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Admissible regions for neglecting parameters in linear statistical models

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Abstract. Large number of parameters in linear statistical models can make their interpretation or subsequent handling difficult. The aim of the paper is to find admissible regions for neglecting a chosen group of parameters.

Introduction

Large number of parameters in linear statistical models can make their interpretation or subsequent handling difficult. For the sake of simplicity, let the parameters which are to be neglected be called as nuisance parameters.

There are several approaches to nuisance parameters. The mostly commonly used approach is based on the model fitting criteria. However there are several others; for more details see [2], [4].

The aim of the paper is to find admissible region for neglecting a chosen group of nuisance parameters, i.e. the set of such values of nuisance parameters which can be neglected.

1. Notation and preliminaries

The original (true) model is denoted as

$$\mathbf{Y} \sim_n \left[(\mathbf{X}, \mathbf{S}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\kappa} \end{pmatrix}, \sigma^2 \mathbf{I} \right],\ \boldsymbol{\beta} \in R^k,\ \boldsymbol{\kappa} \in R^l, \tag{1}$$

where the n-dimensional random vector $\mathbf{Y}$ (observation vector) has the mean value $E(\mathbf{Y})$ equal to $(\mathbf{X}, \mathbf{S}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\kappa} \end{pmatrix}$ and its covariance matrix $\mathrm{Var}(\mathbf{Y})$ is equal to $\sigma^2\mathbf{I}$; $\mathbf{I}$ is the identity matrix and the parameter σ^2 is assumed to be known. The $n \times k$ matrix $\mathbf{X}$ is given and its rank $r(\mathbf{X})$ is equal to $k \leq n$.

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The k-dimensional vector parameter $\boldsymbol{\beta} \in R^k$ (k-dimensional linear vector space) is unknown. The symbol $\boldsymbol{\kappa}$ means the l-dimensional vector parameter which is to be neglected; $\mathbf{S}$ is the $n \times l$ known matrix and due to regularity $r(\mathbf{X}, \mathbf{S}) = k + l \leq n$.

The underparametrized model is

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}),\ \boldsymbol{\beta} \in R^k,\ r(\mathbf{X}) = k \leq n. \tag{2}$$

Let us denote $\mathbf{M}_X = \mathbf{I} - \mathbf{X}\mathbf{X}^+$, where $\mathbf{X}^+$ is the Moore–Penrose generalized inverse of $\mathbf{X}$ (in more detail see [5]).

Notation $\mathbf{A} \leq_L \mathbf{B}$ means that for two positive semidefinite (p.s.d.) matrices $\mathbf{A}$ and $\mathbf{B}$ it holds that $\mathbf{B} - \mathbf{A}$ is p.s.d.

We shall also use the following notation:

$\mathcal{M}(\mathbf{A}_{m,n}) = \{\mathbf{A}\mathbf{u} : \mathbf{u} \in R^n\}$,

$\widehat{\boldsymbol{\beta}}_{true}$ means BLUE in the true model,

$\widehat{\boldsymbol{\beta}}_{under}$ denotes BLUE in the underparametrized model,

$E_{true}(\widehat{\boldsymbol{\beta}}_{under})$ is the mean value under the true model of the BLUE in the undeparameterized model,

$\mathbf{C} = \mathbf{X}'\mathbf{X}$,

$\mathbf{A}^-_{m(N)}$ denotes minimum $\mathbf{N}$-seminorm generalized inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A}$ and $\mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A} = \left[\mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A}\right]'$ ($\mathbf{N}$ is at least p.s.d.); in more detail see [5],

$\chi^2_f(0; 1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with f degrees of freedom.

Lemma 1.1. *The BLUE $\widehat{\boldsymbol{\beta}}_{true}$ of the parameter $\boldsymbol{\beta}$ in the true model (1) is*

$$\widehat{\boldsymbol{\beta}}_{true} = \widehat{\boldsymbol{\beta}}_{under} - \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\left[\mathbf{S}'\mathbf{M}_X\mathbf{S}\right]^{-1}\mathbf{S}'(\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_{under}),$$

where $\widehat{\boldsymbol{\beta}}_{under} = \mathbf{C}^{-1}\mathbf{X}'\mathbf{Y}$ is the BLUE of $\boldsymbol{\beta}$ in the underparametrized model (2). The covariance matrix of $\widehat{\boldsymbol{\beta}}_{true}$ is

$$\begin{aligned}\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) &= \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) + \sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\left[\mathbf{S}'\mathbf{M}_X\mathbf{S}\right]^{-1}\mathbf{S}'\mathbf{X}\mathbf{C}^{-1},\\ \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) &= \sigma^2\mathbf{C}^{-1}.\end{aligned}$$

Proof. The proof is elementary and therefore it is omitted (cf. also [3]). □

Lemma 1.2. *The bias of the estimator $\widehat{\boldsymbol{\beta}}_{under}$ in the true model (1) is*

$$\mathbf{b}_\beta = E_{true}(\widehat{\boldsymbol{\beta}}_{under}) - \boldsymbol{\beta} = \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\boldsymbol{\kappa}.$$

Proof. The proof follows straightforwardly from Lemma 1.1 after taking expectation. □

Now the problem can be formulated as follows: is it possible to determine the set, i.e. the admissible region, for neglecting the parameter $\boldsymbol{\kappa}$,

$$\{\boldsymbol{\kappa} : \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) \geq_L \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) + \mathbf{b}_\beta \mathbf{b}'_\beta\}$$

of such values of the vector parameter $\boldsymbol{\kappa}$ which can be neglected?

The answer is affirmative not only for the considered model but for other linear models as well (see the next section).

Lemma 1.3. *Let us denote*

$$\begin{aligned} \mathbf{T} &= \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) - \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}), \\ \mathbf{b}_\beta &= \mathbf{R}\boldsymbol{\kappa}. \end{aligned}$$

Then

$$\mathcal{M}(\mathbf{R}) \subset \mathcal{M}(\mathbf{T}) \Rightarrow \left(\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) \geq_L \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) + \mathbf{b}_\beta \mathbf{b}'_\beta \Leftrightarrow \boldsymbol{\kappa}'\mathbf{R}'\mathbf{T}^-\mathbf{R}\boldsymbol{\kappa} \leq 1\right).$$

Proof. Since $\mathcal{M}(\mathbf{R}) \subset \mathcal{M}(\mathbf{T})$, also $\mathbf{b}_\beta \in \mathcal{M}(\mathbf{T})$. In view of the Cauchy–Schwarz inequality we have

$$|\mathbf{h}'\mathbf{b}_\beta| \leq \sqrt{\mathbf{h}'\mathbf{T}\mathbf{h}}\sqrt{\mathbf{b}'_\beta \mathbf{T}^- \mathbf{b}_\beta} \quad \forall \mathbf{h} \in R^k$$

and with respect to the Scheffé inequality [6], p. 69, it is valid that

$$|\mathbf{h}'\mathbf{b}_\beta| \leq \sqrt{\mathbf{h}'\mathbf{T}\mathbf{h}} \Leftrightarrow \mathbf{b}'_\beta \mathbf{T}^- \mathbf{b}_\beta \leq 1 \quad \forall \mathbf{h} \in R^k.$$

However,

$$|\mathbf{h}'\mathbf{b}_\beta| \leq \sqrt{\mathbf{h}'\mathbf{T}\mathbf{h}} \Leftrightarrow \mathbf{h}'\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true})\mathbf{h} \geq \mathbf{h}'\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}\mathbf{h} + \mathbf{h}'\mathbf{b}_\beta \mathbf{b}'_\beta \mathbf{h}$$

for any $\mathbf{h} \in R^k$, and thus

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) \geq_L \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) + \mathbf{b}_\beta \mathbf{b}'_\beta \Leftrightarrow \boldsymbol{\kappa}'\mathbf{R}'\mathbf{T}^-\mathbf{R}\boldsymbol{\kappa} \leq 1.$$

□

The matrix $\mathbf{A} = \mathbf{R}'\mathbf{T}^-\mathbf{R}$ is called the criterion matrix, since it determines such a region $\mathcal{N}$,

$$\mathcal{N} = \left\{\boldsymbol{\kappa} : \boldsymbol{\kappa}'\mathbf{A}\boldsymbol{\kappa} \leq 1\right\},$$

in which the vector $\boldsymbol{\kappa}$ can be neglected, i.e. it defines the admissible region for neglecting the nuisance vector parameter $\boldsymbol{\kappa}$.

In our case

$$\begin{aligned} \mathbf{T} &= \sigma^2 \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\left[\mathbf{S}'\mathbf{M}_X\mathbf{S}\right]^{-1}\mathbf{S}'\mathbf{X}\mathbf{C}^{-1}, \\ \mathbf{R} &= \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}, \end{aligned}$$

and thus $\mathcal{M}(\mathbf{R}) \subset \mathcal{M}(\mathbf{T})$,

$$\{\boldsymbol{\kappa} : \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) \geq_L \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) + \mathbf{b}_\beta \mathbf{b}'_\beta\} = \{\boldsymbol{\kappa} : \boldsymbol{\kappa}'\mathbf{A}\boldsymbol{\kappa} \leq 1\},$$

where the criterion matrix $\mathbf{A}$ is

$$\mathbf{A} \;=\; \frac{1}{\sigma^2}\mathbf{S}'\mathbf{X}\mathbf{C}^{-1}\Big\{\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\big[\mathbf{S}'\mathbf{M}_X\mathbf{S}\big]^{-1}\mathbf{S}'\mathbf{X}\mathbf{C}^{-1}\Big\}^{-}\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}. \qquad (3)$$

Example 1.4. Let the values of the function $f(x) = \beta_1 + \beta_2 x + \kappa x^2,\ x \in R^1$, be measured at the points $x = -2,\ -1,\ 0,\ 1,\ 2$. The covariance matrix of the observation vector $\mathbf{Y}$ is $\mathrm{Var}(\mathbf{Y}) = \sigma^2\mathbf{I}$, where σ^2 is known.

Thus

$$\mathbf{X} = \begin{pmatrix} 1, & -2 \\ 1, & -1 \\ 1, & 0 \\ 1, & 1 \\ 1, & 2 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{pmatrix}.$$

The matrix $\mathbf{A}$ is

$$\mathbf{A} = \frac{1}{\sigma^2}\mathbf{S}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\big[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}(\mathbf{S}'\mathbf{M}_X\mathbf{S})^{-1}\mathbf{S}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\big]^{-}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}$$
$$= \frac{1}{\sigma^2}(2,0)\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\frac{1}{17.2}(2,0)\right]^{-}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{1}{\sigma^2}17.2$$

and the admissible region for κ is

$$\{\kappa : -0.2411\sigma \le \kappa \le 0.2411\sigma\}.$$

In this case

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{true}) = \sigma^2(\mathbf{I},0)\begin{pmatrix} \mathbf{X}'\mathbf{X}, & \mathbf{X}'\mathbf{S} \\ \mathbf{S}'\mathbf{X}, & \mathbf{S}'\mathbf{S})^{-1} \end{pmatrix}^{-1}$$
$$= \sigma^2\left[\begin{pmatrix} 5, & 0 \\ 0, & 10 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \end{pmatrix}\frac{1}{34}(10,0)\right]^{-1} = \sigma^2\begin{pmatrix} 0.4857, & 0 \\ 0, & 0.1 \end{pmatrix}$$
$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}\kappa = \begin{pmatrix} 2 \\ 0 \end{pmatrix}\kappa, \quad \mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) = \sigma^2\begin{pmatrix} 0.2, & 0 \\ 0, & 0.1 \end{pmatrix}.$$

Thus

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}_{under}) + \mathbf{b}\mathbf{b}' = \sigma^2\begin{pmatrix} 0.2000, & 0 \\ 0, & 0.1000 \end{pmatrix} + \sigma^2\begin{pmatrix} 0.4822^2, & 0 \\ 0,0 & \end{pmatrix}$$
$$= \sigma^2\begin{pmatrix} 0.4325, & 0 \\ 0, & 0.1000 \end{pmatrix}.$$

The problem is whether such criterion matrix $\mathbf{A}$ exists in other linear models and if so, then of what form.

2. Models with constraints

The following models are considered.

Model with the constraints (i):

true

$$\mathbf{Y} \sim_n \left[(\mathbf{X},\mathbf{S})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\kappa}\end{pmatrix}, \sigma^2\mathbf{I}\right],\ \mathbf{b}+(\mathbf{B},\mathbf{G})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\kappa}\end{pmatrix} = \mathbf{0},\ r(\mathbf{X},\mathbf{S}) = k+l \leq n,$$
$$r(\mathbf{B},\mathbf{G}) = q < k+l,$$

underparametrized

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}),\ \mathbf{b}+\mathbf{B}\boldsymbol{\beta} = \mathbf{0},\ r(\mathbf{X}) = k \leq n,\ r(\mathbf{B}) = q < k.$$

Model with the constraints (ii):

true

$$\mathbf{Y} \sim_n \left[(\mathbf{X},\mathbf{S})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\kappa}\end{pmatrix}, \sigma^2\mathbf{I}\right],\ \mathbf{b}+\mathbf{B}\boldsymbol{\beta} = \mathbf{0},\ r(\mathbf{X},\mathbf{S}) = k+l \leq n,\ r(\mathbf{B}) = q < k,$$

underparametrized

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}),\ \mathbf{b}+\mathbf{B}\boldsymbol{\beta} = \mathbf{0},\ r(\mathbf{X}) = k \leq n,\ r(\mathbf{B}) = q < k.$$

Model with the constraints (iii):

true

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}),\ \mathbf{b}+(\mathbf{B},\mathbf{G})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\kappa}\end{pmatrix} = \mathbf{0},\ r(\mathbf{X}) = k \leq n,\ r(\mathbf{B},\mathbf{G}) = q < k+l,$$

underparametrized

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})\ \mathbf{b}+\mathbf{B}\boldsymbol{\beta} = \mathbf{0},\ r(\mathbf{X}) = k \leq n,\ r(\mathbf{B}) = q < k.$$

Assumptions on regularity, e.g. in the true model with the constraints (iii), impose a restriction on the dimension of the parameter $\boldsymbol{\kappa}$. When we enlarge the dimension of the model we end up with a singular model. The analysis of singular models is beyond the scope of this paper.

3. Admissible regions

In this section a matrix $\mathbf{A}$ (a criterion matrix) with the property

$$\boldsymbol{\kappa}'\mathbf{A}\boldsymbol{\kappa} \leq 1 \quad \Rightarrow \quad \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}_{true}) \geq \mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}_{under}) + \mathbf{b}_\beta \mathbf{b}'_\beta$$

is given.

Theorem 3.1. *In the model with the constraints (i) the admissible region is* $\{\boldsymbol{\kappa} : \boldsymbol{\kappa}'\mathbf{A}\boldsymbol{\kappa} \le 1\}$, *where*

$$\mathbf{A} = \frac{1}{\sigma^2}\big[\mathbf{S}'\mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+} + \mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\big]$$
$$\Big\{\big[\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U} - \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\big]\big[\mathbf{S}'\mathbf{M}_X\mathbf{S} + \mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}$$
$$\times\big[\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} - \mathbf{S}'\mathbf{X}\mathbf{C}^{-1}\big]\Big\}^{-} \tag{4}$$
$$\times\big[(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{-1}\mathbf{X}'\mathbf{S} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big],$$
$$\mathbf{U} = \mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{S} - \mathbf{G}$$

and

$$\mathbf{b}_\beta = E_{true}\left(\widehat{\widehat{\boldsymbol{\beta}}}_{under}\right) = (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\mathbf{S}\boldsymbol{\kappa} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\boldsymbol{\kappa},$$
$$\mathrm{Var}\left(\widehat{\widehat{\boldsymbol{\beta}}}_{true}\right) - \mathrm{Var}\left(\widehat{\widehat{\boldsymbol{\beta}}}_{under}\right) = \sigma^2\big[\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U} - \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\big]$$
$$\times\big[\mathbf{S}'\mathbf{M}_X\mathbf{S} + \mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\big[\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} - \mathbf{S}'\mathbf{X}\mathbf{C}^{-1}\big].$$

Proof. The proof is given in Appendix. □

Remark 3.2. In the model with the constraints (i) it is valid that

$$\mathbf{b} + \mathbf{B}\widehat{\widehat{\boldsymbol{\beta}}}_{under} = \mathbf{0},$$

and thus

$$\mathbf{b} + \mathbf{B}\widehat{\widehat{\boldsymbol{\beta}}}_{under} + \mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true} \neq \mathbf{0}.$$

Therefore, it should be investigated whether $\mathbf{G}\boldsymbol{\kappa}$ differs significantly from $\mathbf{0}$.

Here,

$$\begin{aligned}
\widehat{\widehat{\boldsymbol{\kappa}}}_{true} &= \mathbf{Z}^{-1}\mathbf{S}'\mathbf{v} + \mathbf{Z}^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}' + \mathbf{U}\mathbf{Z}^{-1}\mathbf{U}')^{-1}(\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{w}\\
&\quad + \mathbf{G}\mathbf{Z}^{-1}\mathbf{S}'\mathbf{v} + \mathbf{b}),\\
\mathbf{Z} &= \mathbf{S}'\mathbf{M}_X\mathbf{S},\ \mathbf{U} = \mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{S} - \mathbf{G},\\
\mathbf{v} &= \mathbf{Y} - \mathbf{X}\mathbf{C}^{-1}\mathbf{X}'\mathbf{Y}, \quad \mathbf{w} = \mathbf{Y} - \mathbf{S}\mathbf{Z}^{-1}\mathbf{S}'\mathbf{Y},\\
\mathrm{Var}(\widehat{\widehat{\boldsymbol{\kappa}}}_{true}) &= \sigma^2\Big\{\mathbf{Z}^{-1} - \mathbf{Z}^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}' + \mathbf{U}\mathbf{Z}^{-1}\mathbf{U}')^{-1}\mathbf{U}\mathbf{Z}^{-1}\Big\}.
\end{aligned}$$

If we compare the realization of the random variable

$$\widehat{\widehat{\boldsymbol{\kappa}}}'_{true}\mathbf{G}'\big[\mathrm{Var}(\mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true})\big]^{-}\mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true}$$

with the value $\chi^2_{r[Var(G\widehat{\widehat{\kappa}}_{true})]}(0; 1-\alpha)$, then we can judge whether a breach of constraints can be tolerated or not.

Remark 3.3. The criterion matrix (3) for the model without constraints can be obtained from (4) if the terms with $\mathbf{B}$ and $\mathbf{G}$ are omitted and $(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}$ is substituted by $\mathbf{C}^{-1}$.

Corollary 3.4. *If* $\mathbf{G} = \mathbf{0}$*, then the model with the constraints (ii) is obtained. In this case the criterion matrix is*

$$\mathbf{A} = \frac{1}{\sigma^2}\mathbf{S}'\mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\Big\{(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\mathbf{S}\big[\mathbf{S}'\mathbf{M}_X\mathbf{S}\big]^{-1} \times\mathbf{S}'\mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{C'})^{+}\Big\}^{-}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\mathbf{S}.$$

Remark 3.5. In the model with the constraints (iii) it is valid that

$$\mathbf{b}_\beta = E_{true}(\widehat{\widehat{\boldsymbol{\beta}}}_{under}) - \boldsymbol{\beta} = \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\boldsymbol{\kappa}$$

and, obviously,

$$\mathrm{Var}(\widehat{\widehat{\boldsymbol{\beta}}}_{true}) = \mathrm{Var}(\widehat{\widehat{\boldsymbol{\beta}}}_{under}) + \sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big[\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big]^{-1} \times\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}.$$

Thus the criterion matrix is

$$\mathbf{A} = \frac{1}{\sigma^2}\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\Big\{\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big[\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big]^{-1} \times\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\Big\}^{-}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}.$$

Since $\mathbf{b} + \mathbf{B}\widehat{\widehat{\boldsymbol{\beta}}}_{under} = \mathbf{0}$, one has $\mathbf{b} + \mathbf{B}\widehat{\widehat{\boldsymbol{\beta}}}_{under} + \mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true} \neq \mathbf{0}$. Thus it is reasonable to compare the realization of the random variable

$$\widehat{\widehat{\boldsymbol{\kappa}}}'_{true}\mathbf{G}'\big[\mathrm{Var}(\mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true})\big]^{-}\mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true}$$

with the $(1-\alpha)$-quantile $\chi^2_{r[Var(G\widehat{\widehat{\kappa}}_{true})]}(0; 1-\alpha)$ of the central chi-squared random variable with $r\big[\mathrm{Var}(\mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true})\big]$ degrees of freedom. Here

$$\widehat{\widehat{\boldsymbol{\kappa}}}_{true} = -\big[(\mathbf{G}')^{-}_{m(BC^{-1}B')}\big]'(\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{Y} + \mathbf{b}),$$

and with respect to the assumption $r(\mathbf{B}) = q < k, r(\mathbf{G}) = l < k$, it is valid that

$$\mathrm{Var}(\widehat{\widehat{\boldsymbol{\kappa}}}_{true}) = \big[\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big]^{-1}.$$

If $\mathbf{G}\boldsymbol{\kappa} = \mathbf{0}$, then

$$\widehat{\widehat{\boldsymbol{\kappa}}}'_{true}\mathbf{G}'\Big\{\big[\mathbf{G}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big]^{-1}\mathbf{G}'\Big\}^{-1}\mathbf{G}\widehat{\widehat{\boldsymbol{\kappa}}}_{true} \sim \chi^2_l(0).$$

Thus the decision whether $\mathbf{G}\boldsymbol{\kappa}$ can be neglected can be easily made.

4. Appendix

Proof of Theorem 3.1. The BLUE $\widehat{\widehat{\boldsymbol{\beta}}}_{under}$ in the underparametrized model is

$$\widehat{\widehat{\boldsymbol{\beta}}}_{under} = \mathbf{C}^{-1}\mathbf{X}'\mathbf{Y} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{Y} + \mathbf{b}),$$

and thus

$$\mathbf{b}_{\beta} = E_{true}\left(\widehat{\widehat{\boldsymbol{\beta}}}_{under}\right) - \boldsymbol{\beta} = (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{C'})^{+}\mathbf{X}'\mathbf{S}\boldsymbol{\kappa} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\boldsymbol{\kappa},$$

where we made use of the facts that

$$E_{true}(\mathbf{C}^{-1}\mathbf{X}'\mathbf{Y}) = \mathbf{C}^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{S}\boldsymbol{\kappa}) = \boldsymbol{\beta} + \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\boldsymbol{\kappa}$$

and

$$\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} = (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}.$$

The covariance matrix is

$$\begin{aligned}\mathrm{Var}\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}}}_{true}\\ \widehat{\widehat{\boldsymbol{\kappa}}}_{true}\end{pmatrix} &= \sigma^2\begin{pmatrix}\mathbf{C}, & \mathbf{X}'\mathbf{S}\\ \mathbf{S}'\mathbf{X}, & \mathbf{S}'\mathbf{S}\end{pmatrix}^{-1} - \sigma^2\begin{pmatrix}\mathbf{C}, & \mathbf{X}'\mathbf{S}\\ \mathbf{S}'\mathbf{X}, & \mathbf{S}'\mathbf{S}\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}\mathbf{B}'\\ \mathbf{G}'\end{pmatrix}\left[(\mathbf{B},\mathbf{G})\begin{pmatrix}\mathbf{C}, & \mathbf{X}'\mathbf{S}\\ \mathbf{S}'\mathbf{X}, & \mathbf{S}'\mathbf{S}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{B}'\\ \mathbf{G}'\end{pmatrix}\right]^{-1}(\mathbf{B},\mathbf{G})\\ &\times\begin{pmatrix}\mathbf{C}, & \mathbf{X}'\mathbf{S}\\ \mathbf{S}'\mathbf{X}, & \mathbf{S}'\mathbf{S}\end{pmatrix}^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Var}(\widehat{\widehat{\boldsymbol{\beta}}}_{true}) &= \sigma^2\mathbf{C}^{-1} + \sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\mathbf{Z}^{-1}\mathbf{S}'\mathbf{X}\mathbf{C}^{-1} - \sigma^2\big[\mathbf{C}^{-1}\mathbf{B}' + \mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\mathbf{Z}^{-1}\mathbf{U}'\big]\\ &\times\big[\mathbf{B}\mathbf{C}^{-1}\mathbf{B}' + \mathbf{U}\mathbf{Z}^{-1}\mathbf{U}'\big]^{-1}\big[\mathbf{B}\mathbf{C}^{-1} + \mathbf{U}\mathbf{Z}^{-1}\mathbf{S}'\mathbf{X}\mathbf{C}^{-1}\big],\end{aligned}$$

where

$$\mathbf{Z} = \mathbf{S}'\mathbf{M}_X\mathbf{S}.$$

Since

$$\begin{aligned}&(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}' + \mathbf{U}\mathbf{Z}^{-1}\mathbf{U}')^{-1} =\\ &= (\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} - (\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z} + \mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1},\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{true}) = {}& \sigma^2\mathbf{C}^{-1} - \sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} + \sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1}\\ &-\sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} + \sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\\ &\times\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} + \sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'\\ &\times(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1}\\ &-\sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1} - \sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\\ &\times\mathbf{S}'\mathbf{XC}^{-1} + \sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\\ &\times\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1} + \sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\\ &\times\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1}.\end{aligned}$$

Now the following equalities will be utilized

$$\begin{aligned}(\mathbf{BC}^{-1}\mathbf{B}'+\mathbf{UZ}^{-1}\mathbf{U}')^{-1}\mathbf{UZ}^{-1} &= (\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\\ &= (\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1} - (\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'\\ &\quad\times(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1},\end{aligned}$$

hence

$$\begin{aligned}&\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{S}'\mathbf{XC}^{-1}\\ &= \mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\\ &\quad\times\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1},\end{aligned}$$

and

$$\begin{aligned}&\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1}\\ &= \mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} - \mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\\ &\quad\times\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1}.\end{aligned}$$

Thus it can be written that

$$\begin{aligned}\mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{true}) = {}& \sigma^2\mathbf{C}^{-1} - \sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} + \underline{\sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1}}\\ &-\sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1}\\ &+\sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1}\\ &-\sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{S}'\mathbf{XC}^{-1}\\ &-\underline{\sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1}}\\ &+\underline{\sigma^2\mathbf{C}^{-1}\mathbf{X}'\mathbf{SZ}^{-1}\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}}\\ &\times\underline{\mathbf{U}'(\mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{UZ}^{-1}\mathbf{S}'\mathbf{XC}^{-1}}.\end{aligned}$$

The underlined terms can be rewritten by the help of the following equalities

$$\begin{aligned}
&\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1} = \\
&= \mathbf{Z}^{-1}-\mathbf{Z}^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}'+\mathbf{U}'\mathbf{Z}^{-1}\mathbf{U})^{-1}\mathbf{U}\mathbf{Z}^{-1}\\
&= \mathbf{Z}^{-1}-\mathbf{Z}^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\mathbf{Z}^{-1}+\mathbf{Z}^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\\
&\quad\times\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\mathbf{Z}^{-1}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{true}) &= \mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{under})+\sigma^2\mathbf{C}^{-1}\mathbf{X}'\boldsymbol\Sigma^{-1}\mathbf{S}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{S}'\boldsymbol\Sigma^{-1}\\
&\times\mathbf{X}\mathbf{C}^{-1}-\sigma^2\mathbf{C}^{-1}\mathbf{X}'\boldsymbol\Sigma^{-1}\mathbf{S}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\\
&-\sigma^2\mathbf{C}^{-1}(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{S}'\boldsymbol\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\\
&+\sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{C}^{-1}\\
&= \mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{under})+\sigma^2\big[\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}-\mathbf{C}^{-1}\mathbf{X}'\boldsymbol\Sigma^{-1}\mathbf{S}\big]\\
&\times\big[\mathbf{Z}+\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}\big]^{-1}\big[\mathbf{U}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}-\mathbf{S}'\boldsymbol\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\big],
\end{aligned}$$

since

$$\mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{true}) = \sigma^2\mathbf{C}^{-1}-\sigma^2\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C})^{-1}.$$

Now it is sufficient to prove the relation

$$\begin{aligned}
&\mathcal{M}\big[(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\boldsymbol\Sigma^{-1}\mathbf{S}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\big]\\
&\qquad\subset\mathcal{M}\big[\mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{true})-\mathrm{Var}(\widehat{\widehat{\boldsymbol\beta}}_{true})\big],
\end{aligned}$$

which is implied by the following equalities

$$\begin{aligned}
&(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\mathbf{S}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\\
&=\mathbf{C}^{-1}\mathbf{X}\mathbf{S}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{S}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{G}\\
&=\mathbf{C}^{-1}\mathbf{X}\mathbf{S}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{U}.
\end{aligned}$$

Thus the criterion matrix $\mathbf{A}$ exists and is given by the relation (4). $\square$

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