ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 15, Number 1, 2011
Available online at www.math.ut.ee/acta/

# Admissibility of linear predictor in the extended growth curve model

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Abstract. In the present paper, we first give the definition of the extended growth curve model, then according to the definition of admissible linear predictor and some matrix properties, obtain the necessary and sufficient conditions for a linear predictor to be admissible in the classes of homogenous and inhomogeneous linear predictors, respectively.

## 1. Introduction

The growth curve model (GCM) was introduced by Potthoff and Roy (1964) and equals

$$Y_{n\times q} = X_{n\times m}B_{m\times p}Z_{p\times q} + \varepsilon_{n\times q},\ \mathbb{E}(\varepsilon) = 0,\ \mathbb{C}ov(\overrightarrow{\varepsilon}) = \Sigma_{q\times q} \otimes I_{n\times n}. \quad (1.1)$$

The first paper on growth curves was presented by Wishart (1938), who recommended that a general regression model should be fitted to each curve and that the effects of the experimental treatments should be evaluated by analyzing the coefficients of the model. Since then different aspects of the model were analyzed by many authors including Khatri (1966), Krishnaiah (1969), Gleser and Olkin (1970), Srivastava and Khatri (1979) and von Rosen (1989). Von Rosen (1991) showed a very interesting and detailed review on the GCM which included methods of estimating the parameters in the model, tests, covariance structures, confidence intervals, results based on a canonical version, restrictions to the parameter space of $B$, incomplete data, Bayesian approaches and the use of the GCM in longitudinal studies. Lee (1988) performed prediction and estimation of the growth curves with special covariance structures. Jiang and Su (2003) presented two optimal

Received July 30, 2010.
2010 *Mathematics Subject Classification.* 62C15, 62F10.
*Key words and phrases.* Extended growth curve model, linear predictor, loss function, admissibility.
Foundation item: supported by Huazhong Agricultural University Doctoral Fund (52204-02067), Interdisciplinary Fund (2008xkjc008) and Torch Plan Fund (2009XH003).

linear predictions of the growth curve model. Zhang and Qin (2010) showed the admissibility in the GCM with respect to the restricted parameter sets under matrix loss function.

Verbyla and Venables (1988) added profiles to the GCM with some concrete problems and got the Extended Growth Curve Model (EGCM) which was defined as follows:

$$Y_{n\times q} = \sum_{i=1}^{k} X_i B_i Z_i + \varepsilon_{n\times q},\ \mathbb{E}(\varepsilon) = 0,\ \mathbb{C}ov(\overrightarrow{\varepsilon}) = \Sigma_{q\times q} \otimes I_{n\times n}. \tag{1.2}$$

An algorithm which was used to obtain maximum likelihood estimators was presented in their paper, some examples were given to illustrate how the model was created and some remarks about the applications of the model were made too. Von Rosen (1989) also introduced a model under a nested subspace condition and gave the explicit form of the maximum likelihood estimators of the parameters in the model. He also indicated that the nested structure was necessary and sufficient to obtain explicit expressions for the estimators. Von Rosen (1990) showed that the maximum likelihood estimator of $B_i$ was unbiased and gave the expressions for $D[\hat{B}_i]$ and $E(\hat{\Sigma})$. Xiao and Zhu (2006) showed the admissibility of the parameters in the EGCM under some special restrictions.

In this paper, we define an admissible linear predictor of the EGCM (1.2), and get the necessary and sufficient conditions for a linear predictor of the linear predictable variable to be admissible in the classes of homogeneous and inhomogeneous linear predictors, respectively.

## 2. Notation and main results

In the EGCM (1.2), let $X_i : n \times m_i$ and $Z_i : p_i \times q$ be the given design matrix and structure matrix, respectively, let $B_i : p_i \times m_i$ be an unknown parameter matrix for $i = 1, \cdots, k$. Let $Y$ be an $n \times q$ observation matrix and $\varepsilon$ be an $n \times q$ random error matrix. Denote $A'$ the transpose of the matrix $A$. Partition $Y$ and $\varepsilon$ into $Y = (y_1, \ \cdots, y_n)'$ and $\varepsilon = (\varepsilon_1, \ \cdots, \varepsilon_n)'$, where $y_1, \cdots, y_n$ are $n$ independent observations. We make the following assumptions:

$$\mathbb{E}(\varepsilon) = 0,\ \mathbb{C}ov(\varepsilon) \equiv \mathbb{C}ov(\overrightarrow{\varepsilon}) = \Sigma \otimes I_n,$$

where $\overrightarrow{\varepsilon}$ denotes a vector consisting of the columns of $\varepsilon$, $A_1 \otimes A_2$ is the Kronecker product of $A_1$ and $A_2$, $\Sigma(\geq 0)$ is an unknown $q \times q$ covariance matrix and $I_n$ is the $n^{th}$ identity matrix. The notation $A \geq 0$ means that $A$ is a nonnegative definite symmetric matrix.

In the EGCM (1.2), we will use an observation matrix $Y$ to predict the unknown observation matrix $Y_0 = \sum_{i=1}^{k} X_{i0} B_i Z_{i0} + \varepsilon_0$. Let $\mathbb{E}(\varepsilon_0) = 0$, $\mathbb{C}ov(\varepsilon_0) \equiv \mathbb{C}ov(\overrightarrow{\varepsilon_0}) = \Sigma \otimes I_m$. Let $X = (X_1, X_2, \cdots, X_k)$ and $X_0 = (X_{10}, X_{20}, \cdots, X_{k0})$, we usually want to predict $KY_0L$, where $K$ and $L$ are known $s \times m$ and $q \times l$ matrices. It is easy to prove that if $\mu(X_0'K') \subseteq \mu(X')$ and $\mu(Z_0L) \subseteq \mu(Z)$, then $KY_0L$ is linearly predictable, where $\mu(\cdot)$ denotes the vector space spanned by the columns of a matrix.

Without loss of generality, let $k = 2$ in the EGCM (1.2). Define the matrix loss function and risk function of $d(Y)$ as follows:

$$L(d(Y), B_1, B_2, \Sigma) = (d(Y) - KY_oL)(d(Y) - KY_oL)',$$

and

$$R(d(Y), B_1, B_2, \Sigma) = \mathbb{E}[(d(Y) - KY_oL)(d(Y) - KY_oL)'].$$

Denote the parameter space $\Theta = \{(B_1, B_2, \Sigma) : \Sigma \geq 0, B_i$ is a real matrix, $i = 1, 2\}$. The definitions of admissibility will be presented in the following.

**Definition 2.1.** The homogeneous linear predictable class of predictable function $KY_0L$ is

$$\begin{aligned}\pounds_0 = \{&DYF : D \textit{ and } F \textit{ are } s \times n \textit{ and } q \times l \textit{ constant matrices},\\ &Z_iF = Z_{i0}L = H_i,\ i = 1, 2\}.\end{aligned}$$

where $Z_{i0}$ and $H_i$ are $p_i \times q$ and $p_i \times l$ known matrices, $i = 1, 2$.

**Definition 2.2.** The predictor $d_1(Y)$ of $KY_0L$ is called consistently superior to another linear predictor $d_2(Y)$, if $R(d_1(Y), B_1, B_2, \Sigma) \leq R(d_2(Y), B_1, B_2, \Sigma)$ for all $(B_1, B_2, \Sigma) \in \Theta$, and there exists at least one point $(B_{10}, B_{20}, \Sigma_0) \in \Theta$ such that

$$R(d_1(Y), B_{10}, B_{20}, \Sigma_0) - R(d_2(Y), B_{10}, B_{20}, \Sigma_0) \neq 0.$$

**Definition 2.3.** For $DYF \in \pounds_0$, if there does not exist any linear predictor $D_1YF \in \pounds_0$ which is consistently superior to $DYF$, then $DYF$ is called the admissible predictor of $KY_0L$ in the class $\pounds_0$. For convenience, we denote $DYF \overset{\pounds_0}{\sim} KY_0L$.

Let $\widetilde{D} = DX(X'X)^+X'$, $U = (DX - KX_0)(X'X)^+(DX - KX_0)'$ and $A = DX(X'X)^+X'D' - KX_0(X'X)^+X_0'K'$, where $(X'X)^+$ denotes the Moore–Penrose inverse of $X'X$.

**Theorem 2.1.** *Suppose that $DYF \in \pounds_0$ and $KY_0L$ is linearly predictable. Then $DYF \overset{\pounds_0}{\sim} KY_0L$ if and only if*

(1) $DD' = \widetilde{D}\widetilde{D}'$;

(2) $DX = KX_0$; or $DX \neq KX_0$, and for any $a \in (-1, 1)$, the matrix $aU + A$ is not positive semi-definite.

However, sometimes one may consider a predictor with an intercept included in $KY_0L$, so the following notation will be introduced.

**Definition 2.4.** The inhomogeneous class of linear predictors of the predictable function $KY_0L$ is given by

$$\pounds = \{DYF + M : DYF \in \pounds_0,\ M \textit{ is a } s \times l \textit{ constant matrix}\}.$$

**Definition 2.5.** For $DYF + M \in \pounds$, if there does not exist any linear predictor $D_1YF + M_* \in \pounds$ which is consistently superior to $DYF + M$, then $DYF + M$ is called an admissible predictor of $KY_0L$ in the class $\pounds$. For convenience, we denote $DYF + M \overset{\pounds}{\sim} KY_0L$.

**Theorem 2.2.** *Suppose that $DYF + M \in \pounds$ and $KY_0L$ is linearly predictable. Then $DYF + M \overset{\pounds}{\sim} KY_0L$ if and only if*
(1) $DD' = \widetilde{D}\widetilde{D}'$;
(2) $DX = KX_0$, and $M = 0$; or $DX \neq KX_0$, and for any $a \in (-1, 1)$, the matrix $aU + A$ is not positive semi-definite.

From a statistical perspective, the importance of Theorem 2.1 and Theorem 2.2 stems from the fact that one only needs to find $D$ and easily check if it can satisfy conditions (1) and (2) whether one wants to get a good predictor of $KY_0L$.

*Remark* 2.1. (1) From a decision theoretic perspective, the admissibility is an essential property. In the viewpoint, a good predictor describes that the predictor makes less loss than the other predictors.
(2) The model in the present paper has no restrictions to the probability distribution, whereas the model in Verbyla and Venables (1988) requires a normal distribution assumption.
(3) It is difficult to obtain these results if we cancel the restriction $Z_iF = Z_{i0}L = H_i$ for $i = 1, 2$ in $\pounds_0$. The problem can be regarded as an open problem which needs further discussion.

## 3. Proof of Theorem 2.1

In order to prove Theorem 2.1, we first give the following lemmas.

**Lemma 3.1.** *If $\xi$ and $\xi_0$ are $n \times q$ and $m \times q$ stochastic matrices, respectively, with $\mathbb{E}\xi = M$, $\mathbb{C}ov(\overrightarrow{\xi}) = \Sigma \otimes I_n$, $\mathbb{E}\xi_0 = M_0$, $\mathbb{C}ov(\overrightarrow{\xi_0}) = \Sigma \otimes I_m$ and $\mathbb{C}ov(\overrightarrow{\xi}, \overrightarrow{\xi_0}) = O$, where $O$ is a matrix in which all elements are null. Then, for any $q \times q$ constant matrix $W$, we have*
(i) $\mathbb{E}(\xi W \xi') = tr(W\Sigma)I_n + MWM'$;
(ii) $\mathbb{E}(\xi_0 W \xi_0') = tr(W\Sigma)I_m + M_0WM_0'$;
(iii) $\mathbb{E}(\xi W \xi_0') = MWM_0'$.

**Lemma 3.2.** *Under the EGCM (1.2), for any $DYF \in \pounds_0$, we have*

$$\begin{aligned}
\text{(i) } R(DYF, B_1, B_2, \Sigma) &= \textstyle\sum_{i=1}^{2}(DX_iB_iZ_iF - KX_{i0}B_iZ_{i0}L) \\
&\quad \times \textstyle\sum_{i=1}^{2}(DX_iB_iZ_iF - KX_{i0}B_iZ_{i0}L)' \\
&\quad + tr(F'\Sigma F)DD' + tr(L'\Sigma L)KK' \\
&= \textstyle\sum_{i=1}^{2}(DX_i - KX_{i0})(B_iZ_iF) \\
&\quad \times \textstyle\sum_{i=1}^{2}(B_iZ_iF)'(DX_i - KX_{i0})' \\
&\quad + tr(F'\Sigma F)DD' + tr(L'\Sigma L)KK';
\end{aligned}$$

(ii) $R(\widetilde{D}YF, B_1, B_2, \Sigma) = \sum_{i=1}^{2}(DX_i - KX_{i0})(B_iZ_iF)$
$\times \sum_{i=1}^{2}(B_iZ_iF)'(DX_i - KX_{i0})'$
$+tr(F'\Sigma F)DX(X'X)^+X'D' + tr(L'\Sigma L)KK'$;

(iii) $R(DYF, B_1, B_2, \Sigma) \geq R(\widetilde{D}YF, B_1, B_2, \Sigma)$ *for all* $(B_1, B_2, \Sigma) \in \Theta$, *and the equality holds if and only if* $DD' = DX(X'X)^+X'D'$.

**Lemma 3.3.** *Let* $DYF \in \pounds_0$ *and* $DD' = DX(X'X)^+X'D'$. *For any* $D_1YF \in \pounds_0$, *if there does not exist any constant* $a_1$ *such that* $D_1X_1 - KX_{10} = a_1(DX_1 - KX_{10})$ *or there does not exist any constant* $a_2$ *such that* $D_1X_2 - KX_{20} = a_2(DX_2 - KX_{20})$, *then* $D_1YF$ *is not consistently superior to* $DYF$.

*Proof.* By assertion (iii) of Lemma 3.2, in order to obtain the assertion of the present Lemma, we need only to show that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$. We now consider the following cases.

**Case 1:** $DX - KX_0 = 0$. It follows from the assumption of the present Lemma that $D_1X - KX_0 \neq 0$. Let $\Sigma \to 0$, and the conclusion is obvious.

**Case 2:** $DX_1 - KX_{10} \neq 0$ or $DX_2 - KX_{20} \neq 0$. In the present case, we also can see that $D_1X - KX_0 \neq 0$. Otherwise, $D_1X - KX_0 = 0 \cdot (DX - KX_0)$ contradicts the assumption of the present Lemma. Furthermore, we again consider the following three subcases.

**Case 2_1:** There does not exist constants $a_1$ and $a_2$ such that $D_1X_1 - KX_{10} = a_1(DX_1 - KX_{10})$ and $D_1X_2 - KX_{20} = a_2(DX_2 - KX_{20})$. Suppose that the ranks of $(DX_1 - KX_{10})$ and $(DX_2 - KX_{20})$ are $r_1$ and $r_2$, respectively. We perform a singular value decompositions of $(DX_1 - KX_{10})$ and $(DX_2 - KX_{20})$ as

$$DX_1 - KX_{10} = P_1 \begin{pmatrix} \wedge_1 & 0 \\ 0 & 0 \end{pmatrix} Q_1, \ DX_2 - KX_{20} = P_2 \begin{pmatrix} \wedge_2 & 0 \\ 0 & 0 \end{pmatrix} Q_2,$$

where $P_i$ and $Q_i$ are $s \times s$ and $m_i \times m_i$, $i = 1, 2$, orthogonal matrices. $\wedge_1 = diag(\lambda_{11}, \cdots, \lambda_{1r_1})$ and $\wedge_2 = diag(\lambda_{21}, \cdots, \lambda_{2r_2})$ are diagonal matrices, here $\lambda_{1j} > 0$, $j = 1, \cdots, r_1$, are the singular values of $(DX_1 - KX_{10})$ and $\lambda_{2j} > 0$, $j = 1, \cdots, r_2$, are the singular values of $(DX_2 - KX_{20})$. Let $P = (P_1 \ P_2)$ and $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, and denote

$$Z_{2s \times (m_1+m_2)} = P'(D_1X_1 - KX_{10} \ D_1X_2 - KX_{20})Q'.$$

Then,

$$(D_1X_1 - KX_{10} \ D_1X_2 - KX_{20}) = \frac{1}{2}PZQ.$$

(i) There exists at least $z_{ij} \neq 0$, where $i \neq j$ or $r_1 < i = j < m_1$ or $m_1 + r_2 < i = j < m_1 + m_2$, if $Z \neq \begin{pmatrix} G & O \\ O & O \end{pmatrix}$, where $G = diag(g_1, \cdots, g_r)$ is a $r^{th}$ diagonal matrix. Without loss of generality, we may assume $i \neq j$ and $i, j < m_1$. Let $e_i$ denote the $i^{th}$ column of the $2s^{th}$ identity matrix $I_{2s}$, $d_j$

denote the $j^{th}$ column of the $m_1^{th}$ identity matrix $I_{m_1}$, $f_j$ denote the $j^{th}$ column of the $l^{th}$ identity matrix $I_l$. Let $\alpha = Pe_i$, $B_1(m)H_1 = mQ_1'd_jf_j'$ and $B_2H_2 = 0$. We have

$$\begin{aligned}
&\alpha'[R(\widetilde{D}_1YF, B_1(m), B_2, I_{q\times q}) - R(DYF, B_1(m), B_2, I_{q\times q})]\alpha \\
&\quad = tr(F'F)\alpha'(DD' - D_1X(X'X)^+X'D_1')\alpha \\
&\qquad + \frac{1}{4}\alpha' PZQ \begin{pmatrix} B_1(m)H_1 \\ B_2H_2 \end{pmatrix} \begin{pmatrix} B_1(m)H_1 \\ B_2H_2 \end{pmatrix}' (PZQ)'\alpha - 0 \\
&\quad = tr(F'F)\alpha'(DD' - D_1X(X'X)^+X'D_1')\alpha + \frac{1}{4}(mz_{ij} + c_1)^2 \\
&\quad \to +\infty, \text{ as m} \to +\infty,
\end{aligned}$$

where $c_1$ is a constant.

(ii) If $Z = \begin{pmatrix} G & O \\ O & O \end{pmatrix}$ with $G = diag(g_1, \cdots, g_r)$, we can see that $G \neq 0$, otherwise, $D_1X_1 - KX_{10} = 0$ and $D_1X_2 - KX_{20} = 0$, what contradicts the assumption. Obviously $r \geq 2$. We again take the following two subcases into consideration.

($1^0$) If there exists $g_i = 0$ and $g_j \neq 0$, $j \neq i$. Without loss of generality, we might as well assume $g_1 \neq 0$ and $g_2 = 0$. Taking $\alpha = (1, 1, 0, \cdots, 0)' \in R^{2s}$, $\beta = (1, -\lambda_1/\lambda_2, 0, \cdots, 0)' \in R^{m_1}$ and $\tau \neq 0 \in R^l$, and setting $B_1(m)H_1 = mQ_1'\beta\tau'$ and $B_2H_2 = 0$, we have

$$\begin{aligned}
&\alpha'[R(\widetilde{D}_1YF, B_1(m), B_2, I_{q\times q}) - R(DYF, B_1(m), B_2, I_{q\times q})]\alpha \\
&\quad = tr(F'F)\alpha'(DD' - D_1X(X'X)^+X'D_1')\alpha \\
&\qquad + \frac{1}{4}\alpha' PZQ \begin{pmatrix} B_1(m)H_1 \\ B_2H_2 \end{pmatrix} \begin{pmatrix} B_1(m)H_1 \\ B_2H_2 \end{pmatrix}' (PZQ)'\alpha - 0 \\
&\quad = tr(F'F)\alpha'(DD' - D_1X(X'X)^+X'D_1')\alpha + \frac{1}{4}m^2\tau\tau' \\
&\quad \to +\infty, \text{ as m} \to +\infty.
\end{aligned}$$

($2^0$) If all $g_i \neq 0$ for $i = 1, \cdots, r$, then there exist $i$ and $j$ with $i \neq j$ such that $\frac{g_i}{\lambda_i} \neq \frac{g_j}{\lambda_j}$. We assume $\frac{g_1}{\lambda_1} \neq \frac{g_2}{\lambda_2}$, then

$$\begin{aligned}
&\alpha'[R(\widetilde{D}_1YF, B_1(m), B_2, I_{q\times q}) - R(DYF, B_1(m), B_2, I_{q\times q})]\alpha \\
&\quad = tr(F'F)\alpha'(DD' - D_1X(X'X)^+X'D_1')\alpha \\
&\qquad + \frac{1}{4}\alpha' PZQ \begin{pmatrix} B_1(m)H_1 \\ B_2H_2 \end{pmatrix} \begin{pmatrix} B_1(m)H_1 \\ B_2H_2 \end{pmatrix}' (PZQ)'\alpha - 0 \\
&\quad = tr(F'F)\alpha'(DD' - D_1X(X'X)^+X'D_1')\alpha \\
&\qquad + \frac{1}{4}m^2\tau\tau'(g_1\lambda_2 - g_2\lambda_1)^2/\lambda_2^2 \ \to +\infty, \text{ as m} \to +\infty.
\end{aligned}$$

It can be concluded that $D_1YF$ is not consistently superior to $DYF$.

**Case 2_2:** There does not exist a constant $a_1$ such that $D_1X_1 - KX_{10} = a_1(DX_1 - KX_{10})$, but there exists a constant $a_2$ such that $D_1X_2 - KX_{20} = a_2(DX_2 - KX_{20})$. We can show that $D_1YF$ is not consistently superior to $DYF$, which is similar to the **Case 2_1**.

**Case 2_3:** There does not exist a constant $a_2$ such that $D_1X_2 - KX_{20} = a_2(DX_2 - KX_{20})$, but we have a constant $a_1$ such that $D_1X_1 - KX_{10} = a_1(DX_1 - KX_{10})$. Similarly, it can be seen that $D_1YF$ is not consistently superior to $DYF$.

Hence, the desired result is proved. □

*Proof of Theorem 2.1. Necessity.* Condition (1) holds because of (iii) in Lemma 3.2. We suppose that condition (1) holds, but condition (2) does not hold, that is, $DX \neq KX_0$, and there exists $a_0 \in (-1, 1)$ such that

$$a_0U + A \geq 0. \tag{3.1}$$

Let $D_0 = a_0DX(X'X)^+X' + (1 - a_0)KX_0(X'X)^+X'$. Obviously, $D_0YF \in \pounds_0$, and $D_0$ satisfies condition (1). For convenience, denote $\mathbb{B} = \binom{B_1H_1}{B_2H_2}$. By condition (1), Lemma 3.2 and (3.1), for any $(B_1, B_2, \Sigma) \in \Theta$, we have

$$\begin{aligned} R(DYF, B_1, B_2, \Sigma) - R(D_0YF, B_1, B_2, \Sigma) \\ &= (1 - a_0^2)(DX - KX_0)\mathbb{B}\mathbb{B}'(DX - KX_0)' \\ &\quad + (1 - a_0)tr(F'\Sigma F)(a_0U + A) \geq 0. \end{aligned}$$

Moreover, due to $1 - a_0^2 > 0$ and $DX \neq KX_0$, we know that there exists $\mathbb{B}_0$ such that

$$(1 - a_0^2)(DX - KX_0)\mathbb{B}_0\mathbb{B}_0'(DX - KX_0)' \neq 0.$$

It follows from Definition 2.2 that $D_0YF$ is consistently superior to $DYF$. This contradicts $DYF \overset{\pounds_0}{\sim} KY_0L$, so condition (2) holds.

*Sufficiency.* Suppose that conditions (1) and (2) hold. To prove $DYF \overset{\pounds_0}{\sim} KY_0L$, we need only to prove that for any $D_1YF \in \pounds_0$, $D_1YF$ cannot be consistently superior to $DYF$. From Lemma 3.2 we get that we need only to prove that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$.

If there does not exist a constant $a_i$ such that $D_1X_i - KX_{i0} = a_i(DX_i - KX_{i0})$ for $i = 1, 2$, then from the proof of Lemma 3.3, we know that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$. We now suppose that there exists a constant $a$ such that $D_1X - KX_0 = a(DX - KX_0)$.

If $DX = KX_0$, we have $D_1X = KX_0$. Consequently,

$$R(\widetilde{D}_1YF, B_1, B_2, \Sigma) = R(DYF, B_1, B_2, \Sigma),$$

which implies that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$.

If $DX \neq KX_0$, we have the following cases.

(1) $|a| < 1$. By simple calculations, we obtain

$$\begin{aligned}
&R(DYF, 0, 0, I_{q\times q}) - R(\widetilde{D}_1YF, 0, 0, I_{q\times q}) \\
&\quad = tr(F'F)(DX(X'X)^+X'D' - D_1X(X'X)^+X'D_1') \\
&\quad = tr(F'F)[DX(X'X)^+X'D' - (aDX \\
&\qquad -(a-1)KX_0)(X'X)^+(aDX - (a-1)KX_0)'] \\
&\quad = (1-a)tr(F'F)[(1+a)DX(X'X)^+X'D' - aDX(X'X)^+X_0'K' \\
&\qquad -aKX_0(X'X)^+X'D' - (1-a)KX_0(X'X)^+X_0'K'] \\
&\quad = (1-a)tr(F'F)[DX(X'X)^+X'D' - KX_0(X'X)^+X_0'K' \\
&\qquad +a(DX - KX_0)(X'X)^+(DX - KX_0)'] \\
&\quad = (1-a)tr(F'F)(aU + A).
\end{aligned}$$

The derivation above shows that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$.

(2) $|a| > 1$. It follows from $DX \neq KX_0$ that there exist $\alpha \in \mathbb{R}^s$ and $\mathbb{B}_0$ such that

$$\alpha'(DX - KX_0)\mathbb{B}_0\mathbb{B}_0'(DX - KX_0)'\alpha > 0.$$

Then,

$$\begin{aligned}
&\alpha'[R(\widetilde{D}_1YF, m\mathbb{B}_0, I_{q\times q}) - R(DYF, m\mathbb{B}_0, I_{q\times q})]\alpha \\
&\quad = tr(F'F)\alpha'[D_1X(X'X)^+X'D_1' - DX(X'X)^+X'D']\alpha \\
&\qquad +m^2(a^2-1)\alpha'(DX - KX_0) \\
&\qquad \times\mathbb{B}_0\mathbb{B}_0'(DX - KX_0)'\alpha \to +\infty, \ as\ m \to \infty,
\end{aligned}$$

which means that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$.

(3) $a = 1$. In this case, $R(\widetilde{D}_1YF, \mathbb{B}, \Sigma) = R(DYF, \mathbb{B}, \Sigma)$. Therefore, $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$.

(4) $a = -1$. It is easy to see that the matrix $A$ is not positive semi-definite and there exists $\alpha \in \mathbb{R}^s$ such that $\alpha'A\alpha < 0$. Hence, we have

$$\alpha'[R(DYF, 0, I_{q\times q}) - R(\widetilde{D}_1YF, 0, I_{q\times q})]\alpha = 2tr(F'F)\alpha'(A - U)\alpha < 0,$$

which means that $\widetilde{D}_1YF$ cannot be consistently superior to $DYF$.

Thus, the desired assertion is shown. □

## 4. Proof of Theorem 2.2

In order to complete the proof of Theorem 2.2, we first give two Lemmas as follows.

**Lemma 4.1.** *Under the EGCM (1.4), let $DYF + M \in \pounds$, and we have*

(1) $R(DYF + M, \mathbb{B}, \Sigma)$

$$\begin{aligned}
&= (DX - KX_0)\mathbb{B}\mathbb{B}'(DX - KX_0)' \\
&\quad +tr(F'\Sigma F)DD' + tr(L'\Sigma L)KK' + (DX - KX_0)\mathbb{B}M' \\
&\quad +M\mathbb{B}'(DX - KX_0)' + MM';
\end{aligned}$$

(2) $R(\widetilde{D}YF+M,\mathbb{B},\Sigma)$
$$= (DX-KX_0)\mathbb{B}\mathbb{B}'(DX-KX_0)'+tr(F'\Sigma F)DX(X'X)^{+}X'D' +tr(L'\Sigma L)KK'+(DX-KX_0)\mathbb{B}M' +M\mathbb{B}'(DX-KX_0)'+MM';$$

(3) $R(DYF+M,\mathbb{B},\Sigma)\geq R(\widetilde{D}YF+M,\Sigma,\mathbb{B})$ *for any* $(B_1,B_2,\Sigma)$*, and the equality holds if and only if* $DD'=\widetilde{D}\widetilde{D}'$*.*

**Lemma 4.2.** *Let* $DYF+M\in \pounds$*, and* $DD'=\widetilde{D}\widetilde{D}'$*. For any* $D_1YF+M_*\in\pounds$*, if there does not exist constant* $a_1$ *such that* $D_1X_1-KX_{10}=a_1(DX_1-KX_{10})$*, or there does not exist constant* $a_2$ *such that* $D_1X_2-KX_{20}=a_2(DX_2-KX_{20})$*, then* $D_1YF+M_*$ *is not consistently superior to* $DYF+M$*.*

It is easy to obtain the proofs of the two lemmas by imitating the proofs of Lemma 3.2 and Lemma 3.3, respectively.

*Proof of Theorem 2.2. Necessity.* According to Lemma 4.1, it is clear that condition (1) holds. Suppose that condition (1) holds, but condition (2) does not hold.

If $DX=KX_0$ and $M\neq 0$, we see from Theorem 2.1 that $DYF$ is consistently superior to $DYF+M$. However, it contradicts with $DYF+M\overset{\pounds}{\sim}KY_0L$ and implies $DX=KX_0$ and $M=0$.

If $DX\neq KX_0$, and there exists $a_0\in(-1,1)$ such that $a_0U+A\geq 0$, we have

$$\begin{aligned}
&R(DYF+M,\mathbb{B},\Sigma)-R(D_0YF+a_0M,\mathbb{B},\Sigma)\\
&\quad=(DX-KX_0)\mathbb{B}\mathbb{B}'(DX-KX_0)'\\
&\quad\quad+tr(F'\Sigma F)DX(X'X)^{+}X'D'+tr(L'\Sigma L)KK'\\
&\quad\quad+MM'+(DX-KX_0)\mathbb{B}M'+M\mathbb{B}'(DX-KX_0)'\\
&\quad\quad-[(D_0X-KX_0)\mathbb{B}\mathbb{B}'(D_0X-KX_0)'+tr(F'\Sigma F)D_0D_0'\\
&\quad\quad+tr(L'\Sigma L)KK'+a_0(D_0X-KX_0)\mathbb{B}M'\\
&\quad\quad+a_0M\mathbb{B}'(D_0X-KX_0)'+a_0^2MM']\\
&\quad=(1-a_0^2)(DX-KX_0)\mathbb{B}\mathbb{B}'(DX-KX_0)'\\
&\quad\quad+(1-a_0^2)tr(F'\Sigma F)DX(X'X)^{+}X'D'+(1-a_0^2)(DX-KX_0)\mathbb{B}M'\\
&\quad\quad+(1-a_0^2)M\mathbb{B}'(DX-KX_0)'-a_0(1-a_0)tr(F'\Sigma F)DX(X'X)^{+}X_0'K'\\
&\quad\quad-a_0(1-a_0)tr(F'\Sigma F)KX_0(X'X)^{+}X'D'\\
&\quad\quad-(1-a_0)^2tr(F'\Sigma F)KX_0(X'X)^{+}X_0'K'+(1-a_0^2)MM'\\
&\quad=(1-a_0^2)[(DX-KX_0)\mathbb{B}+M][(DX-KX_0)\mathbb{B}+M]'\\
&\quad\quad+(1-a_0)tr(F'\Sigma F)(a_0U+A)\geq 0,
\end{aligned}$$

where the definition of $D_0$ is the same as that in the proof of Theorem 2.1, that is, $D_0=a_0DX(X'X)^{+}X'+(1-a_0)KX_0(X'X)^{+}X'$. Moreover, there exists $\mathbb{B}_0$ such that

$$(1-a_0^2)[(DX-KX_0)\mathbb{B}_0+M][(DX-KX_0)\mathbb{B}_0+M]'\neq 0.$$

This means by Definition 2.2 that $D_0YF + a_0M$ is consistently superior to $DYF + M$. The above two cases contradict $DYF + M \stackrel{\pounds}{\sim} KY_0L$, and hence condition (2) also holds.

*Sufficiency.* We suppose that conditions (1) and (2) hold. To prove $DYF + M \stackrel{\pounds}{\sim} KY_0L$, we only need to show that for any $D_1YF + M_* \in \pounds$, $D_1YF + M_*$ cannot be consistently superior to $DYF + M$. By Lemma 4.1, we only need to prove that $\widetilde{D}_1YF + M_*$ cannot be consistently superior to $DYF + M$.

If there does not exist a constant $a_i$ such that $D_1X_i - KX_{i0} = a_i(DX_i - KX_{i0})$ for $i = 1, 2$, then from Lemma 4.2 we know that $\widetilde{D}_1YF + M_*$ cannot be consistently superior to $DYF + M$.

If $DX \neq KX_0$, we now discuss the following cases.

(1) $DD' = \widetilde{D}\widetilde{D}'$, $DX = KX_0$ and $M = 0$. In this case, we have

$$R(DYF + M, B_1, B_2, \Sigma) = tr(F'\Sigma F)DX(X'X)^+X'D' + tr(L'\Sigma L)KK'.$$

If $D_1X \neq KX_0$ or $D_1X = KX_0$, but $M_* \neq 0$, then it easily follows that $\widetilde{D}_1YF + M_*$ cannot be consistently superior to $DYF + M$.

If $D_1X = KX_0$ and $M_* = 0$, then we get

$$R(DYF + M, B_1, B_2, \Sigma) \equiv R(\widetilde{D}_1YF + M_*, B_1, B_2, \Sigma),$$

which also means that $\widetilde{D}_1YF + M_*$ cannot be consistently superior to $DYF + M$.

(2) $DD' = \widetilde{D}\widetilde{D}'$, $DX \neq KX_0$, and for any $a \in (-1, 1)$ the matrix $aU + A$ is not positive semi-definite. From Lemma 4.2, we only need to consider the case when there exists a constant $a$ such that $D_1X - KX_0 = a(DX - KX_0)$. We take the following four steps.

(I) $|a| > 1$. Since $DX \neq KX_0$, there exist $\alpha \in \mathbb{R}^s$, $B_{10}$ and $B_{20}$ such that

$$\alpha'(DX - KX_0)\mathbb{B}_0\mathbb{B}_0'(DX - KX_0)'\alpha > 0.$$

Thus we get

$$\begin{aligned}
&\alpha'[R(\widetilde{D}_1YF + M_*, mB_{10}, mB_{20}, I_{q\times q}) \\
&\quad -R(DYF + M, mB_{10}, mB_{20}, I_{q\times q})]\alpha \\
&= tr(F'F)\alpha'(D_1X(X'X)^+X'D_1' - DX(X'X)^+X'D')\alpha \\
&\quad +(a^2-1)m^2\alpha'(DX - KX_0)\mathbb{B}_0\mathbb{B}_0'(DX - KX_0)'\alpha \\
&\quad +2m\alpha'(DX - KX_0)\mathbb{B}_0(aM_* - M)'\alpha \\
&\quad +\alpha'(M_*M_*' - MM')\alpha \ \to +\infty, \ as\ m \ \to \ \infty,
\end{aligned}$$

which implies that $\widetilde{D}_1YF + M_*$ cannot be superior to $DYF + M$.

(II) $|a| < 1$. By condition (2) we know that there exists $\alpha \in \mathbb{R}^s$ such that $\alpha'(aU + A)\alpha < 0$, which means that $\alpha'(DX - KX_0) \neq 0$, and hence there exists $\mathbb{B}_0$ such that $\alpha'(DX - KX_0)\mathbb{B}_0 = -\alpha' M$. Thus,

$$\begin{aligned}
&\alpha'[R(DYF + M, B_{10}, B_{20}, I_{q\times q}) \\
&\qquad -R(\widetilde{D}_1 YF + M_*, B_{10}, B_{20}, I_{q\times q})]\alpha \\
&\quad = tr(F'F)\alpha'[DX(X'X)^+X'D' - D_1X(X'X)^+X'D_1']\alpha \\
&\qquad -\alpha'[(D_1X - KX_0)\mathbb{B}_0 + M_*][(D_1X - KX_0)\mathbb{B}_0 + M_*]'\alpha \\
&\quad = (1-a)tr(F'F)\alpha'(aU + A)\alpha \\
&\qquad -\alpha'[(D_1X - KX_0)\mathbb{B}_0 + M_*][(D_1X - KX_0)\mathbb{B}_0 + M_*]'\alpha < 0.
\end{aligned}$$

Consequently, $\widetilde{D}_1 YF + M_*$ cannot be consistently superior to $DYF + M$.

(III) $a = -1$. By condition (2) we know that there exists $\alpha \in \mathbb{R}^s$ such that $\alpha'(aU + A)\alpha < 0$, which means that $\alpha'(DX - KX_0) \neq 0$, and hence there exists $\mathbb{B}_0$ such that $\alpha'(DX - KX_0)\mathbb{B}_0 = -\alpha' M$. We get

$$\begin{aligned}
&\alpha'[R(DYF + M, B_{10}, B_{20}, I_{q\times q}) \\
&\qquad -R(\widetilde{D}_1 YF + M_*, B_{10}, B_{20}, I_{q\times q})]\alpha \\
&\quad = tr(F'F)\alpha'[DX(X'X)^+X'D' - D_1X(X'X)^+X'D_1']\alpha \\
&\qquad -\alpha'[(D_1X - KX_0)\mathbb{B}_0 + M_*][(D_1X - KX_0)\mathbb{B}_0 + M_*]'\alpha \\
&\quad = 2tr(F'F)\alpha'(aU + A)\alpha \\
&\qquad -\alpha'[(D_1X - KX_0)\mathbb{B}_0 + M_*][(D_1X - KX_0)\mathbb{B}_0 + M_*]'\alpha < 0.
\end{aligned}$$

Therefore, $\widetilde{D}_1 YF + M_*$ cannot be superior to $DYF + M$.

(IV) $a = 1$. We can easily see that

$$\begin{aligned}
&R(DYF + M, B_1, B_2, \Sigma) - R(\widetilde{D}_1 YF + M_*, B_1, B_2, \Sigma) \\
&\quad = [(DX - KX_0)\mathbb{B} + M][(DX - KX_0)\mathbb{B} + M]' \\
&\qquad -[(DX - KX_0)\mathbb{B} + M_*][(DX - KX_0)\mathbb{B} + M_*]' \\
&\quad = MM' - M_*M_*' + (DX - KX_0)\mathbb{B}(M - M_*)' \\
&\qquad +(M - M_*)\mathbb{B}'(DX - KX_0)'.
\end{aligned}$$

If $M = M_*$, then $\widetilde{D}_1 YF + M_*$ cannot be superior to $DYF + M$.

If $M \neq M_*$, since $DX \neq KX_0$, then there exist $\alpha \in \mathbb{R}^s$ and $\mathbb{B}_0$ such that

$$\alpha'(DX - KX_0)\mathbb{B}_0(M - M_*)'\alpha = f \neq 0.$$

Let

$$B_i(m) = \begin{cases} mB_{i0}, & \text{as } f < 0, \\ -mB_{i0}, & \text{as } f > 0, \end{cases} \quad i = 1, 2,$$

where $m$ is a positive constant. It then follows that

$$\begin{aligned}
&\alpha'[R(DYF + M, B_1(m), B_2(m), \Sigma) \\
&\qquad -R(\widetilde{D}_1 YF + M_*, B_1(m), B_2(m), \Sigma)]\alpha \\
&\quad = \alpha'(MM' - M_*M_*')\alpha - 2m|f| \ \to -\infty, \ as\ m \to +\infty,
\end{aligned}$$

which means that $\widetilde{D}_1 YF + M_*$ cannot be consistently superior to $DYF + M$.

Hence, we completed the proof of Theorem 2.2. □

**Acknowledgement.** The authors are grateful to Professor Dietrich von Rosen for constructive suggestions that have improved the presentation of the paper.

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