ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 14, 2010

Orthogonal decompositions in growth curve models

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Dedicated to Professor L. Kubáček on the occasion of his eightieth birthday

Abstract. The article shows advantage of orthogonal decompositions in the standard and extended growth curve models. Using this, distribution of estimators of ρ and σ^2 in the standard GCM with uniform correlation structure is derived. Also, equivalence of Hu and von Rosen conditions in the extended GCM under mild conditions is shown.

1. The standard growth curve model with uniform correlation structure

The basic model that we consider is the following one:

$$Y = XBZ' + \mathbf{e}, \quad \mathrm{Vec}(\mathbf{e}) \sim N\left(0, \Sigma \otimes I_n\right), \quad \Sigma = \theta_1 G + \theta_2 ww', \qquad (1)$$

where $Y_{n\times p}$ is a matrix of independent p-variate observations, $X_{n\times m}$ is an ANOVA design matrix, $Z_{p\times r}$ is a regression variables matrix, and $\mathbf{e}$ is a matrix of random errors. As for the unknown parameters, $B_{m\times r}$ is an location parameters matrix, and θ_1, θ_2 are (scalar) variance parameters. Matrix $G_{p\times p} > 0$ and vector $w \in \mathbb{R}^p$ are known. The Vec operator stacks elements of a matrix into a vector column-wise.

Assumed correlation structure is called generalized uniform correlation structure. It was studied in the context of the growth curve model (GCM) in [6], and recently in [4]. A special case was studied also in [3].

As for the estimation of unknown parameters, Žežula in [6] used directly model (1), whereas Ye and Wang [4] used modified model with orthogonal

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Received March 15, 2010.

2010 *Mathematics Subject Classification.* Primary 62H10; Secondary 62J05.

Key words and phrases. Growth curve model, extended growth curve model, orthogonal decomposition, uniform correlation structure.

The research was supported by grants VEGA MŠ SR 1/0131/09, VEGA MŠ SR 1/0325/10, and by the Agency of the Slovak Ministry of Education for the Structural Funds of the EU, under project ITMS:26220120007.

decomposition:

$$\begin{aligned}
YG^{-\frac{1}{2}} &= Y_1 + Y_2, \\
Y_1 = YG^{-\frac{1}{2}}P_F &= XBZ'G^{-\frac{1}{2}}P_F + e_1, \\
Y_2 = YG^{-\frac{1}{2}}M_F &= XBZ'G^{-\frac{1}{2}}M_F + e_2,
\end{aligned} \tag{2}$$

where $F = G^{-\frac{1}{2}}w$, P_F is the orthogonal projection matrix onto the column space $\mathcal{R}(F)$ of F, and $M_F = I - P_F$ onto its orthogonal complement.

Let us denote

$$S = \frac{1}{n - r(X)}\, Y'M_XY,$$

$$W_1 = P_FG^{-\frac{1}{2}}SG^{-\frac{1}{2}}P_F\,, \quad W_2 = M_FG^{-\frac{1}{2}}SG^{-\frac{1}{2}}M_F\,.$$

The estimators of Žežula are

$$\hat{\theta}_1 = \frac{(\mathbf{1}'w)^2\,\mathrm{Tr}(S) - \mathbf{1}'S\mathbf{1}\,w'w}{(\mathbf{1}'w)^2\,\mathrm{Tr}(G) - \mathbf{1}'G\mathbf{1}\,w'w}\,, \quad \hat{\theta}_2 = \frac{\mathbf{1}'S\mathbf{1}\,\mathrm{Tr}(G) - \mathbf{1}'G\mathbf{1}\,\mathrm{Tr}(S)}{(\mathbf{1}'w)^2\,\mathrm{Tr}(G) - \mathbf{1}'G\mathbf{1}\,w'w}\,, \tag{3}$$

and the estimators of Ye and Wang are

$$\begin{aligned}
\hat{\theta}_1^* = \frac{\mathrm{Tr}(W_2)}{p-1} &= \frac{w'G^{-1}w.\,\mathrm{Tr}\left(G^{-1}S\right) - w'G^{-1}SG^{-1}w}{(p-1)\left(w'G^{-1}w\right)}\,, \\
\hat{\theta}_2^* = \frac{(p-1)\,\mathrm{Tr}(W_1) - \mathrm{Tr}(W_2)}{(p-1)w'G^{-1}w} &= \frac{p.w'G^{-1}SG^{-1}w - w'G^{-1}w.\,\mathrm{Tr}\left(G^{-1}S\right)}{(p-1)\left(w'G^{-1}w\right)^2}\,.
\end{aligned} \tag{4}$$

These pairs of estimators are both unbiased, but different. Naturally, we would like to know the variances. Since $S \sim \mathcal{W}_p\left(n - r(X), \frac{1}{n-r(X)}\Sigma\right)$, it is easy to establish that

$$\mathrm{Var}(\mathrm{Vec}\,S) = \frac{1}{n - r(X)}\left(I_{p^2} + K_{pp}\right)(\Sigma \otimes \Sigma)\,, \tag{5}$$

where K_{pp} is the commutation matrix, see e.g. [5]. This immediately implies

$$\begin{aligned}
\mathrm{Var}\left[\mathrm{Tr}\left(G^{-1}S\right)\right] &= \frac{2}{n - r(X)}\,\mathrm{Tr}\left(G^{-1}\Sigma G^{-1}\Sigma\right), \\
\mathrm{Var}\left[w'G^{-1}SG^{-1}w\right] &= \frac{2}{n - r(X)}\left(w'G^{-1}\Sigma G^{-1}w\right)^2, \\
\mathrm{Cov}\left[\mathrm{Tr}\left(G^{-1}S\right), w'G^{-1}SG^{-1}w\right] &= \frac{2}{n - r(X)}\,w'G^{-1}\Sigma G^{-1}\Sigma G^{-1}w\,.
\end{aligned}$$

Necessary formulas for $\operatorname{Var}[\operatorname{Tr}(S)]$, $\operatorname{Var}\mathbf{1}'S\mathbf{1}$, and $\operatorname{Cov}[\operatorname{Tr}(S), \mathbf{1}'S\mathbf{1}]$ are special cases. Short computation gives

$$\operatorname{Var}\hat{\theta}_1 = \frac{2}{n-r(X)} \cdot \frac{(\mathbf{1}'w)^4 \operatorname{Tr}\left(\Sigma^2\right) - 2(\mathbf{1}'w)^2 w'w.\mathbf{1}'\Sigma^2\mathbf{1} + (w'w)^2(\mathbf{1}'\Sigma\mathbf{1})^2}{\left[(\mathbf{1}'w)^2 \operatorname{Tr}(G) - \mathbf{1}'G\mathbf{1}\, w'w\right]^2}, \tag{6}$$

$$\operatorname{Var}\hat{\theta}_2 = \frac{2}{n-r(X)} \cdot \frac{\left[\operatorname{Tr}(G)\mathbf{1}'\Sigma\mathbf{1}\right]^2 - 2\operatorname{Tr}(G)\mathbf{1}'G\mathbf{1}\mathbf{1}'\Sigma^2\mathbf{1} + (\mathbf{1}'G\mathbf{1})^2 \operatorname{Tr}(\Sigma^2)}{\left[(\mathbf{1}'w)^2 \operatorname{Tr}(G) - \mathbf{1}'G\mathbf{1}\, w'w\right]^2}, \tag{7}$$

and

$$\operatorname{Var}\hat{\theta}_1^* = \frac{2}{n-r(X)} \cdot \frac{1}{(p-1)^2\left(w'G^{-1}w\right)^2}\Big[\left(w'G^{-1}w\right)^2 \operatorname{Tr}\left(G^{-1}\Sigma G^{-1}\Sigma\right) + \\ + \left(w'G^{-1}\Sigma G^{-1}w\right)^2 - 2\left(w'G^{-1}w\right)\left(w'G^{-1}\Sigma G^{-1}\Sigma G^{-1}w\right)\Big], \tag{8}$$

$$\operatorname{Var}\hat{\theta}_2^* = \frac{2}{n-r(X)} \cdot \frac{1}{(p-1)^2\left(w'G^{-1}w\right)^4}\left[(w'G^{-1}w)^2 \operatorname{Tr}\left(G^{-1}\Sigma G^{-1}\Sigma\right) + \right. \\ \left. + p^2(w'G^{-1}\Sigma G^{-1}w)^2 - 2p(w'G^{-1}w)\left(w'G^{-1}\Sigma G^{-1}\Sigma G^{-1}w\right)\right]. \tag{9}$$

Analytical comparison of these quantities is quite difficult. Few simulations performed suggest that in general Ye and Wang's estimators tend to have smaller variance.

Very important special case of the previous model is the model with

$$\Sigma = \sigma^2\left[(1-\rho)I_p + \rho\mathbf{1}\mathbf{1}'\right]. \tag{10}$$

This correlation structure is called the uniform correlation structure or the intraclass correlation structure. It is the case with $G = I_p$ and $w = \mathbf{1}$, slightly reparametrized. It must hold

$$-\frac{1}{p-1} \le \rho \le 1.$$

As a special case of (1), estimators of σ^2 and ρ can be then obtained by a simple transformation of $\hat{\theta}_1$ and $\hat{\theta}_2$:

$$\hat{\sigma}^2 = \hat{\theta}_1 + \hat{\theta}_2 \qquad \text{and} \qquad \hat{\rho} = \frac{\hat{\theta}_2}{\hat{\theta}_1 + \hat{\theta}_2}. \tag{11}$$

This implies the following form of estimators due to Žežula:

$$\hat{\sigma}_Z^2 = \frac{\operatorname{Tr}(S)}{p}, \quad \hat{\rho}_Z = \frac{1}{p-1}\left(\frac{\mathbf{1}'S\mathbf{1}}{\operatorname{Tr}(S)} - 1\right), \tag{12}$$

and due to Ye and Wang:

$$\hat{\sigma}_{YW}^2 = \frac{\operatorname{Tr}(V_1) + \operatorname{Tr}(V_2)}{p}, \quad \hat{\rho}_{YW} = 1 - \frac{p\operatorname{Tr}(V_2)}{(p-1)\left(\operatorname{Tr}(V_1) + \operatorname{Tr}(V_2)\right)}, \tag{13}$$

where

$$V_1 = P_{\mathbf{1}} S P_{\mathbf{1}}, \quad V_2 = M_{\mathbf{1}} S M_{\mathbf{1}} \, .$$

Ye and Wang recognized that $\hat{\sigma}^2_Z = \hat{\sigma}^2_{YW}$, but they failed to recognize that also $\hat{\rho}_Z = \hat{\rho}_{YW}$.

Lemma 1. *$\hat{\sigma}^2_Z = \hat{\sigma}^2_{YW}$ and $\hat{\rho}_Z = \hat{\rho}_{YW}$ for any Y.*

Proof. Trivially,

$$\mathrm{Tr}\,(V_1) = \mathrm{Tr}\,(P_{\mathbf{1}} S P_{\mathbf{1}}) = \mathrm{Tr}\,(S P_{\mathbf{1}}) = \frac{1}{p}\,\mathrm{Tr}(S\mathbf{1}\mathbf{1}') = \frac{1}{p}\mathbf{1}'S\mathbf{1},$$

and

$$\mathrm{Tr}\,(V_2) = \mathrm{Tr}\,(M_{\mathbf{1}} S M_{\mathbf{1}}) = \mathrm{Tr}\,(S M_{\mathbf{1}}) = \mathrm{Tr}(S) - \mathrm{Tr}\,(S P_{\mathbf{1}}) = \mathrm{Tr}(S) - \frac{1}{p}\mathbf{1}'S\mathbf{1}.$$

Substituting these values into (13), we easily get (12). □

Thus, in the following we can write only $\hat{\sigma}^2$ and $\hat{\rho}$.

This orthogonal decomposition is very useful for derivation of the distribution of the estimators.

Lemma 2. *Let $H \sim \mathcal{W}_p(\ell, \Xi)$, $\Xi > 0$, and let $T_{k\times p}$ be an arbitrary matrix. Then,*

$$\mathrm{Tr}(THT') \sim \sum_{i=1}^{r(T)} \lambda_i \chi^2_\ell,$$

where $\lambda_1, \ldots, \lambda_{r(T)}$ are all positive eigenvalues of $T\Xi T'$ and $r(T)$ is the rank of T. In particular,

$$\mathrm{E}\ \mathrm{Tr}(THT') = \ell \sum_{i=1}^{r(T)} \lambda_i\,, \quad \mathrm{Var}\,\mathrm{Tr}(THT') = 2\ell \sum_{i=1}^{r(T)} \lambda_i^2\,.$$

Proof. There must exist independent r.v. $X_1, \ldots, X_\ell$ distributed as $N_p(0, \Xi)$ such that $H = \sum_{i=1}^{\ell} X_i X_i' = G'G$, where $G' = (X_1, \ldots, X_\ell)$. Then, $TX_i \sim N_k\,(0, T\Xi T')\ \ \forall\, i$ (which may be singular). According to the Theorem in [1] it holds

$$\mathrm{Tr}(THT') = \mathrm{Tr}\,\big((GT')'(GT')\big) \sim \sum_{i=1}^{r(T)} \lambda_i \chi^2_\ell,$$

where $\lambda_1, \ldots, \lambda_{r(T)}$ are all positive eigenvalues of $T\Xi T'$. Since χ^2-statistics are independent, the claims about mean and variance are trivial. □

The results concerning distributions of $\mathrm{Tr}\,(V_1)$ and $\mathrm{Tr}\,(V_2)$ can be found in [4], but without proof. We extend these results to the parameters of interest.

Theorem 3. *Distributions of* $\mathrm{Tr}\,(V_1)$ *and* $\mathrm{Tr}\,(V_2)$ *are independent,*

$$\mathrm{Tr}\,(V_1) \sim \frac{\sigma^2[1+(p-1)\rho]}{n-r(X)}\,\chi^2_{n-r(X)},$$
$$\mathrm{Tr}\,(V_2) \sim \frac{\sigma^2(1-\rho)}{n-r(X)}\,\chi^2_{(p-1)(n-r(X))},$$

so that

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{p(n-r(X))}\left[(1+(p-1)\rho)\chi^2_{n-r(X)} + (1-\rho)\chi^2_{(p-1)(n-r(X))}\right],$$
$$\frac{1-\rho}{1+(p-1)\rho}\left[\frac{1+(p-1)\hat{\rho}}{1-\hat{\rho}}\right] \sim F_{n-r(X),(p-1)(n-r(X))}\,.$$

Proof. It is well known that under normality

$$S \sim \mathcal{W}_p\left(n-r(X), \frac{1}{n-r(X)}\,\Sigma\right).$$

(see e.g. Theorem 3.8 in [5]). We want to make use of Lemma 2 with $\ell = n-r(X)$, $\Xi = \dfrac{1}{n-r(X)}\,\Sigma$, $T = P_{\mathbf{1}}$, and also with $T = M_{\mathbf{1}}$. Since

$$P_{\mathbf{1}}\left(\frac{1}{n-r(X)}\,\Sigma\right)P_{\mathbf{1}} = \frac{\sigma^2[1+(p-1)\rho]}{n-r(X)}\,P_{\mathbf{1}}$$

is a multiple of idempotent matrix of rank 1, its only positive eigenvalue is equal to $\sigma^2[1+(p-1)\rho]/(n-r(X))$. Similarly, since $M_{\mathbf{1}}$ is idempotent with rank $p-1$ and

$$M_{\mathbf{1}}\left(\frac{1}{n-r(X)}\,\Sigma\right)M_{\mathbf{1}} = \frac{\sigma^2(1-\rho)}{n-r(X)}\,M_{\mathbf{1}}\,,$$

it has $p-1$ positive eigenvalues which are all equal to $\sigma^2(1-\rho)/(n-r(X))$. Now the results for $\mathrm{Tr}\,(V_1)$ and $\mathrm{Tr}\,(V_2)$ follow from Lemma 2, perpendicularity of $M_{\mathbf{1}}$ and $P_{\mathbf{1}}$, and properties of χ^2-distribution.

This, together with (13), immediately implies the result for $\hat{\sigma}^2$. The second formula in (13) can be transformed to

$$\frac{1+(p-1)\hat{\rho}}{1-\hat{\rho}} = \frac{(p-1)\,\mathrm{Tr}\,(V_1)}{\mathrm{Tr}\,(V_2)}\,.$$

Because the distributions of $\mathrm{Tr}\,(V_1)$ and $\mathrm{Tr}\,(V_2)$ are independent, clearly

$$\frac{\sigma^2(1-\rho)(n-r(X))}{\sigma^2[1+(p-1)\rho](n-r(X))}\cdot\frac{(p-1)\,\mathrm{Tr}\,(V_1)}{\mathrm{Tr}\,(V_2)} \sim F_{n-r(X),(p-1)(n-r(X))}\,.$$

□

This result is not very useful with respect to $\hat{\sigma}^2$, since its distribution depends on both σ^2 and ρ, but enables us to test for any specific value of ρ. Using simple transformation, we can even derive directly the probability density function of $\hat{\rho}$:

$$f(x) = \left(\frac{1-\rho}{(p-1)[1+(p-1)\rho]}\right)^{\frac{n-r(X)}{2}} \frac{\Gamma\left(\frac{p\,(n-r(X))}{2}\right)}{\Gamma\left(\frac{n-r(X)}{2}\right)\Gamma\left(\frac{(p-1)(n-r(X))}{2}\right)}\times$$
$$\times\left(1+\frac{1-\rho}{(p-1)[1+(p-1)\rho]}\frac{1+(p-1)x}{1-x}\right)^{-\frac{p(n-r(X))}{2}}\times$$
$$\times\left(\frac{1+(p-1)x}{1-x}\right)^{\frac{n-r(X)}{2}-1}\frac{p}{(1-x)^2}\,.$$

Also, $1-\alpha$ confidence interval for ρ is given by

$$\left(\frac{1-c_1}{1+(p-1)c_1}\,;\frac{1-c_2}{1+(p-1)c_2}\right), \tag{14}$$

where

$$c_1 = \frac{1-\hat{\rho}}{1+(p-1)\hat{\rho}}\,F_{n-r(X),(p-1)(n-r(X))}\left(1-\frac{\alpha}{2}\right)$$

and

$$c_2 = \frac{1-\hat{\rho}}{1+(p-1)\hat{\rho}}\,F_{n-r(X),(p-1)(n-r(X))}\left(\frac{\alpha}{2}\right).$$

Figures 1–4 below show histograms and theoretical densities of $\hat{\rho}$ for a special case of the model (quadratic growth in three groups, 2500 simulations) for various true values of unknown parameter.

Example 4. Let us consider random sample from bivariate normal distribution with the same variances in both dimensions. It can be formally written as GCM with the uniform correlation structure:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ \vdots & \vdots \\ Y_{n1} & Y_{n2} \end{pmatrix} = \mathbf{1}_n\,(\mu_1,\mu_2)\,I_2 + e\,, \quad e \sim N_{n\times 2}\left(0_{n\times 2}, \sigma^2\begin{pmatrix} 1 & \rho \\ \rho & 1\end{pmatrix}\otimes I_n\right).$$

Using the above mentioned estimator we get

$$\hat{\rho} = \frac{2s_{12}}{s_1^2+s_2^2}\,,$$

where s_{12} is sample covariance of the two variables, and s_1^2 and s_2^2 are sample variances. This estimator is slightly more effective than the standard sample correlation coefficient (in the sense of MSE).

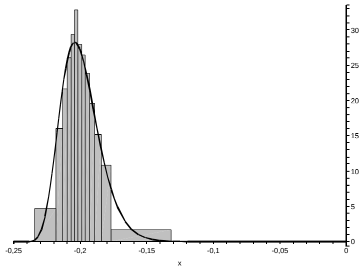

FIGURE 1. $\rho = -0,2$

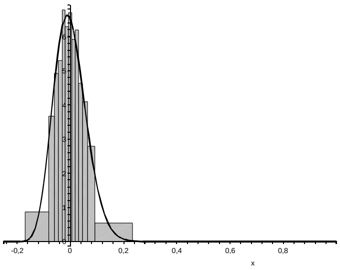

FIGURE 2. $\rho = 0$

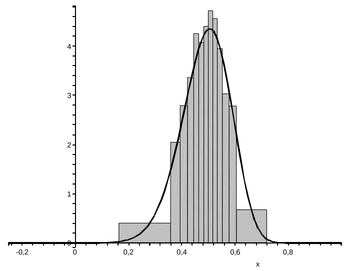

FIGURE 3. $\rho = 0,5$

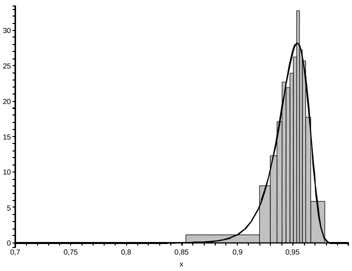

FIGURE 4. $\rho = 0,95$

2. The extended growth curve model

The extended growth curve model (ECGM) with fixed effects, called also the sum-of-profiles model, is

$$Y = \sum_{i=1}^{k} X_i B_i Z_i' + e\,, \quad e \sim N_{n\times p}\left(0, \Sigma \otimes I_n\right). \tag{15}$$

The dimensions of matrices X_i, B_i, and Z_i are $n \times m_i$, $m_i \times r_i$, and $p \times r_i$, respectively. Usually it is supposed that column spaces of X_i's are ordered,

$$\mathcal{R}\left(X_k\right) \subseteq \cdots \subseteq \mathcal{R}\left(X_1\right)\,, \tag{16}$$

while nothing is said about different matrices Z_i. Recently, Hu (see [2]) came up with modification of the model, assuming

$$X_i' X_j = 0 \;\; \forall\, i \neq j. \tag{17}$$

His idea is to separate groups rather then models. We will show that the two models are under certain conditions equivalent.

Example 5. Let us consider EGCM with two groups with different growth patterns – linear and quadratic:

$$Y_{ij} = \begin{cases} \beta_1 + \beta_2 t_j + e_{ij}\,, & i = 1,\ldots,n_1\,,\ j = 1,\ldots,p\,,\\ \beta_3 + \beta_4 t_j + \beta_5 t_j^2 + e_{ij}\,, & i = n_1+1,\ldots,n_1+n_2\,,\ j = 1,\ldots,p\,. \end{cases}$$

This model can be written as

$$Y = \begin{pmatrix} \mathbf{1}_{n_1} & 0 \\ 0 & \mathbf{1}_{n_2} \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ t_1 & \dots & t_p \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{1}_{n_2} \end{pmatrix} \beta_5 \begin{pmatrix} t_1^2 & \dots & t_p^2 \end{pmatrix} + e,$$

or, by the new way, as

$$Y = \begin{pmatrix} \mathbf{1}_{n_1} \\ 0 \end{pmatrix} (\beta_1, \beta_2) \begin{pmatrix} 1 & \dots & 1 \\ t_1 & \dots & t_p \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{1}_{n_2} \end{pmatrix} (\beta_3, \beta_4, \beta_5) \begin{pmatrix} 1 & \dots & 1 \\ t_1 & \dots & t_p \\ t_1^2 & \dots & t_p^2 \end{pmatrix} + e.$$

Note that in the second form in the previous example $\mathcal{R}(Z_1) \subset \mathcal{R}(Z_2)$. This leads to the idea that we can consider a model in which the column spaces of all matrices Z_i are nested, which naturally arises in situations when different groups use polynomial regression functions of different order.

Let us consider model (15) with condition (16), such that

$$n - p \geq r(X_1). \tag{18}$$

Since X_i are 0-1 matrices whose columns are indicators of different groups, without loss of generality we can assume that all columns of X_1 are mutually perpendicular, and columns of every X_{i+1} are a subset of columns of X_i. Let us define $X_k^* = X_k$ and $X_i^* = X_i \setminus X_{i+1}$, $i = 1, \dots, k-1$, where the symbol $X_i \setminus X_{i+1}$ denotes the matrix consisting of those columns of X_i which are not in X_{i+1}. It is easy to see, that $X_i = (X_i^*, \dots, X_k^*)$ and $P_{X_i} - P_{X_{i+1}} = P_{X_{i+1}^{\perp} \cap X_i} = P_{X_i^*}$.

Then, we can reformulate the model (15) with von Rosen's condition (16) in the following way:

$$\begin{aligned} \mathrm{E}\, Y &= \sum_{i=1}^{k} X_i B_i Z_i{}' = \sum_{i=1}^{k} (X_i^*, \dots, X_k^*) \begin{pmatrix} B_{ii}^* \\ \vdots \\ B_{ik}^* \end{pmatrix} Z_i' = \sum_{i=1}^{k} \sum_{j=i}^{k} X_j^* B_{ij}^* Z_i' = \\ &= \sum_{j=1}^{k} \sum_{i=1}^{j} X_j^* B_{ij}^* Z_i' = \sum_{j=1}^{k} X_j^* \left(B_{1j}^*, \dots, B_{jj}^* \right) \begin{pmatrix} Z_1' \\ \vdots \\ Z_j' \end{pmatrix} \overset{\mathrm{df}}{=} \sum_{j=1}^{k} X_j^* B_j^* Z_j^{*\prime} \end{aligned} \tag{19}$$

(matrices X_j^* have dimensions $n \times m_j^*$ and B_{ij}^* $m_j^* \times r_i$, where $m_i = \sum_{j=i}^{k} m_j^*$). It is now easy to see that model (19) satisfies Hu's condition:

$$X_i^{*\prime} X_j^* = 0 \quad \forall\, i \neq j. \tag{20}$$

Moreover, now we have

$$Z_i^* = (Z_1, \dots, Z_i), \quad \forall\, i = 1, \dots, k,$$

which implies $\mathcal{R}(Z_1^*) \subset \dots \subset \mathcal{R}(Z_k^*)$.

ECGM with Hu's condition is much easier to handle. If all matrices X_i^* and Z_i^* are of full rank, then all B_i^* are estimable, and unbiased LSE $\hat{B}_i^*$ depend only on X_i^* and Z_i^*:

$$\hat{B}_i^* = \left(X_i^{*\prime} X_i^*\right)^{-1} X_i^{*\prime} Y \Sigma^{-1} Z_i^* \left(Z_i^{*\prime} \Sigma^{-1} Z_i^*\right)^{-1}, \tag{21}$$

see [2]. Such a closed form was difficult to obtain in the von Rosen model. Even for two components the estimators are rather complicated:

$$\begin{aligned} \hat{B}_1 &= \left(X_1{}'X_1\right)^{-1} X_1{}'Y\Sigma^{-1}Z_1 \left(Z_1{}'\Sigma^{-1}Z_1\right)^{-1} \\ &\quad - \left(X_1{}'X_1\right)^{-1} X_1{}'P_{X_2}Y \left(P_{Z_2}^{\Sigma^{-1}M_{Z_1}^{\Sigma^{-1}}}\right)' \Sigma^{-1}Z_1 \left(Z_1{}'\Sigma^{-1}Z_1\right)^{-1}, \\ \hat{B}_2 &= \left(X_2{}'X_2\right)^{-1} X_2{}'Y\Sigma^{-1}Z_2 \left(Z_2{}'\Sigma^{-1}M_{Z_1}^{\Sigma^{-1}} Z_2\right)^{-1}, \end{aligned}$$

see [7]. Each $\hat{B}_1$ and $\hat{B}_2$ depends on both Z_1 and Z_2, and $\hat{B}_1$ even on X_2.

Estimator of common variance matrix can be split into perpendicular pieces:

$$\hat{\Sigma} = \frac{1}{n - r(X_1)} Y' M_{X_1} Y = \frac{1}{n - \sum_{i=1}^k r(X_i^*)} Y' \left(I - \sum_{i=1}^k P_{X_i^*}\right) Y. \tag{22}$$

In the last expression, the left-hand side term is the estimator using von Rosen's model and right-hand side one using Hu's model. It is easy to see, that the estimators are equivalent, since $X_1 = (X_1^*, \ldots, X_k^*)$.

The situation in Hu's model is much easier also for a special correlation structure $\Sigma = \sigma^2 R$ with R known. The unbiased estimator of residual variance σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n - \sum_{i=1}^k r(X_i) \operatorname{Tr}\left(P_{Z_i}^{R^{-1}} R\right)} \operatorname{Tr}\left[(Y - \hat{Y})'(Y - \hat{Y})\right],$$

where $\hat{Y} = \sum_{i=1}^k P_{X_i} Y \left(P_{Z_i}^{R^{-1}}\right)'$ is the unbiased estimator of $\mathrm{E}\, Y$. For comparison, in the von Rosen model the unbiased estimator of residual variance is

$$\hat{\sigma}^2 = \frac{1}{m} \operatorname{Tr}\left[(Y - \hat{Y})'(Y - \hat{Y})\right],$$

where

$$m = (n - r(X_1)) \operatorname{Tr}(R) + \sum_{i=1}^{k-1} (r(X_i) - r(X_{i+1})) \operatorname{Tr}\left(M_{(Z_1,\ldots,Z_i)}^{R^{-1}} R\right) + \\ + r(X_k) \operatorname{Tr}\left(M_{(Z_1,\ldots,Z_k)}^{R^{-1}} R\right) = \\ = n - \sum_{i=1}^{k-1} (r(X_i) - r(X_{i+1})) \operatorname{Tr}\left(P_{(Z_1,\ldots,Z_i)}^{R^{-1}} R\right) - r(X_k) \operatorname{Tr}\left(P_{(Z_1,\ldots,Z_k)}^{R^{-1}} R\right),$$

see [7].

3. Conclusion

The method of orthogonal decomposition is very promising in complex models. Many tasks, which are very difficult or impossible to handle in basic models, can be done with ease in models consisting of mutually orthogonal components. As it is shown above, simple transformation can change a model into an equivalent which allows to determine explicit forms of estimators and/or their distribution. We hope the method will prove even more useful in the future, either in the models investigated here or in some others.

References

[1] Glueck D. H. and Muller K. E. (1998), *On the trace of a Wishart*, Comm. Statist. Theory Methods **27(9)**, 2137–2141.

[2] Hu J. (2009), *Properties of the explicit estimators in the extended growth curve model*, Statistics **44(5)**, 477–492.

[3] Klein D. and Žežula I. (2007), *On uniform correlation structure*; In: Mathematical Methods In Economics And Industry, Herľany, Slovakia, pp. 94–100.

[4] Ye R. D. and Wang S. G. (2009), *Estimating parameters in extended growth curve models with special covariance structures*, J. Statist. Plann. Inference **139**, 2746–2756.

[5] Žežula I. (1993), *Covariance components estimation in the growth curve model*, Statistics **24**, 321–330.

[6] Žežula I. (2006), *Special variance structures in the growth curve model*, J. Multivariate Anal. **97**, 606–618.

[7] Žežula I. (2008), *Remarks on unbiased estimation of the sum-of-profiles model parameters*, Tatra Mt. Math. Publ. **39**, 45–52.

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