Polynomial expansions via embedded Pascal's triangles

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Abstract. An expansion is given for polynomials of the form $(\omega + \lambda_1) \cdots (\omega + \lambda_n)$. The coefficients of the resulting polynomials are related to their roots, and a system of equations that enables one to numerically determine the roots in terms of the coefficients is specified. The case where all the roots λ_i are equal is considered as well. A multinomial extension to polynomials of the form $(x_1 + \cdots + x_I)^n$ is then provided. As it turns out, the coefficients of the monomials contained in the resulting polynomial expansion can be determined in terms of the coefficients of the monomials included in the expansion of $(x_1 + \cdots + x_{I-1})^n$ and the rows of embedded Pascal's triangles of successive orders. An algorithm is provided for generating and concatenating these rows, with the particulars of its implementation by means of the symbolic computation software *Mathematica* being discussed as well. Potential applications of such expansions to combinatorics and genomics are also suggested.

1. Introduction

A generalized binomial expansion that has previously been introduced by [9] and [8] for polynomials of the form $\prod_{i=1}^{n}(\omega + \lambda_i)$, is presented in Section 2. As mentioned in [6] and explained in Section 3, when all the roots are equal, this expansion gives rise to an inverse problem, in the sense that given the multiple root of the polynomial, one can directly determine each of the coefficients of its expansion, and conversely, given any of the coefficients, the root can be readily obtained. In the case of distinct roots λ_i, the coefficients are obtained by summing all the combinations of the products of k roots λ_i, $k = 0, \ldots, n$. The number of such combinations is determined by the binomial coefficient, $T_{n,k} = \binom{n}{k}$, where n represents the

degree of the polynomial and k, the number of roots λ_i involved. $T_{n,k}$ is in fact the $(k + 1)$-th entry of the $(n + 1)$-th row of Pascal's triangle.

As it turns out, successive applications of the binomial expansion lead to a general polynomial expansion of $(x_1 + \cdots + x_I)^n$ whose coefficients can be determined by taking the scalar product of those associated with the expansion of $(x_1 + \cdots + x_{I-1})^n$ and a vector of concatenated rows of embedded Pascal's triangles. These results are summarized in [7] and developed in Section 4.

For an exhaustive account of the properties of Pascal's triangle, the reader is referred to [5]. Several results of interest are also included in [10] and [3]. Certain recurrence relationships are discussed, for instance, in [4] and [2], and some extensions, presented in [1].

2. A generalization of the binomial expansion

An expansion of the product

$$\prod_{i=1}^{n}(\omega + \lambda_i)$$

is obtained in this section.

For $n = 1, 2$ and 3, one has

$$\prod_{i=1}^{1}(\omega + \lambda_i) = w + \lambda_1\,,$$

$$\prod_{i=1}^{2}(\omega + \lambda_i) = (\omega + \lambda_1)(\omega + \lambda_2) = \omega^2 + (\lambda_1 + \lambda_2)\omega + \lambda_1\,\lambda_2\,,$$

and

$$\prod_{i=1}^{3}(\omega + \lambda_i) = (\omega + \lambda_1)(\omega + \lambda_2)(\omega + \lambda_3)$$

$$= \omega^3 + (\lambda_1 + \lambda_2 + \lambda_3)\,\omega^2 + (\lambda_1\,\lambda_2 + \lambda_1\,\lambda_3 + \lambda_2\,\lambda_3)\,\omega + \lambda_1\,\lambda_2\,\lambda_3\,,$$

respectively. For a product of n terms, the following generalization, referred to as the Guelph expansion in [8], applies:

$$\prod_{i=1}^{n}(\omega + \lambda_i) = \sum_{k=0}^{n}\omega^{n-k}\sum_{T_{n,k}}\lambda\ldots^{k}\ldots\lambda\,, \tag{1}$$

where $\sum_{T_{n,k}}\lambda\ldots^{k}\ldots\lambda$ denotes the sum of the products of each and every possible combinations of k elements of the set $\lambda_1,\ \ldots,\ \lambda_n$. The number of such combinations is in fact the binomial coefficient given by

$$T_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}\ .$$

For example, consider $\sum_{T_{4,3}} \lambda \ldots^3 \ldots \lambda$. Since in this case, $n = 4$ and $k = 3$, the number of possible combinations of three of the four elements of the set $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is $T_{4,3} = \binom{4}{3} = \frac{4!}{3!\,(1)!} = 4$, which yields

$$\sum_{T_{4,3}} \lambda \ldots^3 \ldots \lambda = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1. \tag{2}$$

Then according to Equation (1), one has for instance

$$\prod_{i=1}^{4}(\omega + \lambda_i) = \sum_{k=0}^{n} \omega^{4-k} \sum_{T_{4,k}} \lambda \ldots^k \ldots \lambda$$

$$= \omega^4 \sum_{T_{4,0}} \lambda \ldots^0 \ldots \lambda \;+\; \omega^3 \sum_{T_{4,1}} \lambda \ldots^1 \ldots \lambda$$

$$+ \omega^2 \sum_{T_{4,2}} \lambda \ldots^2 \ldots \lambda \;+\; \omega \sum_{T_{4,3}} \lambda \ldots^3 \ldots \lambda$$

$$+ \omega^0 \sum_{T_{4,4}} \lambda \ldots^4 \ldots \lambda$$

where, in addition to (2),

$$\sum_{T_{4,0}} \lambda \ldots^0 \ldots \lambda = 1,$$

$$\sum_{T_{4,1}} \lambda \ldots^1 \ldots \lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

$$\sum_{T_{4,2}} \lambda \ldots^2 \ldots \lambda = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4,$$

$$\sum_{T_{4,4}} \lambda \ldots^4 \ldots \lambda = \lambda_1\lambda_2\lambda_3\lambda_4 \,,$$

that is,

$$\prod_{i=1}^{4}(\omega + \lambda_i) = \omega^4 + \omega^3(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$$

$$+ \omega^2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)$$
$$+ \omega\,(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1) + \lambda_1\lambda_2\lambda_3\lambda_4 \,.$$

Clearly, for single-valued roots λ_i this polynomial has the expansion

$$(\omega + \lambda)^4 = \omega^4 + 4\,\lambda\,\omega^3 + 6\lambda^2\omega^2 + 4\lambda^3\,\omega + \lambda^4 \,,$$

and for any positive integer n, one can expand $(\omega + \lambda)^n$ as follows:

$$(\omega + \lambda)^n \; = \; \sum_{k=0}^{n} \omega^{n-k} \underbrace{\sum \lambda \ldots \overset{k}{\ldots} \ldots \lambda}_{T_{n,k}}$$

$$= \; \sum_{k=0}^{n} T_{n,k} \, \lambda^k \omega^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda^k \omega^{n-k} \,, \qquad (3)$$

which is the well-known binomial expansion.

The entries of Pascal's triangle, that is, the binomial coefficients $T_{n,k}$, are displayed in Table 1 for $n = 0, 1, 2, 3, 4$.

TABLE 1. Pascal's triangle (first five rows)

$T_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$	k=0	k=1	k=2	k=3	k=4
n=0	1				
n=1	1	1			
n=2	1	2	1		
n=3	1	3	3	1	
n=4	1	4	6	4	1

For example, one can readily expand $(x + y)^4$ as follows by making use of the last row of Table 1:

$$\sum_{k=0}^{4} T_{4,k} \, y^k x^{4-k} = x^4 + 4 \, y \, x^3 + 6 \, y^2 \, x^2 + 4 y^3 \, x + y^4.$$

3. From polynomial expansion to root identification

Consider the binomial expansion in ω specified by Equation (3), that is,

$$(\omega + \lambda)^n = \sum_{k=0}^{n} a_{n-k} \, \omega^{n-k} = \sum_{k=0}^{n} T_{n,k} \, \lambda^k \omega^{n-k} \,. \qquad (4)$$

In this case, $-\lambda$ is a root of multiplicity n. Thus, for given n and $k = 0, 1, 2, \ldots, n$, one has the identity,

$$a_{n-k} = T_{n,k} \, \lambda^k \,, \qquad (5)$$

which provides a means of determining the coefficients a_{n-k} in the expansion of $(\omega + \lambda)^n$ in terms of the k-th power of the multiple root. Conversely, the following relationship expresses λ in terms of a given coefficient a_{n-k}:

$$\lambda = \left(\frac{a_{n-k}}{T_{n,\,k}} \right)^{\frac{1}{k}}. \tag{6}$$

For example, consider $(\omega + 4)^5$, in which case $\lambda = 4$ and $n = 5$. According to Equation (5), the coefficients of the expansion are

$$
\begin{aligned}
&a_5 = \tbinom{5}{0} 4^0 = \tfrac{5!}{0!(5-0)!} = 1, &\quad &a_4 = \tbinom{5}{1} 4^1 = \tfrac{5!}{1!(5-1)!}\, 4 = 20, \\
&a_3 = \tbinom{5}{2} 4^2 = \tfrac{5!}{2!(5-2)!}\, 16 = 160, &\quad &a_2 = \tbinom{5}{3} 4^3 = \tfrac{5!}{3!(5-3)!}\, 64 = 640, \\
&a_1 = \tbinom{5}{4} 4^4 = \tfrac{5!}{4!(5-4)!}\, 256 = 1280, &\quad &a_0 = \tbinom{5}{5} 4^5 = \tfrac{5!}{5!(5-5)!}\, 1024 = 1024.
\end{aligned}
$$

The resulting expansion in ω is

$$
\begin{aligned}
(\omega + 4)^5 &= a_5\,\omega^5 + a_4\,\omega^4 + a_3\,\omega^3 + a_2\,\omega^2 + a_1\,\omega + a_0 \\
&= \omega^5 + 20\,\omega^4 + 160\,\omega^3 + 640\,\omega^2 + 1280\,\omega + 1024\,.
\end{aligned}
$$

Table 2 summarizes the calculations for determining the coefficients a_{n-k} in terms of $T_{n,k}$ and λ^k when $\lambda = 4$ (the root being -4).

TABLE 2. Calculations for expanding $(\omega + 4)^5$: $n = 5$, $\lambda = 4$

k	0	1	2	3	4	5
$T_{n,k} = \tbinom{n}{k} = \frac{n!}{k!(n-k)!}$	1	5	10	10	5	1
λ^k	1	4	16	64	256	1024
$a_{n-k} = \sum_{T=\binom{n}{k}} \lambda \ldots\ ^k \ldots \lambda = T_{n,k}\,\lambda^k$	1	20	160	640	1280	1024

Now, making use of Equation (6), one can determine λ from any of the coefficients a_{n-k}, the multiple root of the polynomial being $-\lambda$. Table 3 summarizes the calculations.

TABLE 3. Calculations for the determination of $-\lambda$ the root of $(\omega + 4)^5 = \omega^5 + 20\,\omega^4 + 160\,\omega^3 + 640\,\omega^2 + 1280\,\omega + 1024$

k	1	2	3	4	5
$T_{n,k} = \tbinom{n}{k} = \frac{n!}{k!(n-k)!}$	5	10	10	5	1
$a_{n-k} = \sum_{T=\binom{n}{k}} \lambda \ldots\ ^k \ldots \lambda = T_{n,k}\,\lambda^k$	20	160	640	1280	1024
$\lambda = \sqrt[k]{\frac{a_{n-k}}{T_{n,k}}},\quad k \neq 0$	4	4	4	4	4

For expansions of the type $\prod_{i=1}^{n}(\omega + \lambda_i)$, one can utilize the generalized binomial expansion given in Equation (1) in order to identify the coefficients, that is,

$$a_{n-k} = \sum_{T_{n,k}} \lambda \ldots {}^k \ldots \lambda, \quad k = 0, \ldots, n. \tag{7}$$

Thus, in this case, if the roots λ_i are known, one can readily make use of Equation (7) to determine the coefficients a_{n-k}. Conversely, when the coefficients a_{n-k} are known, one obtains a system of algebraic equations in the roots $-\lambda_1, \ldots, -\lambda_n$. For example, when $\lambda_1 = 0.2$, $\lambda_2 = 10$, $\lambda_3 = -2$, $\lambda_4 = 0.5$, the coefficients of the resulting polynomial can be determined as indicated in Table 4.

TABLE 4. Determination of the coefficients of the expansion when $n = 4$ and the roots are $\lambda_1 = 0.2$, $\lambda_2 = 10$, $\lambda_3 = -2$, $\lambda_4 = 0.5$

k	$\sum_{T_{n,k}} \lambda \ldots {}^k \ldots \lambda$	Coefficients in terms of the λ_i's	a_{4-k}
0	$\sum_{T_{4,0}} \lambda \ldots {}^0 \ldots \lambda$	1	1
1	$\sum_{T_{4,1}} \lambda \ldots {}^1 \ldots \lambda$	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$	-8.7
2	$\sum_{T_{4,2}} \lambda \ldots {}^2 \ldots \lambda$	$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$	-14.3
3	$\sum_{T_{4,3}} \lambda \ldots {}^3 \ldots \lambda$	$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1$	13.2
4	$\sum_{T_{4,4}} \lambda \ldots {}^4 \ldots \lambda$	$\lambda_1\lambda_2\lambda_3\lambda_4$	-2

Accordingly, the expansion is

$$\omega^4 - 8.7\,\omega^3 - 14.3\,\omega^2 + 13.2\,\omega - 2.$$

Conversely, if one wishes to determine the roots of this polynomial, one could refer to Table 4, and attempt to solve the following system of equations specified by Equation (7) (excluding $k = 0$):

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & = -8.7 \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 & = -14.3 \\ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 & = 13.2 \\ \lambda_1\lambda_2\lambda_3\lambda_4 & = -2. \end{cases}$$

Such a system of equations has to be solved numerically since it *cannot* be represented as a linear system of the form

$$A\lambda = b\,,$$

where A is an $n \times n$ constant matrix, λ denotes the vector of unknown roots λ_i and b is the vector of polynomial coefficients.

Making use of the *solve* command in *MathLab*, one can obtain the roots as follows:

[l1 l2 l3 l4] = solve('l1+l2+l3+l4 = −8.7, l1*l2+l1*l3+l1*l4+l2*l3+l2*l4

+l3*l4 = −14.3, l1*l2*l3+l1*l2*l4+l2*l3*l4+l3*l4*l1= +13.2, l1*l2*l3*l4

=−2 ');

r1=l1';

r2=l2';

r3=l3';

r4=l4';

% solve for one solution of possible ones, i,e; r1(1), r2(1),r3(1),r4(1)

root1=−r1(1)

root2=−r2(1)

root3=−r3(1)

root4=−r4(1)

As expected, the solution is -10, -0.5, -0.2 and 2. Note that since the system of equations to be solved is not linear, this approach may not be practical for determining the roots of polynomials of degrees greater than five.

4. Polynomial expansions via embedded Pascal's triangles

The expansion given in Equation (3) is now extended to expressions consisting of powers of sums involving I terms. The resulting expansion for $\left(\sum_{i=1}^{I} x_i\right)^n$ is

$$\sum_{\substack{k \\ k' \\ k'' \\ \vdots \\ k^{(I-2)}}} (T_{n,k}T_{k,k'}T_{k',k''}\cdots T_{k^{(I-3)},k^{(I-2)}})x_1^{n-k}x_2^{k-k'}x_3^{k'-k''}\cdots x_{I-1}^{k^{(I-3)}-k^{(I-2)}}x_I^{k^{(I-2)}} \quad (8)$$

where $k, k', k'', k''', \cdots, k^{(I-2)}$ take on the values $k = 0, \ldots, n$; $k' = 0, \ldots, k$; $k'' = 0, \ldots, k'$; $k''' = 0, \ldots, k''$; $\ldots$; $k^{(I-2)} = 0, \ldots, k^{(I-3)}$. The expansion given in Equation (8) is obtained by successive applications of the binomial expansion specified by Equation (4), that is,

$$(\omega + \lambda)^n = \sum_{k=0}^{n} T_{n,k} \, \omega^{n-k} \lambda^k \,.$$

For instance, when $I = 4$, one would first expand $(x_1 + (x_2 + x_3 + x_4))^n$, then expand $(x_2 + (x_3 + x_4))^k$ and finally, $(x_3 + x_4)^{k'}$, resulting in a representation expressed in terms of a triple sum involving the products $(T_{n,k} T_{k,k'} T_{k',k''})$. The range of the subscripts of such products can be represented by so-called k-Tree diagrams. Figure 1 contains a k-Tree diagram which shows the array of possible values of the subscripts k, k' and k'' for $n = 0, 1, 2, 3, 4$. Figure 2 presents symbolically the required entries of Pascal's triangle, namely, $(T_{n,k}, T_{k,k'}, T_{k',k''})$, which form a so-called T-Tree, the numerical values of these binomial coefficients being shown in Figure 3. Letting r_k denote a vector representing the $(k + 1)$-th row of Pascal's triangle, a triangle that comprises $r_0, r_1, \ldots, r_c$ as its rows shall be referred to as a Pascal's triangle of order c, and denoted $\triangle_c$. Observe that the rows appearing in T-Trees are composed of the rows of embedded Pascal's triangles of successive orders.

Table 5 illustrates how the rows of embedded Pascal's triangles are appended in order to determine the coefficients of the expansions. Note that a basic Pascal's triangle is often generated vertically and that the components of r_n are in fact the coefficients of the binomial expansion of $(x+y)^n$. As seen from Table 5, when n is equal to 3 or 4, the expansions for trinomials and quadrinomials raised to the power n, that is, $(x+y+z)^n$ and $(x+y+z+w)^n$, are obtained by making use of the concatenated rows of Pascal's triangles of orders $1, 2, \ldots, n$. Consider for instance $(x + y + z + w)^3$. Observe that the entries appearing on the upper line are $(r_0, r_0, r_1, r_0, r_1, r_2, r_0, r_1, r_2, r_3) \equiv u_3$ and that this vector contains the appended rows of the embedded triangles, $\triangle_0, \triangle_1, \triangle_2, \triangle_3$. The coefficients of the monomials in the expansion of $(x + y + z + w)^3$, which appear on the lower line, are then obtained by multiplying the ten coefficients of the expansion of $(x + y + z)^3$ by the components of the ten corresponding subvectors comprising u_3, that is, $(1 \times r_0, 3 \times r_0, 3 \times r_1, 3 \times r_0, 6 \times r_1, 3 \times r_2, 1 \times r_0, 3 \times r_1, 3 \times r_2, 1 \times r_3) = (1, 3, 3, 3, 3, 6, 6, 3, 6, 3, 1, 3, 3, 3, 6, 3, 1, 3, 3, 1)$.

For the following quadrinomial expression raised to the power 4, one has

$$(x + y + z + w)^4 = \sum_{\substack{k=0,\ldots,4 \\ k'=0,\ldots,k \\ k''=0,\ldots,k'}} (T_{4,k} \, T_{k,k'} \, T_{k',k''}) \, x^{4-k} y^{k-k'} z^{k'-k''} w^{k''} . \tag{9}$$
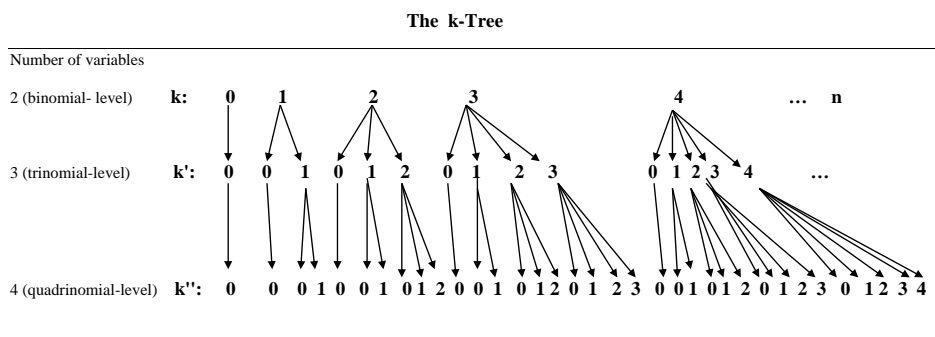
The k-Tree

Number of variables

2 (binomial- level)	**k:**	**0**	**1**	**2**	**3**		**4**		**... n**
3 (trinomial-level)	**k':**	**0**	**0 1**	**0 1 2**	**0 1 2 3**		**0 1 2 3 4**		**...**
4 (quadrinomial-level)	**k'':**	**0**	**0 0 1**	**0 0 1 0 1 2**	**0 0 1 0 1 2 0 1 2 3**		**0 0 1 0 1 2 0 1 2 3 0 1 2 3 4**		

FIGURE 1. *The k-Tree diagram*

The T-Tree (symbolically)

Number of variables

2 (binomial-level)	**T$_{n,k}$:**	**T$_{n,0}$**	**T$_{n,1}$**	**T$_{n,2}$**	**T$_{n,3}$**	**T$_{n,4}$**	**... T$_{n,n}$**
3 (trinomial-level)	**T$_{k;k'}$:**	**T$_{00}$**	**T$_{10}$ T$_{11}$**	**T$_{20}$ T$_{21}$ T$_{22}$**	**T$_{30}$ T$_{31}$ T$_{32}$ T$_{33}$**	**T$_{40}$ T$_{41}$ T$_{42}$ T$_{43}$ T$_{44}$**	**...**
4 (quadrinomial-level)	**T$_{k';k''}$:**	**T$_{00}$**	**T$_{00}$ T$_{10}$ T$_{11}$**	**T$_{00}$ T$_{10}$ T$_{11}$ T$_{20}$ T$_{21}$ T$_{22}$**	**T$_{00}$ T$_{10}$ T$_{11}$ T$_{20}$ T$_{21}$ T$_{22}$ T$_{30}$ T$_{31}$ T$_{32}$ T$_{33}$**	**T$_{00}$ T$_{10}$ T$_{11}$**	
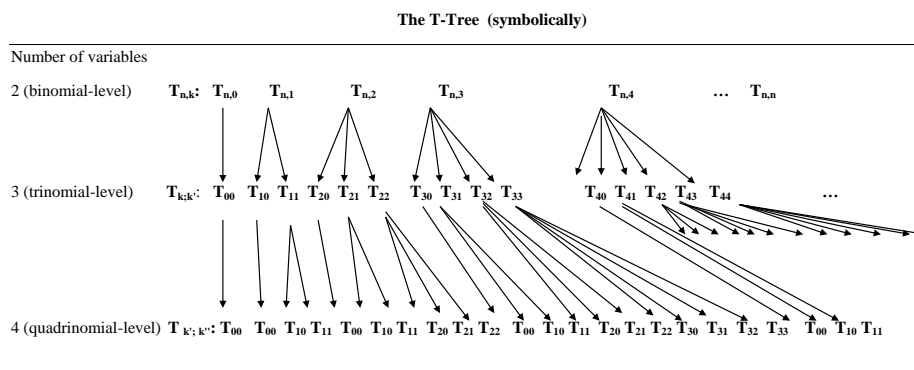
FIGURE 2. *The T-Tree diagram (symbolically)*

The T-Tree (numerical values)

Number of variables

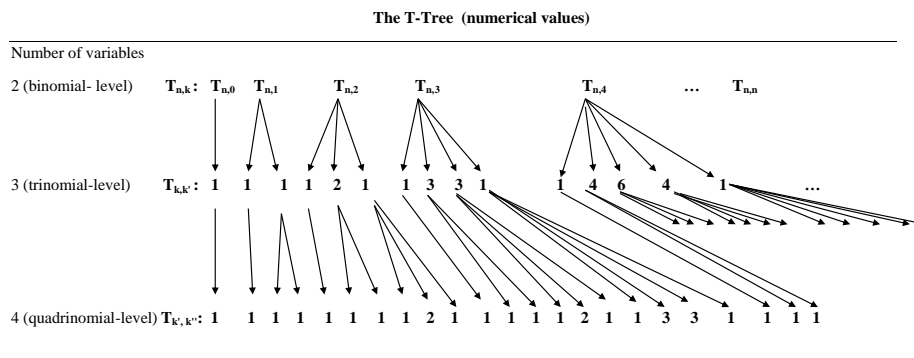|								
2 (binomial- level)	**T$_{n,k}$:**	**T$_{n,0}$**	**T$_{n,1}$**	**T$_{n,2}$**	**T$_{n,3}$**	**T$_{n,4}$**	**... T$_{n,n}$**
3 (trinomial-level)	**T$_{k,k'}$:**	**1**	**1 1**	**1 2 1**	**1 3 3 1**	**1 4 6 4 1**	**...**
4 (quadrinomial-level)	**T$_{k',k''}$:**	**1**	**1 1 1**	**1 1 1 1 2 1**	**1 1 1 1 2 1 1 3 3 1**	**1 1 1 1**	

FIGURE 3. *The T-Tree diagram (numerical values)*

TABLE 5. Embedded Pascal's triangles and the determination of the coefficients of the polynomial expansions

Expansion	Embedded Pascal's Triangles (white background) and Polynomial Coefficients
$(x+y)^0$	1
$(x+y+z)^0$	1
$(x+y+z+w)^0$	1
$(x+y)^1$	1　1
$(x+y+z)^1$	1　1　1
$(x+y+z+w)^1$	1　1　1　1
$(x+y)^2$	1　2　1
$(x+y+z)^2$	1　1　1 / 1　2　2　2　1
$(x+y+z+w)^2$	1　1　1　1 / 1　2　2　2　1　2　1
$(x+y)^3$	1　3　3　1
$(x+y+z)^3$	1　1　1 / 1　3　3　3 / 1　3　6　3　3　1
$(x+y+z+w)^3$	1　1　1　1 / 1　3　3　3　3　3 / 1　3　6　3　6　6　3　3　1
$(x+y)^4$	1　4　6　4　1
$(x+y+z)^4$	1　1　1 / 1　4　4　4 / 1　6　12　6　6 / 1　4　12　24　12　4 / 1　4　6　4　1
$(x+y+z+w)^4$	1　1　1　1 / 1　4　4　4　4　4 / 1　6　12　6　12　12　6　6 / 1　4　12　12　12　12　4　12　12　4 / 1　4　6　4　1

TABLE 6. Detailed tabular coding of the expansion of $(x + y + z + w)^4$ and the probabilities corresponding to the coefficients

n	k	k'	k''	T_{nk}	$T_{kk'}$	$T_{k'k''}$	Coefficients $T_i = T_{nk}T_{kk'}T_{k'k''}$	Terms of the expansion: $x^{n-k}\,y^{k-k'}\,z^{k'-k''}\,w^{k''}$	Probability $P_i = T_i/\Sigma_i T_i$
4	0	0	0	1	1	1	1	x^4	0.00390625
	1	0	0	4	1	1	4	x^3y	0.015625
		1	0	4	1	1	4	x^3z	0.015625
			1	4	1	1	4	x^3w	0.015625
	2	0	0	6	1	1	6	x^2y^2	0.023438
		1	0	6	2	1	12	x^2yz	0.046875
			1	6	2	1	12	x^2yw	0.046875
		2	0	6	1	1	6	x^2z^2	0.023438
			1	6	1	2	12	x^2zw	0.046875
			2	6	1	1	6	x^2w^2	0.023438
	3	0	0	4	1	1	4	xy^3	0.015625
		1	0	4	3	1	12	xy^2z	0.046875
			1	4	3	1	12	xy^2w	0.046875
		2	0	4	3	1	12	xyz^2	0.046875
			1	4	3	2	24	$xyzw$	0.09375
			2	4	3	1	12	xyw^2	0.046875
		3	0	4	1	1	4	xz^3	0.015625
			1	4	1	3	12	xz^2w	0.046875
			2	4	1	3	12	xzw^2	0.046875
			3	4	1	1	4	xw^3	0.015625
	4	0	0	1	1	1	1	y^4	0.00390625
		1	0	1	4	1	4	y^3z	0.015625
			1	1	4	1	4	y^3w	0.015625
		2	0	1	6	1	6	y^2z^2	0.023438
			1	1	6	2	12	y^2zw	0.046875
			2	1	6	1	6	y^2w^2	0.023438
		3	0	1	4	1	4	yz^3	0.015625
			1	1	4	3	12	yz^2w	0.046875
			2	1	4	3	12	yzw^2	0.046875
			3	1	4	1	4	yw^3	0.015625
		4	0	1	1	1	1	z^4	0.00390625
			1	1	1	4	4	z^3w	0.015625
			2	1	1	6	6	z^2w^2	0.023438
			3	1	1	4	4	zw^3	0.015625
			4	1	1	1	1	w^4	0.00390625

Table 6 provides a tabular coding for the expansion given in Equation (9) as well as numerical values for the coefficients $(T_{4,k}T_{k,k'}T_{k',k''}) \equiv T_i$. Naturally, the same coefficients also appear on the last row of Table 5. Note that

the values of $T_{4,k}$, $T_{k,k'}$, and $T_{k',k''}$ given in Table 6 respectively correspond to the values appearing in the $(x + y)^4$ row, the upper $(x + y + z)^4$ row and the upper $(x + y + z + w)^4$ row of Table 5. Thus, appending rows of embedded Pascal's triangles provides a systematic approach for determining the coefficients $T_{i,j}$ appearing in Equation (8).

The probabilities associated with the coefficients, which appear in the last column of Table 6, may be interpreted in terms of an experiment in which four tetrahedral dice bearing the letters (x, y, z, w) on each of their sides are rolled simultaneously. One can determine the probability that the dice will fall on a certain combination of faces by noting that the exponent of a given letter appearing in a certain monomial in the expansion indicates how many times the dice fell on the face bearing that letter. The associated coefficients given in Table 6, once divided by the sum of all of the coefficients, that is, $T_i / \sum_i T_i = P_i$, provide such probabilities. Equivalently, $P_i = T_i / I^n$ where I is the number of variables or faces ($I = 4$), and n is the power of the expansion ($n = 4$) corresponding to the number of dice being rolled, so that in this example, $I^n = 256$. Alternatively, these probabilities can be calculated by multiplying each coefficient by the corresponding monomial in the expansion after letting $x = y = z = w = 1/4$. For example, the second probability in the table can be calculated as follows: $P_2 = T_2(x^3y) = 4[(1/4)^3(1/4)] = 4/256 = 0.015625$.

The above example can readily be extended to an experiment in which the dice have b faces. In such a case, one would expand $(x_1 + x_2 + \cdots + x_b)^n$, where n represents the number of dice being rolled simultaneously. As an application to genetics, one may consider the expansion of $(x + y)^n$ where x and y represent the respective probabilities that a boy or a girl be born, and n denotes the number of children born in a given family. For example, if one needs to determine the probabilities of certain combinations of boys and girls being born out four children, first, one would expand $(x + y)^4$. Then, noting that the coefficients of the expansion correspond to the entries appearing in the fifth row of Pascal's triangle, that is, [1 4 6 4 1], one would have the expansion $x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,xy^3 + y^4$. Thus, if one assumes that $x = y = 1/2$, it follows that the probability that two of the four children are girls is $6/16 = 37.5\%$. Applications in genomics such as the structuring of a single DNA strand are also worth investigating. This would involve the expansion of $(A + T + C + G)^n$ where A, T, C and G are the DNA nucleotides and n is the length of the DNA string. Even though n could be quite large in this context, the coefficients of the resulting multinomial expansions can be efficiently calculated by making use of a recursive algorithm that generates and concatenates rows of embedded Pascal's triangles, as is explained in the next section.

5. Evaluation of the polynomial coefficients in terms of embedded Pascal's triangles

The following recursive algorithm generates embedded Pascal's triangles whose rows are appended into single strings, from which the coefficients of the expansion of $(\sum_{i=1}^{n} a_i)^d$ can be determined.

Notation and algorithm

n is the number of terms being summed;
a_i denotes the i-th term in the sum;
d is the degree to which the sum is being raised $(d \geq 2)$;
r_k is a vector representing the $(k+1)$-th row of Pascal's triangle, $k = 0, 1 \ldots$.

Denoting the i-th component of r_k by $r_k[[i]]$, $i = 1, \ldots, k+1$, and letting $r_k[[1]] = r_k[[k+1]] = 1$, one has $r_k[[i]] = r_{k-1}[[i]] + r_{k-1}[[i+1]]$ for $i = 1, \ldots, k$;
$vs_{d,t}$ denotes the appended rows of embedded Pascal's triangles;
$vs_{d,t}$ is defined recursively as follows in terms of $vs_{d,t-1}$
with $vs_{d,2} = (r_d, r_{d-1}, \ldots, r_0)$
(note the reverse ordering of the $d+1$ subvectors $r_0, r_1, \ldots, r_d$):
$vs_{d,t} = \{\{vs_{d,t-1}[[1]], \ldots, vs_{d,t-1}[[d+1]]\}, \{vs_{d,t-1}[[2]], \ldots, vs_{d,t-1}[[d+1]]\}, \ldots, \{vs_{d,t-1}[[d]], vs_{d,t-1}[[d+1]]\}, vs_{d,t-1}[[d+1]]\}$
where $vs_{d,t-1}[[i]]$ denotes the i-th subvector of $vs_{d,t-1}$;
$cr_{d,s}$ denotes a vector involved in one of the steps leading to the determination of the coefficients of $(\sum_{i=1}^{n} a_i)^d$.

Letting $cr_{d,2} = r_d$,
$cr_{d,s}$ is obtained as follows in terms of $cr_{d,s-1}$ and $vs_{d,s-1}$:
$cr_{d,s} = \{ cr_{d,s-1}[[1]] \bullet vs_{d,s-1}[[1]], \ldots, cr_{d,s-1}[[d+1]] \bullet vs_{d,s-1}[[d+1]]\}$
where for example $cr_{d,s-1}[[1]] \bullet vs_{d,s-1}[[1]]$ is understood to be equal to a vector whose j-th subvector is the product of the j-th component of $cr_{d,s-1}[[1]]$ with the j-th subvector of $vs_{d,s-1}[[1]]$.
$cr_{d,n}$ is the required vector under reverse lexicographic ordering.

This algorithm was implemented in *Mathematica*. Given its simplicity, other computational packages such as *Maple* or *MathLab* could possibly also be utilized. Note that the default sorting procedure used by *Mathematica* for polynomial terms (or monomials) in an expansion corresponds to the negative lexicographic ordering, the variables being sorted in the reverse order. This is commonly known as the reverse lexicographic ordering. This explains why the vectors vs and the vector of coefficients cr are in reverse order with respect to the ordering specified in Section 4. To obtain the vector of the coefficients of $(\sum_{i=1}^{n} a_i)^d$ in the lexicographic order, it suffices to remove the levels (brackets) from $cr_{d,n}$ and then to reverse the order of the resulting components. This yields $c_{d,n}$, the vector of the coefficients under

lexicographic ordering. Note that the *Mathematica* code given below does not involve any divisions or combinatorial quantities.

Mathematica code

$r_0 = 1$;
$r_{k:} = r_k = Flatten[Prepend[Append[Table[r_{k-1}[[i]] + r_{k-1}[[i+1]], \{i, 1, k - 1\}], 1], 1]]$;
$cr_{d,2} := cr_{d,2} = r_d$;
$vs_{d,2} := vs_{d,2} = Table[r_{d-i}, \{i, 0, d\}]$;
$vs_{d,t} := vs_{d,t} = Table[Table[vs_{d,t-1}[[i]], \{i, j, d+1\}], \{j, 1, d+1\}]$;
$cr_{d,s} := cr_{d,s} = Table[cr_{d,s-1}[[i]]vs_{d,s-1}[[i]], \{i, 1, d+1\}]$;
$c_{d,n} := c_{d,n} = Reverse[Flatten[cr_{d,n}]]$;

After specifying n and d, the required vector of coefficients is readily determined in lexicographic order. For instance, on typing $c_{4,4}$, one directly obtains $\{1, 4, 4, 4, 6, 12, 12, 6, 12, 6, 4, 12, 12, 12, 24, 12, 4, 12, 12, 4, 1, 4, 4, 6, 12, 6, 4, 12, 12, 4, 1, 4, 6, 4, 1\}$, as given in Table 6.

Some steps are illustrated below for the case $n = 4$ and $d = 4$. In this case the code yields

r_4
$\{1, 4, 6, 4, 1\}$

$vs_{4,2}$
$\{\{1, 4, 6, 4, 1\}, \{1, 3, 3, 1\}, \{1, 2, 1\}, \{1, 1\}, 1\}$

$vs_{4,3}$
$\{\{\{1, 4, 6, 4, 1\}, \{1, 3, 3, 1\}, \{1, 2, 1\}, \{1, 1\}, 1\}, \{\{1, 3, 3, 1\}, \{1, 2, 1\}, \{1, 1\}, 1\},$
$\{\{1, 2, 1\}, \{1, 1\}, 1\}, \{\{1, 1\}, 1\}, \{1\}\}$

$cr_{4,2}$
$\{1, 4, 6, 4, 1\}$

$cr_{4,3}$
$\{\{1, 4, 6, 4, 1\}, \{4, 12, 12, 4\}, \{6, 12, 6\}, \{4, 4\}, 1\}$
where
$cr_{4,3} = \{cr_{4,2}[[1]] \bullet vs_{4,2}[[1]], \cdots, cr_{4,2}[[5]] \bullet vs_{4,2}[[5]]\}$
$= \{1 \bullet \{1, 4, 6, 4, 1\}, 4 \bullet \{1, 3, 3, 1\}, 6 \bullet \{1, 2, 1\}, 4 \bullet \{1, 1\}, 1 \bullet 1\}$

$cr_{4,4}$
$\{\{\{1, 4, 6, 4, 1\}, \{4, 12, 12, 4\}, \{6, 12, 6\}, \{4, 4\}, 1\}, \{\{4, 12, 12, 4\}, \{12, 24, 12\},$
$\{12, 12\}, 4\}, \{\{6, 12, 6\}, \{12, 12\}, 6\}, \{\{4, 4\}, 4\}, \{1\}\}$
where
$cr_{4,4} = \{cr_{4,3}[[1]] \bullet vs_{4,3}[[1]], \ldots, cr_{4,3}[[5]] \bullet vs_{4,3}[[5]]\}$
$= \{\{1, 4, 6, 4, 1\} \bullet \{\{1, 4, 6, 4, 1\}, \{1, 3, 3, 1\}, \{1, 2, 1\}, \{1, 1\}, 1\},$
$\{4, 12, 12, 4\}\} \bullet \{\{1, 3, 3, 1\}, \{1, 2, 1\}, \{1, 1\}, 1\}, \{6, 12, 6\}\} \bullet \{\{1, 2, 1\}, \{1, 1\}, 1\},$
$\{4, 4\} \bullet \{\{1, 1\}, 1\}, 1 \bullet 1\},$

$cr_{4,4}$ being the required vector of coefficients for $(a_1+a_2+a_3+a_4)^4$ under reverse lexicographic ordering. On removing the levels within the vector $cr_{4,4}$ and reversing the order of the components, one obtains a vector whose components are the coefficients of the monomials sorted in lexicographic order as given in Table 6. This vector can be directly obtained by simply typing $c_{4,4}$, which produces

$\{1, 4, 4, 4, 6, 12, 12, 6, 12, 6, 4, 12, 12, 12, 24, 12, 4, 12, 12, 4, 1, 4, 4, 6, 12, 6,$
$4, 12, 12, 4, 1, 4, 6, 4, 1\}$

The computational power that is currently available can easily accommodate polynomials of higher orders containing more terms.

Note that the expansion of $(a_1 + a_2 + a_3 + a_4)^4$ can also be determined from

$$\sum_{i=0}^{4}\sum_{j=0}^{4-i}\sum_{k=0}^{4-i-j} \frac{4!}{i!\,j!\,k!\,(4-i-j-k)!}\, a_1^i\, a_2^j\, a_3^k\, a_4^{4-i-j-k}\,,$$

which yields

$a_1^4 + 4\,a_1^3 a_2 + 6\,a_1^2 a_2^2 + 4\,a_1 a_2^3 + a_2^4 + 4\,a_1^3 a_3 + 12\,a_1^2 a_2 a_3 + 12\,a_1 a_2^2 a_3$
$+4\,a_2^3 a_3 + 6\,a_1^2 a_3^2 + 12\,a_1 a_2 a_3^2 + 6\,a_2^2 a_3^2 + 4\,a_1 a_3^3 + 4\,a_2 a_3^3 + a_3^4 + 4 a_1^3 a_4$
$+12\,a_1^2 a_2 a_4 + 12\,a_1 a_2^2 a_4 + 4\,a_2^3 a_4 + 12\,a_1^2 a_3 a_4 + 24\,a_1 a_2 a_3 a_4 + 12\,a_2^2 a_3 a_4$
$+12\,a_1 a_3^2 a_4 + 12\,a_2 a_3^2 a_4 + 4\,a_3^3 a_4 + 6\,a_1^2 a_4^2 + 12\,a_1 a_2 a_4^2 + 6\,a_2^2 a_4^2 + 12\,a_1 a_3 a_4^2$
$+12\,a_2 a_3 a_4^2 + 6\,a_3^2 a_4^2 + 4\,a_1 a_4^3 + 4\,a_2 a_4^3 + 4\,a_3 a_4^3 + a_4^4$

in *reverse* lexicographic ordering.

Observe that the coefficient of $a_1^i\, a_2^j\, a_3^k\, a_4^{4-i-j-k}$ is the multinomial coefficient,

$$\binom{4}{i,j,k} = \frac{4!}{i!j!k!(4-i-j-k)!} = \binom{i+j}{j}\binom{i+j+k}{k}\binom{4}{4-i-j-k}$$
$$= T_{i+j,j}\, T_{i+j+k,k}\, T_{4,4-i-j-k}\,,$$

which clearly can be expressed as a product of binomial coefficients. This produces a representation equivalent to that given in Equation (9). Similarly, the product of binomial coefficients appearing in Equation (8) can be expressed as a single multinomial coefficient. Thus, the proposed methodology provides an alternative approach for determining multinomial coefficients in terms of the products of certain entries of Pascal's triangle.

6. Conclusions

First, a representation of a generalized binomial expansion was presented, and some relationships between the roots and the coefficients of the monomials appearing in the expansions were pointed out. The case of binomial

expansions wherein all the roots are equal has also been treated, and generalizations to powers of multinomial expressions have been considered as well. It was pointed out that the coefficients of the monomials resulting from the expansion of $(\sum_{i=1}^{I} x_i)^n$ can be expressed in terms of those of $(\sum_{i=1}^{I-1} x_i)^n$ and a certain vector of concatenated rows of Pascal's triangles of successive orders. A simple and efficient recursive algorithm is provided for the determination of these coefficients. Unlike the usual approach which relies on multinomial coefficients, that herein advocated only involves products. The proposed polynomial expansion methodology, which was shown to have combinatorial applications, is also potentially applicable to certain types of genomic studies as well as other fields of scientific investigations.

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