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On the first derivative of the sums of trigonometric series with quasi-convex coefficients of higher order

Xhevat Z. Krasniqi

Abstract. In this paper, for the sum of sine or cosine series with quasi-convex coefficients of higher order, the representation of their first derivatives are found in terms of the r-th differences of coefficients of the series obtained by formal differentiation. Also some estimates in terms of coefficients of the series are obtained for the integrals of the absolute values of those derivatives.

1. Introduction and preliminaries

Let us consider the trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

and

$$\sum_{k=1}^{\infty} a_k \sin kx, \tag{2}$$

whose coefficients tend to zero, in other words,

$$\lim_{k\to\infty} a_k = 0. \tag{3}$$

A numerical sequence $\{a_k\}$ is said to be quasi-convex if

$$\sum_{k=1}^{\infty} k|\triangle^2 a_k| < \infty, \tag{4}$$

where $\triangle a_k = a_k - a_{k+1}$, $\triangle^2 a_k = \triangle(\triangle a_k)$.

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It is a well-known fact that conditions (3) and (4) are satisfied if and only if the sequence $\{a_k\}$ can be expressed as a difference of two convex sequences ($\triangle^2 a_k \geq 0$) that tend to zero (see [6], page 129). Likewise, it is known that the series (1) and (2) with convex coefficients that tend to zero, converge uniformly on each interval $[\varepsilon, \pi]$, $\varepsilon > 0$, and their sums are continuously differentiable on $(0, \pi]$ (see [6], page 129). So, under conditions (3) and (4), the series (1) and (2) possess these characteristics too. We shall denote with $f(x)$ and $g(x)$ the sums of the series (1) and (2), respectively.

S. A. Telyakovskiĭ [4] has investigated the estimates of integrals of $|f'(x)|$ and $|g'(x)|$ on intervals that are inside the interval $(0, \pi]$. Firstly, he studied the aspects of how the integrals of $|f'(x)|$ and $|g'(x)|$ increase on intervals $[\varepsilon, \pi]$, when $\varepsilon \to +0$, if these functions are integrable over their period, and secondly, he studied the aspects of how these integrals decrease on intervals $[0, \varepsilon]$, when $\varepsilon \to +0$, if these functions are integrable over their period. In fact, he studied these integrals over the intervals of the form $[\pi/(m+1), \pi/\ell]$, where $\ell \leq m$ are natural numbers. Then putting $\ell = 1$ and letting $m \to \infty$, he obtained the estimates mentioned above.

In this paper, using differences of higher order of the coefficients of the series (1) and (2), we shall prove some statements that generalize Telyakovskiĭ's results proved in [4]. Before doing this we need some definitions and notation.

A null sequence $\{a_k\}$ is said to be r-fold monotone, with $r \in \mathbb{N}$, if $\triangle^i a_k \geq 0$ for all k and $i = 1, 2, \ldots, r$ (see [3]). Clearly, if $a_k = o(1)$ and $\triangle^r a_k \geq 0$ for any r, then the sequence $\{a_k\}$ is r-fold monotone.

We say that $\{a_k\}$ is a quasi-convex sequence of order r if it tends to zero and satisfies the condition

$$\sum_{k=1}^{\infty} k^r |\triangle^{r+1} a_k| < \infty, \tag{5}$$

where $\triangle^r a_k = \triangle\left(\triangle^{r-1} a_k\right)$.

We note that for $r = 1$ the concept of quasi-convexity of order r reduces to the standard concept of quasi-convexity of a sequence. In addition, condition (5) always implies that $\sum_{k=1}^{\infty} k|\triangle^{r+1} a_k| < \infty$, which is essential in this paper.

The structure of a quasi-convex sequence of order r is characterized by the following result (see [1]): a null sequence $\{a_k\}$ is quasi-convex of order r if and only if it can be represented as a difference of two r-fold monotone sequences.

Throughout this paper O-symbols contain positive constants that may be, in general, different in different estimates.

The rest of the paper is organized as follows. Section 2 contains some helpful lemmas that are needed to prove main results. Section 3 includes the main results. We conclude with Section 4, where we provide a few corollaries of the main results.

Finally, we would like to mention the nice technique by Telyakovskiĭ [4] that encouraged the author to obtain results of the present paper.

2. Helpful lemmas

The following result is well known (see, for example, Bromwich [2]).

Lemma 1. *If* $a_k = O(1)$ *and the series*

$$\sum_{k=1}^{\infty} k^r |\triangle^{r+1} a_k|$$

converges, where r *is a natural number, then the series*

$$\sum_{k=1}^{\infty} k^i |\triangle^{i+1} a_k|$$

converges for $i = 1, 2, \dots, r$.

We denote

$$\lambda_k := k a_k, \quad (k = 1, 2, \dots).$$

Lemma 2. *Let* a_k *be real numbers such that*

$$\triangle^r a_k \to 0 \quad \text{as} \quad k \to \infty, \quad r \in \mathbb{N}. \tag{6}$$

If the condition

$$\sum_{k=1}^{\infty} k |\triangle^{r+1} a_k| < \infty \tag{7}$$

holds, then the condition

$$\sum_{k=1}^{\infty} |\triangle^{r+1} \lambda_k| < \infty \tag{8}$$

holds as well.

Proof. Mathematical induction can be used to prove, with regard to r, that the identity

$$\triangle^{r+1} \lambda_k = k \triangle^{r+1} a_k - (r+1) \triangle^r a_{k+1}$$

holds for all natural numbers r. Therefore

$$\sum_{k=1}^{\infty} \left|\triangle^{r+1} \lambda_k\right| \leq \sum_{k=1}^{\infty} k \left|\triangle^{r+1} a_k\right| + (r+1) \sum_{k=2}^{\infty} \left|\triangle^r a_k\right|. \tag{9}$$

From (6), we have

$$\triangle^r a_k = \sum_{i=k}^{\infty} \left(\triangle^{r+1} a_i\right),$$

thus, using (9), we get

$$\begin{aligned}\sum_{k=1}^{\infty}|\triangle^{r+1}\lambda_k| &\leq \sum_{k=1}^{\infty}k|\triangle^{r+1}a_k|+(r+1)\sum_{k=2}^{\infty}\sum_{i=k}^{\infty}|\triangle^{r+1}a_i|\\ &\leq \sum_{k=1}^{\infty}k|\triangle^{r+1}a_k|+(r+1)\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}1\right)\left|\triangle^{r+1}a_k\right|\\ &\leq (r+2)\sum_{k=1}^{\infty}k|\triangle^{r+1}a_k|.\end{aligned}$$

In this way (8) follows from (7). □

Lemma 3. *Let a_k be real numbers that satisfy conditions (3) and (5). Then for $\ell=1,2,\dots,r$, the following holds:*

$$\triangle^{\ell}\lambda_k \to 0 \quad as \quad k\to\infty. \tag{10}$$

Proof. We have

$$\triangle^{\ell}\lambda_k = k\triangle^{\ell}a_k-\ell\triangle^{\ell-1}a_{k+1},\quad \ell=1,2,\dots,r,$$

and since

$$\left|k\triangle^{\ell}a_k\right|=\left|k\sum_{i=k}^{\infty}\left(\triangle^{\ell+1}a_i\right)\right|\leq\sum_{i=k}^{\infty}i\left|\triangle^{\ell+1}a_i\right|\to 0 \quad \text{as} \quad k\to\infty,$$

(10) follows. □

Remark 1. We note that for $r=1$, Lemma 2 and Lemma 3 partially reduce to the Lemmas 1 and 2, respectively, proved in [4].

3. Main results

Let us first denote

$$\begin{aligned}B_0^1(x) &= \frac{1}{2},\\ B_k^1(x) &= \frac{1}{2}+\cos x+\cdots+\cos kx \quad \text{for} \quad k\geq 1,\\ B_k^r(x) &= \sum_{\nu=0}^{k}B_{\nu}^{r-1}(x) \quad \text{for} \quad r=2,3,\dots \ \text{and} \ k\geq 0;\\ \widetilde{B}_k^1(x) &= \sin x+\cdots+\sin kx \quad \text{for} \quad k\geq 1,\\ \widetilde{B}_k^r(x) &= \sum_{\nu=0}^{k}\widetilde{B}_{\nu}^{r-1}(x) \quad \text{for} \quad r=2,3,\dots \ \text{and} \ k\geq 1.\end{aligned}$$

Theorem 4. *If the coefficients of the series (2) tend to zero and satisfy condition (5), then for the first derivative of its sum the following equality holds:*

$$g'(x) = \sum_{k=0}^{\infty} \triangle^{r+1} \lambda_k B_k^{r+1}(x), \qquad 0 < x \leq \pi. \tag{11}$$

Proof. By Lemma 1 (for $i = 1$), the series (2) converges uniformly on $[\varepsilon, \pi]$, $\varepsilon > 0$, therefore its $(C, 1)$-means

$$\sigma_n(g; x) := \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k \sin kx$$

converge uniformly on $[\varepsilon, \pi]$ as well.

Let us prove that $\sigma_n'(g; x)$ converge uniformly to $g'(x)$ on $[\varepsilon, \pi]$. Indeed, if we denote

$$\beta_k := \left(1 - \frac{k}{n+1}\right) \lambda_k, \quad k = 0, 1, \ldots, n+1,$$

then

$$\sigma_n'(g; x) = \frac{\beta_0}{2} + \sum_{k=1}^{n} \beta_k \cos kx \quad (\beta_0 = \beta_{n+1} = 0).$$

Applying $(r+1)$-times Abel's transformation, we obtain

$$\sigma_n'(g; x) = \sum_{k=0}^{n-r} \triangle^{r+1} \beta_k B_k^{r+1}(x) + \sum_{j=1}^{r} \triangle^j \beta_n B_n^{j+1}(x).$$

It is easily shown that

$$\triangle^{r+1} \beta_k = \left(1 - \frac{k}{n+1}\right) \triangle^{r+1} \lambda_k + \frac{r+1}{n+1} \triangle^r \lambda_{k+1} \quad (k + r \leq n),$$

therefore

$$\sigma_n'(g; x) = \sum_{k=0}^{n-r} \triangle^{r+1} \lambda_k B_k^{r+1}(x) + \tau_n(x),$$

where

$$\begin{aligned} \tau_n(x) \quad = \quad & -\frac{1}{n+1} \sum_{k=1}^{n-r} k \triangle^{r+1} \lambda_k B_k^{r+1}(x) + \frac{r+1}{n+1} \sum_{k=0}^{n-r} \triangle^r \lambda_{k+1} B_k^{r+1}(x) \\ & + \frac{1}{n+1} \sum_{j=1}^{r} \left(\triangle^j \lambda_n + j \triangle^{j-1} \lambda_{n+1}\right) B_n^{j+1}(x). \end{aligned}$$

Since, by Lemma 2, the series $\sum_{k=0}^{\infty} |\triangle^{r+1} \lambda_k|$ converges, and

$$B_k^{r+1}(x) \leq \frac{C}{x^{r+1}},$$

where C is a constant independent of k and x (see [5]), the series

$$\sum_{k=0}^{\infty} \triangle^{r+1}\lambda_k B_k^{r+1}(x)$$

converges uniformly on each interval $[\varepsilon, \pi]$, $\varepsilon > 0$. So, our theorem will be proved if we show that $\tau_n(x)$ uniformly tends to zero on $[\varepsilon, \pi]$.

For $x \in [\varepsilon, \pi]$, we have

$$\begin{aligned}|\tau_n(x)| \quad &\leq \quad \frac{C}{\varepsilon^{r+1}}\left\{\frac{1}{n+1}\sum_{k=1}^{n-r} k|\triangle^{r+1}\lambda_k| + \right.\\ &\left. +\frac{r+1}{n+1}\sum_{k=0}^{n-r}|\triangle^{r}\lambda_{k+1}| + \frac{1}{n+1}\sum_{j=1}^{r}\left(|\triangle^{j}\lambda_n| + j|\triangle^{j-1}\lambda_{n+1}|\right)\right\}.\end{aligned}$$

Since $\sum_{k=0}^{\infty}|\triangle^{r+1}\lambda_k| < \infty$,

$$\frac{1}{n+1}\sum_{k=1}^{n-r} k|\triangle^{r+1}\lambda_k| \to 0 \quad \text{as} \quad n \to \infty.$$

This follows by standard arguments. For an arbitrary N, one has

$$\frac{1}{n+1}\sum_{k=1}^{n-r} k|\triangle^{r+1}\lambda_k| \leq \frac{1}{n+1}\sum_{k=1}^{N} k|\triangle^{r+1}\lambda_k| + \sum_{k=N+1}^{\infty}|\triangle^{r+1}\lambda_k|.$$

We first choose, for a given $\varepsilon > 0$, a number $N = N(\varepsilon)$ so that

$$\sum_{k=N+1}^{\infty}|\triangle^{r+1}\lambda_k| < \frac{\varepsilon}{2}.$$

So, for all sufficiently large n, we have

$$\frac{1}{n+1}\sum_{k=1}^{n-r} k|\triangle^{r+1}\lambda_k| < \varepsilon.$$

Also, with help of Lemma 3, we obtain that

$$\frac{r+1}{n+1}\sum_{k=0}^{n-r}|\triangle^{r}\lambda_{k+1}| \to 0$$

and

$$\frac{1}{n+1}\sum_{j=1}^{r}\left(|\triangle^{j}\lambda_n| + j|\triangle^{j-1}\lambda_{n+1}|\right) \to 0 \quad \text{as} \quad n \to \infty,$$

which completes the proof of (11). □

Now we shall prove a similar result concerning the cosine series (1).

Theorem 5. *If the coefficients of the series (1) tend to zero and satisfy condition (5), then for the first derivative of its sum the following equality holds:*

$$f'(x) = -\sum_{k=1}^{\infty} \triangle^{r+1}\lambda_k \widetilde{B}_k^{r+1}(x), \qquad 0 < x \le \pi.$$

Proof. Similarly, by Lemma 1 ($i = 1$), the series (1) converges uniformly on $[\varepsilon, \pi]$, $\varepsilon > 0$, therefore its $(C, 1)$-means

$$\sigma_n(f; x) := \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k \cos kx$$

converge uniformly on $(0, \pi]$ as well.

Keeping same notation as in Theorem 4 and applying Abel's transformation $(r+1)$-times to the equality

$$\sigma_n'(g; x) = -\sum_{k=1}^{n} \beta_k \sin kx,$$

we obtain the equality

$$\sigma_n'(f; x) = -\sum_{k=1}^{n-r-1} \triangle^{r+1}\beta_k \widetilde{B}_k^{r+1}(x) - \sum_{j=1}^{r} \triangle^{j}\beta_n \widetilde{B}_n^{j}(x) - \frac{n}{n+1} a_n \sin nx.$$

Then using the equality

$$\triangle^{r+1}\beta_k = \left(1 - \frac{k}{n+1}\right) \triangle^{r+1}\lambda_k + \frac{r+1}{n+1} \triangle^{r}\lambda_{k+1},$$

we get that

$$\sigma_n'(g; x) = -\sum_{k=1}^{n-r-1} \triangle^{r+1}\lambda_k \widetilde{B}_k^{r+1}(x) + \mu_n(x),$$

where

$$\begin{aligned} \mu_n(x) \;&=\; \frac{1}{n+1}\sum_{k=1}^{n-r-1} k\triangle^{r+1}\lambda_k \widetilde{B}_k^{r+1}(x) - \frac{r+1}{n+1}\sum_{k=1}^{n-r-1} \triangle^{r}\lambda_{k+1} \widetilde{B}_k^{r+1}(x) \\ &\quad - \frac{1}{n+1}\sum_{j=1}^{r} \left(\triangle^{j}\lambda_n + j\triangle^{j-1}\lambda_{n+1}\right) \widetilde{B}_n^{j}(x) - \frac{n}{n+1} a_n \sin nx. \end{aligned}$$

Since

$$\widetilde{B}_k^{r+1}(x) \le \frac{C}{x^{r+1}}$$

(C is a constant independent of k and x), repeating the same reasoning as in the proof of Theorem 3.1, we can show that $\mu_n(x) \to 0$ as $n \to \infty$ (we omit the details). With this we have completed the proof of our theorem. □

The next theorem gives an estimate of the integral of $|g'(x)|$ on the intervals $[\pi/(m+1), \pi/\ell]$, $1 \leq \ell \leq m$.

Theorem 6. *Let the sequence $\{a_k\}$ be quasi-convex of order r and tend to zero. Then*

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell} |g'(x)|dx &= O\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{(k+1)^r}{\ell}|\triangle^r\lambda_k|\right) + \\ &+O\left(\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)k^{r-1}|\triangle^{r+1}\lambda_k|\right).\end{aligned} \tag{12}$$

Proof. By Theorem 4 and Lemma 3, we have

$$\begin{aligned}g'(x) &= \sum_{k=0}^{i-1}\triangle^{r+1}\lambda_k B_k^{r+1}(x) + \sum_{k=i}^{\infty}\triangle^{r+1}\lambda_k B_k^{r+1}(x) \\ &= \sum_{k=0}^{i-1}\triangle^{r}\lambda_k B_k^{r}(x) + \sum_{k=i}^{\infty}\triangle^{r+1}\lambda_k \left[B_k^{r+1}(x) - B_{i-1}^{r+1}(x)\right].\end{aligned}$$

The integral (12) can be written as

$$\int_{\pi/(m+1)}^{\pi/\ell} |g'(x)|dx = \sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}|g'(x)|dx. \tag{13}$$

Therefore

$$\begin{aligned}\int_{\pi/(i+1)}^{\pi/i} |g'(x)|dx &\leq \int_{\pi/(i+1)}^{\pi/i}\sum_{k=0}^{i-1}|\triangle^r\lambda_k||B_k^r(x)|dx + \\ &+\int_{\pi/(i+1)}^{\pi/i}\sum_{k=i}^{\infty}|\triangle^{r+1}\lambda_k||B_k^{r+1}(x)-B_{i-1}^{r+1}(x)|dx.\end{aligned}$$

Applying estimates $|B_k^r(x)| \leq C(k+1)^r$ and $0 \leq B_k^{r+1}(x) \leq C/x^{r+1}$ $(0 < x \leq \pi)$, we obtain

$$\int_{\pi/(i+1)}^{\pi/i} |g'(x)|dx \leq C\sum_{k=0}^{i-1}|\triangle^r\lambda_k|\frac{(k+1)^r}{i(i+1)} + Ci^{r-1}\sum_{k=i}^{\infty}|\triangle^{r+1}\lambda_k|. \tag{14}$$

Thus, from (13) and (14), we get that

$$\int_{\pi/(m+1)}^{\pi/\ell} |g'(x)|dx \leq C\sum_{i=\ell}^{m}\sum_{k=0}^{i-1}|\triangle^r\lambda_k|\frac{(k+1)^r}{i(i+1)} + C\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}k^{r-1}|\triangle^{r+1}\lambda_k|. \tag{15}$$

For the first term of the right-hand side of (15), we have

$$
\begin{aligned}
&\sum_{i=\ell}^{m}\sum_{k=0}^{i-1}|\triangle^r\lambda_k|\frac{(k+1)^r}{i(i+1)} = \\
= \quad & \sum_{i=\ell}^{m}\sum_{k=0}^{\ell-1}|\triangle^r\lambda_k|\frac{(k+1)^r}{i(i+1)} + \sum_{i=\ell+1}^{m}\sum_{k=\ell}^{i-1}|\triangle^r\lambda_k|\frac{(k+1)^r}{i(i+1)} \\
= \quad & \sum_{k=0}^{\ell-1}(k+1)^r|\triangle^r\lambda_k|\left(\frac{1}{\ell}-\frac{1}{m+1}\right) \\
& + \sum_{k=\ell}^{m-1}(k+1)^r|\triangle^r\lambda_k|\left(\frac{1}{k+1}-\frac{1}{m+1}\right) \\
\leq \quad & \frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{(k+1)^r}{\ell}|\triangle^r\lambda_k| + \sum_{k=\ell}^{m}\sum_{j=k}^{\infty}j^{r-1}|\triangle^{r+1}\lambda_j|. \qquad (16)
\end{aligned}
$$

The second term in (15) and (16) can then be written as follows:

$$
\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}k^{r-1}|\triangle^{r+1}\lambda_k| = \sum_{i=\ell}^{m}\sum_{k=i}^{m}k^{r-1}|\triangle^{r+1}\lambda_k| + \sum_{i=\ell}^{m}\sum_{k=m+1}^{\infty}k^{r-1}|\triangle^{r+1}\lambda_k| =
$$

$$
= \sum_{k=\ell}^{m}(k+1-\ell)k^{r-1}|\triangle^{r+1}\lambda_k| + (m+1-\ell)\sum_{k=m+1}^{\infty}k^{r-1}|\triangle^{r+1}\lambda_k|. \qquad (17)
$$

The claim of the theorem follows from (15), (16), and (17). □

Remark 2. A result similar to Theorem 6 can be proved for the cosine series (1).

4. Some corollaries

Putting $r = 1$ in Theorems 4 and 5, respectively, we obtain two useful corollaries. The first was proved in [4] and the second is a version of Theorem 3 proved in the same paper.

Corollary 7. *If the coefficients of the series (2) tend to zero and satisfy condition (5), then for the first derivative of its sum the following equality holds:*

$$
g'(x) = \sum_{k=0}^{\infty}\triangle^2\lambda_k B_k^2(x), \qquad 0 < x \leq \pi.
$$

Corollary 8. *If the coefficients of the series (1) tend to zero and satisfy condition (5), then for the first derivative of its sum the following equality holds:*

$$f'(x) = -\sum_{k=1}^{\infty} \triangle^2 \lambda_k \widetilde{B}_k^2(x), \qquad 0 < x \leq \pi.$$

We note that

$$\triangle^{r+1}\lambda_0 B_0^{r+1}(x) = \triangle^{r-1}\left(\triangle^2\lambda_0\right)\frac{1}{2} = -\triangle^r a_1,$$

therefore (11) can be written as

$$g'(x) + \triangle^r a_1 = \sum_{k=1}^{\infty} \triangle^{r+1}\lambda_k B_k^{r+1}(x).$$

Since the functions $B_k^{r+1}(x)$ are nonnegative for $x \in (0,\pi]$ (see [5]), the following corollary holds true.

Corollary 9. *Let the sequence $\{a_k\}$ be quasi-convex of order r and tend to zero. Then the following assertions hold.*

(1) *If*
$$\triangle^{r+1}\lambda_k \geq 0 \quad \textit{for all} \quad k = 1, 2, \dots,$$
then the function
$$g(x) + \triangle^r a_1 x$$
is non-decreasing on $(0,\pi]$.

(2) *If*
$$\triangle^{r+1}\lambda_k \leq 0 \quad \textit{for all} \quad k = 1, 2, \dots,$$
then the function
$$g(x) + \triangle^r a_1 x$$
is non-increasing on $(0,\pi]$.

(3) *If a_k are zero but $\triangle^r a_1 < 0$, or $\triangle^r a_1 > 0$, then the function $g(x)$ is strictly increasing or, respectively, strictly decreasing on $(0,\pi]$.*

From Corollary 9 we immediately obtain the following result proved in [4].

Corollary 10. *Let the sequence $\{a_k\}$ be quasi-convex and tend to zero. Then the following assertions hold.*

(1) *If*
$$\triangle^2\lambda_k \geq 0 \quad \textit{for all} \quad k = 1, 2, \dots,$$
then the function
$$g(x) + \triangle a_1 x$$
is non-decreasing on $(0,\pi]$.

(2) *If*

$$\triangle^2\lambda_k \le 0 \quad \textit{for all} \quad k = 1, 2, \dots,$$

then the function

$$g(x) + \triangle a_1 x$$

is non-increasing on $(0, \pi]$.

(3) *If all the numbers* a_k *are zero but* $\triangle a_1 < 0$ *or* $\triangle a_1 > 0$, *then the function* $g(x)$ *is strictly increasing or, respectively, strictly decreasing on* $(0, \pi]$.

In the end, let us formulate a corollary proved in [4] that follows directly from Theorem 6 (for $r = 1$).

Corollary 11. *Let the sequence* $\{a_k\}$ *be quasi-convex and tend to zero. Then*

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell} |g'(x)|dx \quad &= \quad O\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{k+1}{\ell}|\triangle\lambda_k|\right)\\ &\qquad + O\left(\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)|\triangle^2\lambda_k|\right).\end{aligned}$$

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Department of Mathematics, University of Prishtina, Avenue "Mother Theresa" 5, Prishtinë 10000, Republic of Kosova

E-mail address: xheki00@hotmail.com