parameters otherwise? If we have some strict rules to review the quality of the $\boldsymbol{\beta}$-estimates (with regard to their closeness to the estimate based on the actual value $\boldsymbol{\theta}^*$), we will have a possibility to ground the $\boldsymbol{\beta}$-estimate on some suitably chosen initial values of the second order parameters $\boldsymbol{\theta}_0$ and check whether such an estimate is suited to our rules. This would allow us to omit the iterative procedure of $\boldsymbol{\theta}$-estimation.

In what follows we mostly use the estimates based on an initial value $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$. These are determined according to (2) using $\boldsymbol{\theta}_0$ in place of $\boldsymbol{\theta}$ and (6) using $\boldsymbol{\theta}_0$ instead of $\widehat{\boldsymbol{\theta}}$. These estimates (and appropriate estimators as well) are denoted by $\widehat{\boldsymbol{\theta}}\,(\boldsymbol{\theta}_0)$ and $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}_0)$. If $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ is under consideration the variance-covariance matrices of the estimators (2) and (3) are

$$\mathrm{Var}_{\boldsymbol{\theta}_0}\left(\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta}_0)\right) = 2\mathbf{S}^{-1}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+}$$

and

$$\mathrm{Var}_{\boldsymbol{\theta}_0}\left(\widehat{\boldsymbol{\beta}}(\boldsymbol{\theta}_0)\right) = \left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}.$$

3. Region of insensitivity for the variance of the estimator of a linear function of the mean parameters in the mixed linear model without constraints

Let us consider a mixed linear model from the previous section. Suppose that the actual value of the variance parameters $\boldsymbol{\theta}$ is $\boldsymbol{\theta}^* \in \boldsymbol{\Theta} \subset \mathbb{R}^r$. In what follows this value is a priori not known. Let us choose an initial value $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$ and vector $\mathbf{h} \in \mathbb{R}^k$. Let us compare the estimators of a linear function $\mathbf{h}'\boldsymbol{\beta}$ that we get using $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}_0$ as the input value for (3). In case $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ we have

$$\widehat{\mathbf{h}'\boldsymbol{\beta}}\,(\boldsymbol{\theta}^*) = \mathbf{h}'\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{Y}, \tag{4}$$

where $\mathbf{\Sigma}_{\boldsymbol{\theta}*}$ denotes a variance-covariance matrix of type (1) with $\boldsymbol{\theta}^*$ instead of $\boldsymbol{\theta}$. Variance of (9) when $\boldsymbol{\theta}_0$ is under consideration is

$$\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}^*\right)\right] = \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{X} \times \left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{h}. \tag{5}$$

The estimator of the same linear function $\mathbf{h}'\boldsymbol{\beta}$ using the initial value $\boldsymbol{\theta}_0$ is

$$\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}_0\right) = \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{Y}. \tag{6}$$

Here $\mathbf{\Sigma}_{\boldsymbol{\theta}_0}$ denotes a matrix of type (1) with $\boldsymbol{\theta}_0$ instead of $\boldsymbol{\theta}$. Variance of (6) is

$$\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}_0\right)\right] = \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X} \times \left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{h}. \tag{7}$$

Remark 1. It is of course possible to simplify (7) into the form

$$\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}_0\right)\right] = \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{h}$$

but (7) is more suitable for the following consideration.

As we can see, the estimators (4) and (6) differ about the vectors multiplying $\mathbf{Y}$. This vector is $\mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}*}{}^{-1}$ in case of (4) and $\mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}$ in case of (6). The same vectors appear also in (5) and (7) once in the same form and once transposed. So if these vectors are different then not just the estimates (4) and (6) but also the variances (5) and (7) differ. It means the difference between the variances (5) and (7) can be a disparity criterion for the estimates. Following definition proceeds from this idea.

Definition 1. Let for the mixed linear model, defined above, $\boldsymbol{\theta}_0 \in \mathbf{\Theta}$, $\mathbf{h} \in \mathbb{R}^k$ be given and let $\varepsilon > 0$. The set $\mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0}$ is called *the region of insensitivity for the variance of the estimator of a linear function* $\mathbf{h}'\boldsymbol{\beta}$ if the set $\mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0}$ consists of all points $\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \delta\boldsymbol{\theta}$ (where $\delta\boldsymbol{\theta}$ denotes a shift from the initial point $\boldsymbol{\theta}_0$) satisfying

$$\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta} \in \mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0} \Rightarrow \sqrt{\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta})\right]} \leq (1+\varepsilon)\sqrt{\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\boldsymbol{\theta}_0)\right]}. \tag{8}$$

The following theorem can be found in [7].

Theorem 1. *Suppose that* $\boldsymbol{\theta}_0 \in \mathbf{\Theta}$, $\mathbf{h} \in \mathbb{R}^k$ *and* $\varepsilon > 0$ *are given. The region of insensitivity for the variance of the estimator of a linear function of the mean parameters of the mixed linear model is a set*

$$\mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0} = \{\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta} : \delta\boldsymbol{\theta}'\mathbf{W}_{\mathbf{h}}\delta\boldsymbol{\theta} \leq (2\varepsilon + \varepsilon^2)\mathrm{Var}_{\boldsymbol{\theta}}[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\boldsymbol{\theta})]\}, \tag{9}$$

here

$$\mathbf{W_h} = \begin{pmatrix} \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}{}^{-1}\mathbf{V}_1 \\ \vdots \\ \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}{}^{-1}\mathbf{V}_r \end{pmatrix} \left(\mathbf{M_X}\mathbf{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{M_X}\right)^{+} \times \begin{pmatrix} \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}{}^{-1}\mathbf{V}_1 \\ \vdots \\ \mathbf{h}'\left(\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{\boldsymbol{\theta_0}}{}^{-1}\mathbf{V}_r \end{pmatrix}'.$$

Lemma 1 (cf. [7])**.** *Let us consider the mixed linear model described in Section 3. Let* $\mathbf{h} \in \mathbb{R}^k$*,* $\boldsymbol{\theta}_0 \in \mathbf{\Theta}$*. Then* $\boldsymbol{\theta}_0$ *is orthogonal to the range space* $\mathcal{M}(\mathbf{W_h})$ *of matrix* $\mathbf{W_h}$ *from Theorem 1.*

Corollary 1. *In compliance with Lemma 1, matrix* $\mathbf{W_h}$ *is always singular.*

Remark 2. In Definition 1 and Theorem 1, a linear function $\mathbf{h}'\boldsymbol{\beta}$ is used instead of the entire vector of the mean parameters $\boldsymbol{\beta}$. The choice of $\mathbf{h} \in \mathbb{R}^k$ can result from some requirements of a particular problem. When there is no such choice of $\mathbf{h}$ it is reasonable to use $\mathbf{h} = \mathbf{e}_i,\ i = 1, 2, \dots, k$. In this case, the region of insensitivity determined according to (9) is a set of all admissible input values $\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \delta\boldsymbol{\theta}$ which causes at most the ε-multiple increase of the standard deviation of the estimator of β_i. The intersection

$$\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0} = \bigcap_{i=1}^{k} \mathcal{N}_{\mathbf{e}_i'\boldsymbol{\beta},\boldsymbol{\theta}_0} \tag{10}$$

gives a set of all input values of the variance parameters, that for all the components $\beta_i,\ i = 1, 2, \dots, k$, keeps the standard deviation of the estimator of β_i less than the $(1+\varepsilon)$-multiple of the standard deviation of the estimator of β_i based on $\boldsymbol{\theta}_0$.

Remark 3. The region of insensitivity given in Theorem 1 is based on an approximation of $\mathbf{h}'\widehat{\boldsymbol{\beta}}\left(\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta}\right)$ by a linear Taylor polynomial, that is used in (8). Such a linearization can break the validity of (8). Present numerical studies indicate an adequacy of a linear approximation. (Cf. e.g. [2]).

With respect to our motivation for the regions of insensitivity for the variance of the estimator of a linear function of the mean parameters the ideal location of the actual value $\boldsymbol{\theta}^*$ of the variance parameters is inside the region of insensitivity. If $\boldsymbol{\theta}^* \in \mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0}$, $\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}^*\right)\right]$ is close to $\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}_0\right)\right]$, and consequently $\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}^*\right)$ is close to $\widehat{\mathbf{h}'\boldsymbol{\beta}}\left(\boldsymbol{\theta}_0\right)$. As we do not know $\boldsymbol{\theta}^*$, we are not able to decide whether it is an element of $\mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0}$ or not. What we can do is to find a confidence region $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}}$ which covers the actual value $\boldsymbol{\theta}^*$ with a given probability close to 1. Subsequently we can transform our requirement into $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}} \subset \mathcal{N}_{\mathbf{h}'\boldsymbol{\beta},\boldsymbol{\theta}_0}$. Or more precisely $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}} \subset \mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0}$. Here $\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0}$ is given by (10).

Remark 4. We need some estimate of $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_r)'$ to determine the confidence region for the variance parameters. Let us consider $\widehat{\boldsymbol{\theta}}\,(\boldsymbol{\theta}_0)$ is the estimate computed according to (2) using an initial value $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$ instead of $\boldsymbol{\theta}$. One way of determining a confidence region is based on Chebyshev's and Bonferroni's inequalities. Let us denote $\boldsymbol{\theta}^* = (\theta_1^*, \theta_2^*, \dots, \theta_r^*)'$ the actual value of the variance parameters. According to Chebyshev's inequality

$$\forall c > 0$$

$$P\left\{\left|\widehat{\theta}_i\,(\boldsymbol{\theta}_0) - \theta_i^*\right| \leq c\sqrt{\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\theta}_i\,(\boldsymbol{\theta}_0)\right]}\right\} \geq 1 - \frac{1}{c^2},\ i = 1, 2, \dots, r.$$

Using (7) and Bonferroni's inequality (cf. [4]), we get

$$P\left\{\forall i = 1, 2, \dots, r : \left|\widehat{\theta}_i\,(\boldsymbol{\theta}_0) - \theta_i^*\right| \leq c\sqrt{\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\theta}_i\,(\boldsymbol{\theta}_0)\right]}\right\} \geq 1 - \frac{r}{c^2}.$$

We are looking for a confidence region – a set which covers the actual value $\boldsymbol{\theta}^*$ with the probability of $1-\alpha$, $\alpha \in (0,1)$. Thus we need $1 - \alpha = 1 - \frac{r}{c^2}$, which implies $c = \sqrt{\frac{r}{\alpha}}$. The resultant confidence region is a set

$$\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}} = \Big\{\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_r)' :$$

$$\left|\widehat{\theta}_i\,(\boldsymbol{\theta}_0) - \theta_i\right| \leq \sqrt{\frac{r}{\alpha}\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\widehat{\theta}_i\,(\boldsymbol{\theta}_0)\right]},\ \forall i = 1, 2, \dots, r\Big\}. \tag{11}$$

The next section describes the utility of the regions.

4. The utilization of the regions of insensitivity

The reasons for a need of insensitivity regions were given by the assessment of the quality of the estimates. Suppose that we have a mixed linear model from Section 3 with both mean and variance parameters unknown. Let us choose some initial value $\boldsymbol{\theta}_0$ of the variance parameters and investigate the quality of mean parameter estimate $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}_0)$. Let us determine the region of insensitivity $\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0}$ according to (9) and (10) and the confidence region $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}}$ according to (11). There are two possible cases of the relative position of the insensitivity region $\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0}$ and the confidence region $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}}$.

(1) $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}} \subset \mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0}$. A relative position like this is advantageous. The estimate $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}_0)$ can be regarded as a quality estimate of the parameter $\boldsymbol{\beta}$. $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}_0)$ is comparable to $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}^*)$ which would be achieved if we knew the actual value $\boldsymbol{\theta}^*$ of the variance parameters.

(2) $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}} \nsubseteq \mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_0}$. This time, $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}_0)$ can be distinct from the estimate based on the actual value $\boldsymbol{\theta}^*$. One way how to improve the estimate of $\boldsymbol{\beta}$ is to start with the iterative procedure (2). Let us suppose that this iterative procedure stops after m steps. Then $\widehat{\boldsymbol{\theta}}\,(\boldsymbol{\theta}_m)$ is used as an estimate and $\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_m}$ and $\mathcal{E}_{\boldsymbol{\theta}_m,\boldsymbol{\theta}}$ are determined according to (11), (9) and (10), using $\boldsymbol{\theta}_m$ in place of $\boldsymbol{\theta}_0$. If the updated confidence region $\mathcal{E}_{\boldsymbol{\theta}_m,\boldsymbol{\theta}}$ is a subset of the new region of insensitivity $\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_m}$ we achieved our purpose: $\widehat{\boldsymbol{\beta}}\,(\boldsymbol{\theta}_m)$ can be understood as a reliable estimate of the parameter $\boldsymbol{\beta}$. If $\mathcal{E}_{\boldsymbol{\theta}_m,\boldsymbol{\theta}}$ is still not imbedded into the region of insensitivity $\mathcal{N}_{\boldsymbol{\beta},\boldsymbol{\theta}_m}$ we can try to change the design of the experiment, if possible, and check the relative position of the confidence and insensitivity region again.

5. Mixed linear model with type II constraints

There are many experiments leading to regression models with type II constraints. Let us have a look at such a model and estimation of its parameters. Let us consider an n-dimensional random vector $\mathbf{Y}$ which satisfies

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}), \tag{12}$$

where $\mathbf{X}$ is a known design matrix of $n \times k_1$ dimension and of a full column rank k_1, $\boldsymbol{\beta}_1 \in \mathbb{R}^{k_1}$ is an unknown parameter, and the variance-covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is the same as the one in Section 3. At the same time, we have the following constraints on the mean parameter $\boldsymbol{\beta}_1$:

$$\mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 + \mathbf{b} = \mathbf{0}, \tag{13}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are given $q \times k_1$ and $q \times k_2$ matrices satisfying the following conditions:

$$r(\mathbf{B}_1, \mathbf{B}_2) = q,$$
$$r(\mathbf{B}_2) = k_2,$$
$$k_2 < q < (k_1 + k_2)\,,$$

and $\mathbf{b}$ is a given q-dimensional vector, $r(\mathbf{A})$ denotes the rank of matrix $\mathbf{A}$. $\boldsymbol{\beta}_1$ is an indirectly observable parameter from the model (12), and $\boldsymbol{\beta}_2$ is also unknown and can be estimated on basis of (13) after an estimate of $\boldsymbol{\beta}_1$ is known. Equations (12), (13) describe a mixed linear model with type II constraints.

The aim is to estimate the unknown parameters $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and $\boldsymbol{\theta}$. Let us transform (12), (13) to an equivalent model without constraints (cf. [6]). We need to find matrices $\mathbf{K}_1$ of $k_1 \times (k_1 + k_2 - q)$ dimension and $\mathbf{K}_2$ of $k_2 \times (k_1 + k_2 - q)$ dimension satisfying condition

$$(\mathbf{B}_1, \mathbf{B}_2) \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix} = \mathbf{0}_{q\times(k_1+k_2-q)}. \tag{14}$$

With respect to (14) the range space of matrix $\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}$ is the same as the range space of matrix $\mathbf{M}_{\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}}$, where $\mathbf{B}_2'\mathbf{B}_2$ is a regular matrix (as we suppose $\mathbf{B}_2$ to be of a full column rank k_2) which means $\mathbf{K}_2$ can be written in form

$$\mathbf{K}_2 = -(\mathbf{B}_2' \cdot \mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{B}_1\mathbf{K}_1. \tag{15}$$

Using (14) and (15) we get

$$\mathbf{0} = \mathbf{B}_1\mathbf{K}_1 - \mathbf{B}_2(\mathbf{B}_2' \cdot \mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{B}_1\mathbf{K}_1 = [\mathbf{I} - \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2']\mathbf{B}_1\mathbf{K}_1 = \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{K}_1.$$

Hence the range space of $\mathbf{K}_1$ is equal to the range space of $\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}$.

The following equality results from the assumptions made for (12) and (13):

$$r(\mathbf{K}_1) = (k_1 + k_2 - q).$$

Let $\boldsymbol{\beta}_1^{(0)}$, $\boldsymbol{\beta}_2^{(0)}$ be any values of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ satisfying condition (13). Then all remaining vectors $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ satisfying this condition can be written in the form

$$\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}_1^{(0)} \\ \boldsymbol{\beta}_2^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}\gamma,\ \gamma \in \mathbb{R}^{k_1+k_2-q}. \tag{16}$$

Mixed linear model (12) with type II constraints (13) is equivalent to the following mixed linear model without constraints:

$$(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_1^{(0)}) \sim_n (\mathbf{X}\mathbf{K}_1\gamma, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}). \tag{17}$$

A MINQUE of $\boldsymbol{\theta}$ is, according to (2) and (17),

$$\widehat{\boldsymbol{\theta}} = \mathbf{S}^{-1}_{(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+} \times$$

$$\begin{pmatrix} (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_1^{(0)})'\,(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+\,\mathbf{V}_1\,(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+\,(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_1^{(0)}) \\ \vdots \\ (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_1^{(0)})'\,(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+\,\mathbf{V}_r\,(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+\,(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_1^{(0)}) \end{pmatrix},$$

where

$$\left\{\mathbf{S}_{(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+}\right\}_{i,j} =$$

$$\mathrm{Tr}\left[\mathbf{V}_i\,(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+\,\mathbf{V}_j\,(\mathbf{M}_{\mathbf{XK_1}}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M}_{\mathbf{XK_1}})^+\right],$$

$$i, j = 1, 2, \ldots, r.$$

A $\widehat{\boldsymbol{\theta}}$-LBLUE of the new parameter $\boldsymbol{\gamma}$ is, according to (3) and (17),

$$\widehat{\gamma}\left(\widehat{\boldsymbol{\theta}}\right) = (\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_1^{(0)}). \tag{18}$$

Hence because of (16)

$$\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) \\ \widehat{\boldsymbol{\beta}}_2\left(\widehat{\boldsymbol{\theta}}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}_1^{(0)} \\ \boldsymbol{\beta}_2^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix} \widehat{\gamma}. \tag{19}$$

In order to derive the region of insensitivity for the variance of the estimator of a linear function of $\boldsymbol{\beta}_2$, let us modify the estimator of this parameter. According to (18), (19) and (15) we have

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_2\left(\widehat{\boldsymbol{\theta}}\right) &= \boldsymbol{\beta}_2^{(0)} + \mathbf{K}_2\widehat{\gamma}\left(\widehat{\boldsymbol{\theta}}\right) = \boldsymbol{\beta}_2^{(0)} - \left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\mathbf{B}_1\mathbf{K}_1\widehat{\gamma}\left(\widehat{\boldsymbol{\theta}}\right) \\
&= \boldsymbol{\beta}_2^{(0)} - \left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\mathbf{B}_1\left[\widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) - \boldsymbol{\beta}_1^{(0)}\right] \\
&= \boldsymbol{\beta}_2^{(0)} - \left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) + \left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\mathbf{B}_1\boldsymbol{\beta}_1^{(0)} \\
&= \boldsymbol{\beta}_2^{(0)} - \left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) + \left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\left(-\mathbf{B}_2\boldsymbol{\beta}_2^{(0)} - \mathbf{b}\right) \\
&= -\left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2'\left[\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) + \mathbf{b}\right].
\end{aligned} \tag{20}$$

According to [3] (p. 337, Lemma A.7.9)

$$\begin{aligned}
\left(\mathbf{B}_2'\right)^-_{m\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1'\right]} &= \left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2 \\
&\times\left\{\mathbf{B}_2'\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2\right\}^{-1}.
\end{aligned} \tag{21}$$

This means we have

$$\left\{\left(\mathbf{B}_2'\right)^-_{m\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1'\right]}\right\}'\mathbf{B}_2 = \mathbf{I},$$

hence

$$\left(\mathbf{B}_2'\mathbf{B}_2\right)^{-1}\mathbf{B}_2' = \left\{\left(\mathbf{B}_2'\right)^-_{m\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1'\right]}\right\}'.$$

In accordance with (20) and (21) we can write

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_2\left(\widehat{\boldsymbol{\theta}}\right) &= -\left[\left(\mathbf{B}_2'\right)^-_{m\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\left(\widehat{\boldsymbol{\theta}}\right)\mathbf{B}_1'\right]}\right]'\left(\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) + \mathbf{b}\right) \\
&= -\left\{\mathbf{B}_2'\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1 + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2\right\}^{-1}\mathbf{B}_2' \\
&\times\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1 + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\left[\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1\left(\widehat{\boldsymbol{\theta}}\right) + \mathbf{b}\right].
\end{aligned}$$

6. Region of insensitivity for the variance of the estimator of a linear function of the mean parameters in the mixed linear model with type II constraints

Let us consider model (12) together with constraints (13) from the previous section. Let us derive the region of insensitivity for the variance of the estimator of a linear function of $\boldsymbol{\beta}_1$ at first. We can use the insensitivity region (9) from Theorem 1 and the transformation of (12), (13) to (17). Let $\mathbf{h} \in \mathbb{R}^{k_1}$ be given. For the appropriate linear combination of $\boldsymbol{\beta}_1$ components we have, according to (16),

$$\mathbf{h}'\boldsymbol{\beta}_1 = \mathbf{h}'\boldsymbol{\beta}_1^{(0)} + \mathbf{h}'\mathbf{K}_1\gamma.$$

Let us choose a fixed $\boldsymbol{\theta}_0 \in \mathbb{R}^r$ and denote $\mathbf{K}_1'\mathbf{h} = \mathbf{t}$. Then, using (19) and the matrix $\mathbf{W_h}$ from Theorem 1, we get for $\mathbf{t}'\boldsymbol{\gamma}$

$$\{\mathbf{W}_{\mathbf{t},\boldsymbol{\gamma}}\}_{i,j} = \mathbf{t}'(\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{V}_i(\mathbf{M}_{\mathbf{X}\mathbf{K}_1}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M}_{\mathbf{X}\mathbf{K}_1})^{+}\mathbf{V}_j$$
$$\times\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1(\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{t}$$
$$= \mathbf{h}'\mathbf{K}_1(\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{V}_i(\mathbf{M}_{\mathbf{X}\mathbf{K}_1}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M}_{\mathbf{X}\mathbf{K}_1})^{+}\mathbf{V}_j$$
$$\times\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1(\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{K}_1'\mathbf{h} = \{\mathbf{W}_{\mathbf{h},\boldsymbol{\beta}_1}\}_{i,j},\ i,j = 1,2,\ldots,r.$$

An analogy of Lemma 1 is valid in case of the mixed linear model with type II constraints.

Lemma 2. *Let model* (12) *with constraints* (13) *be given. Let* $\mathbf{W}_{\mathbf{h},\boldsymbol{\beta}_1}$ *be a matrix with*

$$\{\mathbf{W}_{\mathbf{h},\boldsymbol{\beta}_1}\}_{i,j} = \mathbf{h}'\mathbf{K}_1(\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{V}_i$$
$$\times(\mathbf{M}_{\mathbf{X}\mathbf{K}_1}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M}_{\mathbf{X}\mathbf{K}_1})^{+}\mathbf{V}_j\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1(\mathbf{K}_1'\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\mathbf{K}_1)^{-1}\mathbf{K}_1'\mathbf{h},$$

on its (i,j)*-th position. Then* $\boldsymbol{\theta}_0$ *is orthogonal to the range space* $\mathcal{M}\left(\mathbf{W}_{\mathbf{h},\boldsymbol{\beta}_1}\right)$.

Proof. The proof of this assertion is analogous to the proof of the Lemma 1. We just need to use the matrix product $\mathbf{X}\mathbf{K}_1$ instead of $\mathbf{X}$.

□

According to [3] (p. 198)

$$\mathrm{Var}_{\boldsymbol{\theta}_0}\widehat{\beta}_1\left(\widehat{\boldsymbol{\theta}}\right) = \left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1} - \left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1'$$
$$\times\left[\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right]^{+}\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}. \qquad (22)$$

Let us denote this variance-covariance matrix by $\Sigma(\widehat{\boldsymbol{\beta}}_1)$.

Theorem 2. *Let us consider the mixed linear model* (12) *with type II constraints* (13). *Let* $\mathbf{h} \in \mathbb{R}^{k_1}$ *and* $\boldsymbol{\theta}_0 \in \mathbb{R}^r$ *be given. Then for every* $\varepsilon > 0$ *the region of insensitivity for the variance of the estimator of a linear function* $\mathbf{h}\boldsymbol{\beta}_1$ *takes the form*

$$\mathcal{N}_{\mathbf{h}'\boldsymbol{\beta}_1,\boldsymbol{\theta}_0} = \left\{\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta} : \delta\boldsymbol{\theta}'\mathbf{W}_{\mathbf{h},\boldsymbol{\beta}_1}\delta\boldsymbol{\theta} \le \left(2\varepsilon + \varepsilon^2\right)\sqrt{\mathbf{h}'\Sigma(\widehat{\boldsymbol{\beta}}_1)\mathbf{h}}\right\}.$$

Proof. This theorem is a consequence of Theorem 1, transformation of (12) and (13) to (17) and the form of the variance-covariance matrix (22). □

In some cases, an estimate of $\boldsymbol{\beta}_2$ or some of its components is needed. Then the region of insensitivity for the variance of the estimator of a linear function of $\boldsymbol{\beta}_2$ may be useful.

Theorem 3. *Let us consider a mixed linear model* (12) *with type II constraints* (13). *Let* $\mathbf{h}_2 \in \mathbb{R}^{k_2}$ *and* $\boldsymbol{\theta}_0 \in \mathbb{R}^r$ *be given. Let us denote*

- $\mathbf{S}_{1i} = \left\{[\mathbf{B}_2'\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right])^{-1}\mathbf{B}_2\right\}^{-1}\mathbf{B}_2'$
  $\times\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{V}_i$
  $\times\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1'\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2$
- $\mathbf{S}_{2i} = \left\{\mathbf{B}_2'\left[\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2\right\}^{-1}\mathbf{B}_2'$
  $\times\left[(\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_1\left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}$
  $\times\mathbf{V}_i\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}{}^{-1}$
- $\mathbf{t}$ ... *vector having* $\operatorname{cov}_{\boldsymbol{\theta}_0}\left[\mathbf{h}_2'\widehat{\widehat{\boldsymbol{\beta}}}_2(\boldsymbol{\theta}_0), \mathbf{h}_2'\mathbf{S}_{1i}\widehat{\widehat{\boldsymbol{\beta}}}_2(\boldsymbol{\theta}_0)\right]$, *on its i-th position*
- $\mathbf{T}$ ... *matrix having* $\operatorname{cov}_{\boldsymbol{\theta}_0}\left[\mathbf{h}_2'\mathbf{S}_{1i}\widehat{\widehat{\boldsymbol{\beta}}}_2(\boldsymbol{\theta}_0), \mathbf{h}_2'\mathbf{S}_{1j}\widehat{\widehat{\boldsymbol{\beta}}}_2(\boldsymbol{\theta}_0)\right]$ *on its* (i,j)-*position*
- $\mathbf{U}$ ... *matrix having* $\operatorname{cov}_{\boldsymbol{\theta}_0}(\mathbf{h}_2'\mathbf{S}_{2i}\mathbf{v}, \mathbf{h}_2'\mathbf{S}_{2j}\mathbf{v})$ *on its* (i,j)*-position.*

Let $\mathbf{t} \in \mathcal{M}(\mathbf{T}+\mathbf{U})$, *the range space of* $\mathbf{T}+\mathbf{U}$. *Then for every* $\varepsilon > 0$ *the region of insensitivity for the variance of the estimator of a linear function* $\mathbf{h}_2'\boldsymbol{\beta}_2$ *is of the following form:*

$$\mathcal{N}_{\mathbf{h}_2'\boldsymbol{\beta}_2,\boldsymbol{\theta}_0} = \left\{\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta} : [\delta\boldsymbol{\theta} + (\mathbf{T}+\mathbf{U})^+\mathbf{t}]'(\mathbf{T}+\mathbf{U})[\delta\boldsymbol{\theta} + (\mathbf{T}+\mathbf{U})^+\mathbf{t}]\right.$$
$$\left.\le \mathbf{t}'(\mathbf{T}+\mathbf{U})^+\mathbf{t} + (2\varepsilon+\varepsilon^2)\operatorname{Var}_{\boldsymbol{\theta}_0}[\mathbf{h}_2'\widehat{\widehat{\boldsymbol{\beta}}}_2(\boldsymbol{\theta}_0)]\right\}.$$

Proof. Derivation of this insensitivity region is time-consuming. Its procedure is similar to the proof of Theorem 1 which can be found in [10]. □

The utilization of the insensitivity regions from Theorem 2 and Theorem 3 is the same as described in Section 4. Review of the relative position of the insensitivity region, and the confidence region $\mathcal{E}_{\boldsymbol{\theta}_0,\boldsymbol{\theta}}$, shows whether the estimates $\widehat{\boldsymbol{\beta}}_1(\boldsymbol{\theta}_0)$, $\widehat{\boldsymbol{\beta}}_2(\boldsymbol{\theta}_0)$, respectively, are comparable to $\widehat{\boldsymbol{\beta}}_1(\boldsymbol{\theta}^*)$ and $\widehat{\boldsymbol{\beta}}_2(\boldsymbol{\theta}^*)$.

References

[1] Bohacova, H. (2008), *Insensitivity region for variance components in general linear model*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **47**, 7–22.

[2] Bohacova, H. and Heckenbergerova, J. (2008), *Properties of the insensitivity region for a linear function of the fixed effects parameters in the mixed linear model*, Forum Statisticum Slovacum **7**, 7–12.

[3] Fiserova, E., Kubacek, L., and Kunderova, P, (2007), *Linear Statistical Models: Regularity and Singularities*, Academia, Prague.

[4] Humak, K. M. S. (1977), *Statistischen Methoden der Modellbildung. Band I. Statistische Inferenz für lineare Parameter*, Akademie-Verlag, Berlin.

[5] Kubacek, L. and Kubackova, L. (1999), *Unified approach to determining nonsensitiveness regions*, Tatra Mt. Math. Publ. **17**, 121–128.

[6] Kubacek, L. and Kubackova, L. (2000), *Statistics and Metrology*, Palacký University, Olomouc. (in Czech)

[7] Kubacek, L. et al. (1995), *Statistical Models with Linear Structures*, Veda, Bratislava.

[8] Kubacek, L. et al. (1998), *How the design of an experiment influences the nonsensitiveness regions in models with variance components*, Appl. Math. (Warsaw) **43**, 439–460.

[9] Lesanska, E. (2001), *Insensitivity regions for estimators of mean value parameters in mixed models with constraints*, Tatra Mt. Math. Publ. **22**, 37–39.

[10] Rao, C. R. and Kleffe, J. (1988), *Estimation of Variance Components and Application*, North-Holland, Amsterdam – New York – Oxford – Tokyo.

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