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A note on the equality of the BLUPs for new observations under two linear models

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Abstract. We consider two linear models, $\mathscr{L}_1$ and $\mathscr{L}_2$, say, with new unobserved future observations. We give necessary and sufficient conditions in the general situation, without any rank assumptions, that the best linear unbiased predictor (BLUP) of the new observation under the the model $\mathscr{L}_1$ continues to be BLUP also under the model $\mathscr{L}_2$.

1. Introduction

In the literature the invariance of the the best linear unbiased estimator (BLUE) of fixed effects under the general linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{1.1}$$

where $\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$, $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$, and $\mathrm{cov}(\mathbf{y}) = \mathrm{cov}(\boldsymbol{\varepsilon}) = \mathbf{V}_{11}$, has received much attention; see, for example, the papers by Rao (1967, 1971, 1973), and Mitra and Moore (1973). In particular, the connection between the ordinary least squares estimator and BLUE $(\mathbf{X}\boldsymbol{\beta})$, has been studied extensively; see, e.g., Rao (1967), Zyskind (1967), and the review papers by Puntanen and Styan (1989), and Baksalary, Puntanen and Styan (1990a).

According to our knowledge, the equality of the best linear unbiased predictors (BLUPs) in two models $\mathscr{L}_1$ and $\mathscr{L}_2$, defined below, has received very little attention in the literature. Haslett and Puntanen (2010a,b) considered the equality of the BLUPs of the random factor under two mixed models. In their (2010a) paper they gave, without a proof, necessary and sufficient conditions in the general situation that the BLUP of the new observation under the model $\mathscr{L}_1$ continues to be BLUP also under the model $\mathscr{L}_2$. The purpose of this paper is to provide a complete proof of this result.

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Let us start formally by considering the general linear model (1.1), which can be represented as a triplet

$$\mathscr{F} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}_{11}\}\,. \tag{1.2}$$

Vector $\mathbf{y}$ is an $n \times 1$ observable random vector, $\boldsymbol{\varepsilon}$ is an $n \times 1$ random error vector, $\mathbf{X}$ is a known $n \times p$ model matrix, $\boldsymbol{\beta}$ is a $p \times 1$ vector of fixed but unknown parameters, $\mathbf{V}_{11}$ is a known $n \times n$ nonnegative definite matrix. Let $\mathbf{y}_f$ denote the $q \times 1$ unobservable random vector containing new future observations. The new observations are assumed to follow the linear model

$$\mathbf{y}_f = \mathbf{X}_f\boldsymbol{\beta} + \boldsymbol{\varepsilon}_f\,,$$

where $\mathbf{X}_f$ is a known $q \times p$ model matrix associated with the new observations, $\boldsymbol{\beta}$ is the same vector of unknown parameters as in (1.2), and $\boldsymbol{\varepsilon}_f$ is a $q \times 1$ random error vector associated with new observations. The expectation and the covariance matrix are

$$\mathrm{E}\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix} = \begin{pmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{X}_f\boldsymbol{\beta}\end{pmatrix} = \begin{pmatrix}\mathbf{X}\\ \mathbf{X}_f\end{pmatrix}\boldsymbol{\beta}, \quad \operatorname{cov}\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix} = \begin{pmatrix}\mathbf{V}_{11} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix} = \mathbf{V},$$

where the entire covariance matrix $\mathbf{V}$ is assumed to be known. For brevity, we denote

$$\mathscr{L}_1 = \left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix},\, \begin{pmatrix}\mathbf{X}\\ \mathbf{X}_f\end{pmatrix}\boldsymbol{\beta},\, \begin{pmatrix}\mathbf{V}_{11} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}\right\}.$$

Before proceeding, we may introduce the notation used in this paper. We will denote $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices and $\mathbb{R}^m = \mathbb{R}^{m\times 1}$. We will use the symbols $\mathbf{A}'$, $\mathbf{A}^-$, $\mathbf{A}^+$, $\mathscr{C}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})^\perp$, and $\mathscr{N}(\mathbf{A})$ to denote the transpose, a generalized inverse, the Moore–Penrose inverse, the column space, the orthogonal complement of the column space, and the null space, of the matrix $\mathbf{A}$, respectively. By $(\mathbf{A} : \mathbf{B})$ we denote the partitioned matrix with $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times k}$ as submatrices. By $\mathbf{A}^\perp$ we denote any matrix satisfying $\mathscr{C}(\mathbf{A}^\perp) = \mathscr{N}(\mathbf{A}') = \mathscr{C}(\mathbf{A})^\perp$. Furthermore, we will write $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ to denote the orthogonal projector (with respect to the standard inner product) onto $\mathscr{C}(\mathbf{A})$. In particular, we denote $\mathbf{H} = \mathbf{P}_\mathbf{X}$ and $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_\mathbf{X}$. Notice that one choice for $\mathbf{X}^\perp$ is of course $\mathbf{M}$.

We assume the model $\mathscr{L}_1$ to be consistent in the sense that

$$\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}_{11}) = \mathscr{C}(\mathbf{X} : \mathbf{V}_{11}\mathbf{M})\,, \tag{1.3}$$

i.e., the observed value of $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{X} : \mathbf{V}_{11})$ with probability 1. The corresponding consistency is assumed in all models that we will consider.

The linear predictor $\mathbf{G}\mathbf{y}$ is said to be unbiased for $\mathbf{y}_f$ if the expected prediction error is $\mathbf{0}$: $\mathrm{E}(\mathbf{y}_f - \mathbf{G}\mathbf{y}) = \mathbf{0}$ for all $\boldsymbol{\beta} \in \mathbb{R}^p$. This is equivalent to $\mathbf{G}\mathbf{X} = \mathbf{X}_f$, i.e., $\mathbf{X}_f' = \mathbf{X}'\mathbf{G}'$. The requirement that $\mathscr{C}(\mathbf{X}_f') \subset \mathscr{C}(\mathbf{X}')$ means that $\mathbf{X}_f\boldsymbol{\beta}$ is estimable under $\mathscr{F} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}_{11}\}$. Now an unbiased linear predictor $\mathbf{Gy}$ is the best linear unbiased predictor, BLUP, for $\mathbf{y}_f$, if the Löwner ordering

$$\mathrm{cov}(\mathbf{Gy} - \mathbf{y}_f) \leq_{\mathsf{L}} \mathrm{cov}(\mathbf{Fy} - \mathbf{y}_f)$$

holds for all $\mathbf{F}$ such that $\mathbf{Fy}$ is an unbiased linear predictor for $\mathbf{y}_f$.

The following lemma characterizes the BLUP; for the proof, see, e.g., Christensen (2002, p. 283), and Isotalo and Puntanen (2006, p. 1015).

Lemma 1.1. *Consider the linear model $\mathscr{L}_1$ (with new unobserved future observations), where $\mathbf{X}_f\boldsymbol{\beta}$ is a given vector of estimable parametric functions. The linear predictor $\mathbf{Gy}$ is the best linear unbiased predictor* (BLUP) *for $\mathbf{y}_f$ if and only if $\mathbf{G}$ satisfies the equation*

$$\mathbf{G}(\mathbf{X} : \mathbf{V}_{11}\mathbf{X}^{\perp}) = (\mathbf{X}_f : \mathbf{V}_{21}\mathbf{X}^{\perp}) \tag{1.4}$$

for any given choice of $\mathbf{X}^{\perp}$.

We can get, for example, the following matrices $\mathbf{G}_i$ such that $\mathbf{G}_i\mathbf{y}$ equals the BLUP($\mathbf{y}_f$):

$$\begin{aligned}
\mathbf{G}_1 &= \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- + \mathbf{V}_{21}\mathbf{W}^-[\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-]\,,\\
\mathbf{G}_2 &= \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- + \mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\,,\\
\mathbf{G}_3 &= \mathbf{X}_f(\mathbf{X}'\mathbf{X})^-\mathbf{X}' + [\mathbf{V}_{21} - \mathbf{X}_f(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{V}_{11}]\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\,,
\end{aligned}$$

where $\mathbf{W}$ (and the related $\mathbf{U}$) are any matrices such that

$$\mathbf{W} = \mathbf{V}_{11} + \mathbf{XUX}', \quad \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V}_{11})\,.$$

Notice that the equation (1.4) has a unique solution if and only if $\mathscr{C}(\mathbf{X} : \mathbf{V}_{11}\mathbf{M}) = \mathbb{R}^n$. According to Rao and Mitra (1971, p. 24) the general solution to (1.4) can be written, for example, as

$$\mathbf{G} = \mathbf{G}_i + \mathbf{F}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}\,:\,\mathbf{V}_{11}\mathbf{M})})\,,$$

where the matrix $\mathbf{F}$ is free to vary and $\mathbf{P}_{(\mathbf{X}\,:\,\mathbf{V}_{11}\mathbf{M})}$ denotes the orthogonal projector onto the column space of matrix $(\mathbf{X} : \mathbf{V}_{11}\mathbf{M})$. Even though the multiplier $\mathbf{G}$ may not be unique, the observed value $\mathbf{Gy}$ of the BLUP is unique with probability 1; this is due to the consistency requirement (1.3).

Consider now another linear model $\mathscr{L}_2$, which may differ from $\mathscr{L}_1$ through its covariance matrix and model matrix, i.e.,

$$\mathscr{L}_2 = \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{y}_f \end{pmatrix}, \begin{pmatrix} \underline{\mathbf{X}} \\ \underline{\mathbf{X}}_f \end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix} \underline{\mathbf{V}}_{11} & \underline{\mathbf{V}}_{12} \\ \underline{\mathbf{V}}_{21} & \underline{\mathbf{V}}_{22} \end{pmatrix} \right\}.$$

In the next section we consider the conditions under which the BLUP for $\mathbf{y}_f$ under $\mathscr{L}_1$ continues to be BLUP under $\mathscr{L}_2$. Naturally, as pointed out by an anonymous referee, $\underline{\mathbf{X}}_f\boldsymbol{\beta}$ must be estimable under $\{\mathbf{y}, \underline{\mathbf{X}}\boldsymbol{\beta}, \underline{\mathbf{V}}_{11}\}$; otherwise $\mathbf{y}_f$ does not have a BLUP under $\mathscr{L}_2$.

2. Main results

Theorem 2.1. *Consider the models* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ (*with new unobserved future observations*), *where* $\mathscr{C}(\mathbf{X}_f') \subset \mathscr{C}(\mathbf{X}')$ *and* $\mathscr{C}(\underline{\mathbf{X}}_f') \subset \mathscr{C}(\underline{\mathbf{X}}')$. *Then every representation of the* BLUP *for* $\mathbf{y}_f$ *under the model* $\mathscr{L}_1$ *is also a* BLUP *for* $\mathbf{y}_f$ *under the model* $\mathscr{L}_2$ *if and only if*

$$\mathscr{C}\begin{pmatrix} \underline{\mathbf{X}} & \underline{\mathbf{V}}_{11}\underline{\mathbf{M}} \\ \underline{\mathbf{X}}_f & \underline{\mathbf{V}}_{21}\underline{\mathbf{M}} \end{pmatrix} \subset \mathscr{C}\begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix}. \tag{2.1}$$

Proof. For the proof it is convenient to observe that (2.1) holds if and only if

$$\mathscr{C}\begin{pmatrix} \underline{\mathbf{V}}_{11}\underline{\mathbf{M}} \\ \underline{\mathbf{V}}_{21}\underline{\mathbf{M}} \end{pmatrix} \subset \mathscr{C}\begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix}, \tag{2.2a}$$

and

$$\mathscr{C}\begin{pmatrix} \underline{\mathbf{X}} \\ \underline{\mathbf{X}}_f \end{pmatrix} \subset \mathscr{C}\begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix}. \tag{2.2b}$$

Let us first assume that every representation of the BLUP for $\mathbf{y}_f$ under $\mathscr{L}_1$ continues to be BLUP under $\mathscr{L}_2$. Let $\mathbf{G}_0$ be a general solution to (1.4):

$$\begin{aligned} \mathbf{G}_0 = \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- + \mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M} \\ + \mathbf{F}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}\,:\,\mathbf{V}_{11}\mathbf{M})}), \end{aligned}$$

where $\mathbf{F}$ is free to vary. Then $\mathbf{G}_0$ has to satisfy also the other fundamental BLUP equation:

$$\mathbf{G}_0(\underline{\mathbf{X}} : \underline{\mathbf{V}}_{11}\underline{\mathbf{M}}) = (\underline{\mathbf{X}}_f : \underline{\mathbf{V}}_{21}\underline{\mathbf{M}}), \tag{2.3}$$

where $\underline{\mathbf{M}} = \mathbf{I}_n - \mathbf{P}_{\underline{\mathbf{X}}}$. The $\mathbf{X}$-part of the condition (2.3) is

$$\begin{aligned} &\mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\underline{\mathbf{X}} \\ &\quad + \mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\underline{\mathbf{X}} \\ &\qquad + \mathbf{F}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}\,:\,\mathbf{V}_{11}\mathbf{M})})\underline{\mathbf{X}} \\ &\qquad = \underline{\mathbf{X}}_f. \end{aligned} \tag{2.4}$$

Because (2.4) must hold for all matrices $\mathbf{F}$, we necessarily have

$$\mathbf{F}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}\,:\,\mathbf{V}_{11}\mathbf{M})})\underline{\mathbf{X}} = \mathbf{0} \quad \text{for all } \mathbf{F} \in \mathbb{R}^{n\times n},$$

which further implies $\mathscr{C}(\underline{\mathbf{X}}) \subset \mathscr{C}(\mathbf{X} : \mathbf{V}_{11}\mathbf{M})$, i.e.,

$$\underline{\mathbf{X}} = \mathbf{X}\mathbf{K}_1 + \mathbf{V}_{11}\mathbf{M}\mathbf{K}_2 \quad \text{for some } \mathbf{K}_1 \in \mathbb{R}^{p\times p} \text{ and } \mathbf{K}_2 \in \mathbb{R}^{n\times p}, \tag{2.5}$$

and hence

$$\begin{aligned} \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\underline{\mathbf{X}} &= \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-(\mathbf{X}\mathbf{K}_1 + \mathbf{V}_{11}\mathbf{M}\mathbf{K}_2) \\ &= \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{X}\mathbf{K}_1 + \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{V}_{11}\mathbf{M}\mathbf{K}_2 \\ &= \mathbf{X}_f\mathbf{K}_1 + \mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{W}\mathbf{M}\mathbf{K}_2 \\ &= \mathbf{X}_f\mathbf{K}_1, \end{aligned} \tag{2.6}$$

where we have used $\mathbf{V}_{11}\mathbf{M} = \mathbf{WM}$ and $\mathbf{X}'\mathbf{W}^-\mathbf{W} = \mathbf{X}'$. Note also that the assumption $\mathscr{C}(\mathbf{X}'_f) \subset \mathscr{C}(\mathbf{X}')$ implies

$$\mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{X} = \mathbf{X}_f;$$

for properties of the matrix of type $(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-$, see, e.g., Baksalary and Mathew (1990, Th. 2) and Baksalary, Puntanen and Styan (1990b, Th. 2).

In view of (2.5) we have

$$\mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\underline{\mathbf{X}} = \mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\mathbf{V}_{11}\mathbf{M}\mathbf{K}_2\,.$$

Because $\mathbf{V}$ is nonnegative definite, we have $\mathscr{C}(\mathbf{V}_{12}) \subset \mathscr{C}(\mathbf{V}_{11})$, and thereby

$$\mathscr{C}(\mathbf{M}\mathbf{V}_{12}) \subset \mathscr{C}(\mathbf{M}\mathbf{V}_{11}) = \mathscr{C}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})\,,$$

and so

$$\mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\underline{\mathbf{X}} = \mathbf{V}_{21}\mathbf{M}\mathbf{K}_2\,. \tag{2.7}$$

Combining (2.6), (2.7) and (2.4) shows that $\underline{\mathbf{X}}_f = \mathbf{X}_f\mathbf{K}_1 + \mathbf{V}_{21}\mathbf{M}\mathbf{K}_1$, which together with (2.5) yields the following:

$$\begin{pmatrix} \underline{\mathbf{X}} \\ \underline{\mathbf{X}}_f \end{pmatrix} = \begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}, \tag{2.8}$$

i.e., (2.2b) holds.

The right-hand part of the condition (2.3) is

$$\begin{aligned} &\mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\underline{\mathbf{V}}_{11}\underline{\mathbf{M}} && \text{(2.9a)} \\ &\quad + \mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\underline{\mathbf{V}}_{11}\underline{\mathbf{M}} && \text{(2.9b)} \\ &\quad + \mathbf{F}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}:\mathbf{V}_{11}\mathbf{M})})\underline{\mathbf{V}}_{11}\underline{\mathbf{M}} && \text{(2.9c)} \\ &\quad = \underline{\mathbf{V}}_{21}\underline{\mathbf{M}}\,. && \text{(2.9d)} \end{aligned}$$

Again, because (2.9) must hold for all matrices $\mathbf{F}$, we necessarily have

$$\mathbf{F}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}:\mathbf{V}_{11}\mathbf{M})})\underline{\mathbf{V}}_{11}\underline{\mathbf{M}} = \mathbf{0} \quad \text{for all } \mathbf{F} \in \mathbb{R}^{n\times n},$$

which further implies $\mathscr{C}(\underline{\mathbf{V}}_{11}\underline{\mathbf{M}}) \subset \mathscr{C}(\mathbf{X} : \mathbf{V}_{11}\mathbf{M})$, and hence

$$\underline{\mathbf{V}}_{11}\underline{\mathbf{M}} = \mathbf{X}\mathbf{L}_1 + \mathbf{V}_{11}\mathbf{M}\mathbf{L}_2 \quad \text{for some } \mathbf{L}_1 \in \mathbb{R}^{p\times n} \text{ and } \mathbf{L}_2 \in \mathbb{R}^{n\times n}. \tag{2.10}$$

Substituting (2.10) into (2.9a), gives (proceeding as was done above with the $\mathbf{X}$-part)

$$\mathbf{X}_f(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\underline{\mathbf{V}}_{11}\underline{\mathbf{M}} = \mathbf{X}_f\mathbf{L}_1\,,$$

while the term (2.9b) becomes

$$\mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}_{11}\mathbf{M})^-\mathbf{M}\mathbf{V}_{11}\mathbf{M}\mathbf{L}_2 = \mathbf{V}_{21}\mathbf{M}\mathbf{L}_2\,,$$

and hence (2.9) gives

$$\mathbf{X}_f\mathbf{L}_1 + \mathbf{V}_{21}\mathbf{M}\mathbf{L}_2 = \underline{\mathbf{V}}_{21}\underline{\mathbf{M}}\,. \tag{2.11}$$

Combining (2.10) and (2.11) yields

$$\begin{pmatrix} \underline{\mathbf{V}}_{11}\underline{\mathbf{M}} \\ \underline{\mathbf{V}}_{21}\underline{\mathbf{M}} \end{pmatrix} = \begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix}, \tag{2.12}$$

i.e., (2.2a) holds.

To go the other way, suppose that (2.2a) and (2.2b) hold and that $\mathbf{K}_1$ and $\mathbf{K}_2$, and $\mathbf{L}_1$ and $\mathbf{L}_2$ are defined as in (2.8) and in (2.12), respectively. Moreover, assume that $\mathbf{Gy}$ is the BLUP of $\mathbf{y}_f$ under $\mathscr{L}_1$, i.e.,

$$\mathbf{G}(\mathbf{X} : \mathbf{V}_{11}\mathbf{M}) = (\mathbf{X}_f : \mathbf{V}_{21}\mathbf{M}). \tag{2.13}$$

Postmultiplying (2.13) by $\begin{pmatrix} \mathbf{K}_1 & \mathbf{L}_1 \\ \mathbf{K}_2 & \mathbf{L}_2 \end{pmatrix}$ yields

$$\mathbf{G}(\underline{\mathbf{X}} : \underline{\mathbf{V}}_{11}\underline{\mathbf{M}}) = (\underline{\mathbf{X}}_f : \underline{\mathbf{V}}_{21}\underline{\mathbf{M}}),$$

which confirms that $\mathbf{Gy}$ is also the BLUP of $\mathbf{y}_f$ under $\mathscr{L}_2$. Thus the proof is completed. □

If the both models $\mathscr{L}_1$ and $\mathscr{L}_2$ have the same model matrix part we get the following corollary.

Corollary 2.1. *Consider the same situation as in Theorem* 2.1 *but suppose that the two models* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *have the same model matrix part* $\begin{pmatrix} \mathbf{X} \\ \mathbf{X}_f \end{pmatrix}$. *Then every representation of the* BLUP *for* $\mathbf{y}_f$ *under the model* $\mathscr{L}_1$ *is also a* BLUP *for* $\mathbf{y}_f$ *under the model* $\mathscr{L}_2$ *if and only if*

$$\mathscr{C}\begin{pmatrix} \underline{\mathbf{V}}_{11}\mathbf{M} \\ \underline{\mathbf{V}}_{21}\mathbf{M} \end{pmatrix} \subset \mathscr{C}\begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix}.$$

Moreover, the sets of all representations of BLUP*s for* $\mathbf{y}_f$ *under* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are identical if and only if*

$$\mathscr{C}\begin{pmatrix} \mathbf{X} & \underline{\mathbf{V}}_{11}\mathbf{M} \\ \mathbf{X}_f & \underline{\mathbf{V}}_{21}\mathbf{M} \end{pmatrix} = \mathscr{C}\begin{pmatrix} \mathbf{X} & \mathbf{V}_{11}\mathbf{M} \\ \mathbf{X}_f & \mathbf{V}_{21}\mathbf{M} \end{pmatrix}.$$

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