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On approximating the distribution of indefinite quadratic expressions in singular normal vectors

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Abstract. General representations of quadratic forms and quadratic expressions in singular normal vectors are given in terms of the difference of two positive definite quadratic forms and an independently distributed linear combination of normal random variables. Up to now, only special cases have been treated in the statistical literature. The densities of the quadratic forms are then approximated with gamma and generalized gamma density functions. A moment-based technique whereby the initial approximations are adjusted by means of polynomials is presented. Closed form and integral formulae are provided for the approximate density functions of the quadratic forms and quadratic expressions. A detailed step-by-step algorithm for implementing the proposed density approximation technique is also provided. Two numerical examples illustrate the methodology.

1. Introduction

Numerous distributional results are already available in connection with quadratic forms and quadratic expressions in normal random variables. The latter include a linear term and a constant in addition to the quadratic form. Various representations of the density function of a quadratic form have been derived and several procedures have been proposed for computing percentage points; for instance such results are available from [9], [15], [16], [20], [23], [24] and [26]. Central and noncentral indefinite quadratic forms are respectively discussed in [8] and [25]; however, as explained in [25], the expansions obtained are not practical. A linear combination of chi-square variables having even degrees of freedom was considered in [1] and [7]. Exact distributional results were derived in [2], [13] and [22]. As pointed out in [18], a wide array of test statistics can be expressed in terms of quadratic

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forms in normal random vectors. For example, one may consider the lagged regression residuals developed in [3] and discussed in [21], or certain change point test statistics derived in [17]. The distribution of a certain goodness-of-fit score statistic that is given in terms of a sum of quadratic expressions is discussed in [19] . As pointed out in [14], quadratic expressions are also utilized in connection with analysis of variance methods for unbalanced data. Criteria for the independence of quadratic expressions were derived in [5], [6] and [20] .

An accessible approach is proposed in this paper for approximating the densities of positive definite and indefinite quadratic expressions in possibly singular normal random vectors in terms of gamma and generalized gamma densities. It is explained that such approximants can be combined with polynomial adjustments in order to improve their accuracy.

A decomposition of indefinite quadratic expressions in nonsingular normal vectors is derived in Section 2 where the moments of such quadratic forms are determined from a certain recursive relationship involving their cumulants. An integral representation of the density function of an indefinite quadratic form is also provided in that section. Useful representations of quadratic forms and quadratic expressions in singular normal vectors are respectively obtained in Sections 3 and 4. In Section 5, we propose some approximations to the distribution of quadratic expressions in terms of generalized gamma-type distributions and their polynomially adjusted counterparts. That section includes closed form and integral formulae for the approximate densities. Also provided is a step-by-step algorithm for implementing the proposed density approximation approach. Two numerical examples are presented in the last section. The first one is based on exact percentiles whereas the second makes use of simulated percentiles.

In the context of density approximation, the Monte Carlo and analytical approaches have their own merits and shortcomings. Monte Carlo simulations which generate artificial data wherefrom sampling distributions and moments are estimated, can be implemented with relative ease on a wide array of models and error probability distributions. There are, however, some limitations on the range of applicability of such techniques: the results may be subject to sampling variations or simulation inadequacies and may depend on the assumed parameter values. Recent efforts to cope with these issues are discussed for example in [4] and [10–12] . On the other hand, the analytical approach derives results which hold over the whole parameter space but may find limitations in terms of simplifications on the model, which have to be imposed to make the problem tractable. Even when exact theoretical results can be obtained, the resulting expressions can be fairly complicated. The moment-based approximation procedure advocated in this paper has the merit of producing expressions that yield very accurate distributional results over the entire supports of the distributions being considered.

2. Indefinite quadratic expressions: the nonsingular case

A decomposition of noncentral indefinite quadratic expressions in nonsingular normal vectors is given in terms of the difference of two positive definite quadratic forms whose moments are determined from a certain recursive relationship involving their cumulants. An integral representation of the density function of an indefinite quadratic form is also provided.

We first show that an indefinite quadratic expression in a nonsingular normal random vector can be expressed in terms of standard normal variables. Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu},\, \Sigma)$, $\Sigma > 0$, that is, $\mathbf{X}$ is distributed as a p-variate normal random vector with mean $\boldsymbol{\mu}$ and positive definite covariance matrix Σ. On letting $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$, where I is a $p \times p$ identity matrix, one has $\mathbf{X} = \Sigma^{\frac{1}{2}}\mathbf{Z} + \boldsymbol{\mu}$ where $\Sigma^{\frac{1}{2}}$ denotes the symmetric square root of Σ. Then, in light of the spectral decomposition theorem, the quadratic expression $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where A is a $p \times p$ real symmetric matrix, $\mathbf{a}$ is a p-dimensional constant vector and d is a scalar constant can be expressed as

$$\begin{aligned} Q^*(\mathbf{X}) &= (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) + \mathbf{a}'\Sigma^{\frac{1}{2}}(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) + d \\ &= (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'PP'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}PP'(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) \\ &\quad + \mathbf{a}'\Sigma^{\frac{1}{2}}PP'(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) + d \end{aligned}$$

where P is an orthogonal matrix that diagonalizes $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$, that is,

$$P'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}P = \mathcal{D}iag(\lambda_1, \ldots, \lambda_p),$$

$\lambda_1, \ldots, \lambda_p$ being the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ in *decreasing* order with $\lambda_{r+1} = \ldots = \lambda_{r+\theta} = 0$. Let $\mathbf{v}_i$ denote the *normalized* eigenvector of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ *corresponding* to λ_i, $i = 1, \ldots, p$, (such that $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ *and* $\mathbf{v}_i{}'\mathbf{v}_i = 1$) and $P = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$. Letting $\mathbf{U} = P'\mathbf{Z}$ where $\mathbf{U} = (U_1, \ldots, U_p)' \sim \mathcal{N}_p(\mathbf{0}, I)$, $\mathbf{b} = P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu}$ with $\mathbf{b} = (b_1, \ldots, b_p)'$, $\mathbf{g}' = \mathbf{a}'\Sigma^{\frac{1}{2}}P$ and $c = \mathbf{b}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p)\mathbf{b} + \mathbf{g}'\mathbf{b} + d$, one has

$$\begin{aligned} Q^*(\mathbf{X}) &= (\mathbf{U} + \mathbf{b})'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p)(\mathbf{U} + \mathbf{b}) + \mathbf{a}'\Sigma^{\frac{1}{2}}P(\mathbf{U} + \mathbf{b}) + d \\ &= \mathbf{U}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p)\mathbf{U} + (2\mathbf{b}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p) + \mathbf{g}')\mathbf{U} + c \\ &= \sum_{j=1}^{p} \lambda_j U_j^2 + \sum_{j=1}^{p} k_j U_j + c \\ &= \sum_{j=1}^{r} \lambda_j U_j^2 + \sum_{j=1}^{r} k_j U_j - \sum_{j=r+\theta+1}^{p} |\lambda_j| U_j^2 + \sum_{j=r+\theta+1}^{p} k_j U_j \\ &\quad + \sum_{j=r+1}^{r+\theta} k_j U_j + c \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r} \lambda_j \Big(U_j + \frac{k_j}{2\lambda_j}\Big)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| \Big(U_j + \frac{k_j}{2\lambda_j}\Big)^2 \\
&\quad + \sum_{j=r+1}^{r+\theta} k_j U_j + \Big(c - \sum_{j=1}^{r} \frac{k_j^2}{4\lambda_j} - \sum_{j=r+\theta+1}^{p} \frac{k_j^2}{4\lambda_j}\Big) \\
&\equiv Q_1(\mathbf{V}^+) - Q_2(\mathbf{V}^-) + \sum_{j=r+1}^{r+\theta} k_j U_j + \kappa \\
&\equiv Q_1(\mathbf{V}^+) - Q_2(\mathbf{V}^-) + T, \qquad (1)
\end{aligned}
$$

where $\mathbf{k}' = (k_1, \ldots, k_p) = 2\mathbf{b}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p) + \mathbf{g}'$, $\kappa = \Big(c - \sum_{j=1}^{r}(k_j^2/4\lambda_j) - \sum_{j=r+\theta+1}^{p}(k_j^2/4\lambda_j)\Big)$, $T = (\sum_{j=r+1}^{r+\theta} k_j U_j + \kappa) \sim \mathcal{N}(\kappa\,,\, \sum_{j=r+1}^{r+\theta} k_j^2)$, $Q_1(\mathbf{V}^+)$ and $Q_2(\mathbf{V}^-)$ are positive definite quadratic forms with $\mathbf{V}^+ = (U_1 + k_1/(2\lambda_1), \ldots, U_r + k_r/(2\lambda_r))' \sim \mathcal{N}_r(\mathbf{m}_1, I)$, $\mathbf{V}^- = (U_{r+\theta+1} + k_{r+\theta+1}/(2\lambda_{r+\theta+1}), \ldots, U_p + k_p/(2\lambda_p))' \sim \mathcal{N}_{p-r-\theta}(\mathbf{m}_2, I)$, where $\mathbf{m}_1 = (k_1/(2\lambda_1), \ldots, k_r/(2\lambda_r))'$ and $\mathbf{m}_2 = (k_{r+\theta+1}/(2\lambda_{r+\theta+1}), \ldots, k_p/(2\lambda_p))'$, θ being number of null eigenvalues of $A\,\Sigma$.

In particular, when $\mathbf{a} = \mathbf{0}$ and $d = 0$, one has

$$
\begin{aligned}
Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X} &= \sum_{j=1}^{p} \lambda_j (U_j + b_j)^2 \\
&= \sum_{j=1}^{r} \lambda_j (U_j + b_j)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| (U_j + b_j)^2 \\
&\equiv Q_1(\mathbf{Y}^+) - Q_2(\mathbf{Y}^-)\,, \qquad (2)
\end{aligned}
$$

where $\mathbf{Y}^+ = (U_1 + b_1, \ldots, U_r + b_r)' \sim \mathcal{N}_r(\mathbf{m}_1, I)$, $\mathbf{Y}^- = (U_{r+\theta+1} + b_{r+\theta+1}, \ldots, U_p + b_p)' \sim \mathcal{N}_{p-r-\theta}(\mathbf{m}_2, I)$ with $\mathbf{m}_1 = (b_1, \ldots, b_r)'$, $\mathbf{m}_2 = (b_{r+\theta+1}, \ldots, b_p)'$ and $\mathbf{b} = (b_1, \ldots, b_p)' = P'\Sigma^{-1/2}\boldsymbol{\mu}$.

Thus, a noncentral indefinite quadratic expression, $Q^*(\mathbf{X})$, can be expressed as a difference of independently distributed linear combinations of independent non-central chi-square random variables having one degree of freedom each plus a linear combination of normal random variables, or equivalently, as the difference of two positive definite quadratic forms plus linear combination of normal random variables. In particular, a noncentral indefinite quadratic form can be represented as the difference of two positive definite quadratic forms. It should be noted that the chi-square random variables are central whenever $\boldsymbol{\mu} = \mathbf{0}$. When the matrix A is positive semidefinite, so is $Q(\mathbf{X})$, and then, $Q(\mathbf{X}) \sim Q_1(\mathbf{Y}^+)$ as defined in Equation (3). Moreover, if the matrix A is not symmetric, it suffices to replace it by

$(A+A')/2$ within a quadratic form. Accordingly, it will be assumed without any loss of generality that the matrices of the quadratic forms are symmetric.

Expressions for the characteristic function and the cumulant generating function of a quadratic expression in central normal vectors are given for instance in [6]. The cumulants and moments of quadratic forms and quadratic expressions, which are useful for estimating the parameters of the density approximants, can be determined as follows.

Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, $A = A'$, $\mathbf{a}$ be a $p-$dimensional constant vector, d be a scalar constant, $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X}+\mathbf{a}'\mathbf{X}+d$ and $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$. Then, as shown in [18], the s-th cumulant of $Q^*(\mathbf{X})$ and $Q(\mathbf{X})$ are respectively

$$\begin{aligned} k^*(s) &= 2^{s-1}s!\left\{\frac{\mathrm{tr}(A\Sigma)^s}{s} + \frac{1}{4}\,\mathbf{a}'(\Sigma A)^{s-2}\Sigma\,\mathbf{a} + \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu}\right. \\ &\quad \left. + \,\mathbf{a}'(\Sigma A)^{s-1}A\boldsymbol{\mu}\right\}, \quad \text{for} \quad s \geq 2 \\ &= \mathrm{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d, \qquad \text{for} \quad s = 1\,; \end{aligned} \tag{3}$$

and

$$\begin{aligned} k(s) &= 2^{s-1}s!\left\{\frac{\mathrm{tr}(A\Sigma)^s}{s} + \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu}\right\}, \ \text{for} \quad s \geq 2 \\ &= \mathrm{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} \quad \text{for} \quad s = 1, \end{aligned}$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$,

$$\mathbf{b}' = (b_1, \ldots, b_p) = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$$

and $\mathrm{tr}(\cdot)$ denotes the trace of $(\cdot)$. Note that $\mathrm{tr}(A\Sigma)^s = \sum_{j=1}^{p}\lambda_j^s$.

As explained in [27], the moments of a random variable can be obtained from its cumulants by means of the recursive relationship that is specified by Equation (4). Accordingly, the h-th moment of $Q^*(\mathbf{X})$ is given by

$$\mu^*(h) = \sum_{i=0}^{h-1} \frac{(h-1)!}{(h-1-i)!\,i!}\,k^*(h-i)\,\mu^*(i)\,, \tag{4}$$

where $k^*(s)$ is as given in Equation (3).

One could also make use of Equation (4) to determine the moments of the positive definite quadratic forms, $Q_1(\mathbf{X}) \equiv \mathbf{Y}^+A_1\mathbf{Y}^+$ and $Q_2(\mathbf{X}) \equiv \mathbf{Y}^-A_2\mathbf{Y}^-$, appearing in Equation (2) where $A_1 = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, $A_2 = \mathcal{D}iag(|\lambda_{r+\theta+1}|, \ldots, |\lambda_p|)$, $\mathbf{Y}^+ \sim \mathcal{N}_r(\mathbf{m}_1, I)$ with $\mathbf{m}_1 = (b_1, \ldots, b_r)'$, $\mathbf{Y}^- \sim \mathcal{N}_{\,p-r-\theta}(\mathbf{m}_2, I)$ with $\mathbf{m}_2 = (b_{r+\theta+1}, \ldots, b_p)'$ and $\mathbf{b}' = (b_1, \ldots, b_p) = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$.

3. Quadratic forms in singular normal vectors

Singular covariance matrices occur in many contexts. For example, consider a standard linear regression model $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\mathbf{y} \in \mathbb{R}^n$, X is a non-stochastic $n \times k$ matrix of full column rank and $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(\mathbf{0}, \sigma^2 I_n)$, I_n denoting identity matrix order n. The distribution of the residuals $\boldsymbol{e} = \mathbf{y} - X\hat{\boldsymbol{\beta}} = (I_n - X(X'X)^{-1}X')\mathbf{y}$, where $\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\mathbf{y}$, is

$$\boldsymbol{e} \sim \mathcal{N}_n\Big(\mathbf{0}\,,\, \sigma^2(I_n - X(X'X)^{-1}X')\Big)$$

where the covariance matrix, $\sigma^2(I_n - X(X'X)^{-1}X')$, is of rank $n - k$.

Another example of application of singular covariance matrices pertains to economic data, which may be subject to constraints such as the requirement for a company's profits to equal its turnover expenses. If, for example, the data vector $\mathbf{X} = (X_1, \ldots, X_k)'$ must satisfy the restriction $X_1 + \cdots + X_{k-1} = X_k$, then Σ, the covariance matrix of $\mathbf{X}$, will be singular. When $\Sigma_{p\times p}$ is a singular matrix of rank $r < p$, we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements are positive, the remaining diagonal elements being equal to zero. Next, we let $B^*_{p\times p} = UW^{1/2}$ and remove the $p - r$ last columns of B^*, which are null vectors, to obtain the matrix $B_{p\times r}$. Then, it can be verified that $\Sigma = BB'$.

Let $\mathbf{X}$ be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{X}) = \Sigma$ of rank $r \le p$. Since Σ is positive semidefinite and symmetric, one can write $\Sigma = BB'$ where B is $p \times r$ of rank r. Now, consider the linear transformation

$$\mathbf{X} = \boldsymbol{\mu} + B\,\mathbf{Z}_1 \quad \text{where} \quad \mathbf{Z}_1 \sim \mathcal{N}_r(\mathbf{0}, I)\,;$$

then, one has the following decomposition of the quadratic form $Q(\mathbf{X})$:

$$\begin{aligned} Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = (\boldsymbol{\mu} + B\mathbf{Z}_1)'A(\boldsymbol{\mu} + B\mathbf{Z}_1) \\ &= \boldsymbol{\mu}'A\boldsymbol{\mu} + 2\,\mathbf{Z}_1'B'A\boldsymbol{\mu} + \mathbf{Z}_1'B'AB\mathbf{Z}_1 \quad \text{whenever } A = A'. \end{aligned}$$

Let P be an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, $\lambda_1, \ldots, \lambda_r$ being the eigenvalues of $B'AB$. Note that when $B'AB = O$, the null matrix, $Q(\mathbf{X})$ reduces to a linear form. Then, assuming that $B'AB \neq O$, one has $\mathbf{Z} = P'\mathbf{Z}_1 \sim \mathcal{N}_r(\mathbf{0}, I)$, and

$$Q(\mathbf{X}) = \boldsymbol{\mu}'A\,\boldsymbol{\mu} + 2\mathbf{Z}'P'B'A\,\boldsymbol{\mu} + \mathbf{Z}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_r)\mathbf{Z}\,.$$

Thus, the quadratic form $Q(\mathbf{X}) = \mathbf{X}'\,A\,\mathbf{X}$ has the following representation.

Representation 3.1. Letting $A = A'$, $\mathbf{X}$ be a $p \times 1$ normal vector with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\mathrm{Cov}(\mathbf{X}) = \Sigma \geq 0$, $\mathrm{rank}(\Sigma) = r \leq p$, $\Sigma = BB'$ where B is a $p \times r$ matrix and assuming that $B'AB \neq O$, one has

$$\begin{aligned}
Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X} &= \sum_{j=1}^{r} \lambda_j Z_j^2 + 2\sum_{j=1}^{r} b_j^* Z_j + c^* \\
&= \sum_{j=1}^{r_1} \lambda_j Z_j^2 + 2\sum_{j=1}^{r_1} b_j^* Z_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| Z_j^2 + 2 \sum_{j=r_1+\theta+1}^{r} b_j^* Z_j \\
&\quad + 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j + c^* \\
&= \sum_{j=1}^{r_1} \lambda_j \Big(Z_j + \frac{b_j^*}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \Big(Z_j + \frac{b_j^*}{\lambda_j}\Big)^2 + 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j \\
&\quad + \Big(c^* - \sum_{j=1}^{r_1} \frac{b_j^{*2}}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{b_j^{*2}}{\lambda_j} \Big) \\
&\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j + \kappa^* \\
&\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + T^*, \qquad (5)
\end{aligned}$$

where $Q_1(\mathbf{W}_1)$ and $Q_2(\mathbf{W}_2)$ are positive definite quadratic forms with

$$\begin{aligned}
&\mathbf{W}_1 = (W_1, \ldots, W_{r_1})', \quad \mathbf{W}_2 = (W_{r_1+\theta+1}, \ldots, W_r)', \\
&W_j = Z_j + b_j^*/\lambda_j, \ j = 1, \ldots, r_1, r_1+\theta+1, \ldots, r, \\
&\mathbf{b}^{*'} = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}'ABP \ , \quad \mathbf{Z} = (Z_1, \ldots, Z_r)' \sim \mathcal{N}_r(\mathbf{0}, I), \\
P'B'ABP = \ &\mathcal{D}iag(\lambda_1, \ldots, \lambda_r), \ PP' = I, \ c^* = \boldsymbol{\mu}'A\boldsymbol{\mu}, \\
&T^* = 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j + \kappa^* \sim \mathcal{N}(\kappa^* , 4\sum_{j=r_1+1}^{r_1+\theta} b_j^{*2})
\end{aligned}$$

and $\kappa^* = \Big(c^* - \sum_{j=1}^{r_1} b_j^{*2}/\lambda_j - \sum_{j=r_1+\theta+1}^{r} b_j^{*2}/\lambda_j \Big)$, $\lambda_j > 0$, $j = 1, \ldots, r_1$; $\lambda_j = 0$, $j = r_1+1, \ldots, r_1+\theta$; $\lambda_j < 0$, $j = r_1+\theta+1, \ldots, r$.

4. Quadratic expressions in singular normal vectors

Let the $p \times 1$ random vector $\mathbf{X}$ be a singular p-variate normal random variable with $E(\mathbf{X}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{X}) = \Sigma = BB'$ where B is $p \times r$ of rank $r \leq p$. Consider the quadratic expression

$$Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$$

where $A = A'$, $\mathbf{a}$ is a p-dimensional vector and d is a constant.

A representation of $Q^*(\mathbf{X})$ is derived is Section 4.1 and its cumulants are provided in Sections 4.2.

4.1. A decomposition of $Q^*(\mathbf{X})$. Letting $\mathbf{X} = \boldsymbol{\mu} + B\,\mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}_r(\mathbf{0}, I)$, one can write

$$\begin{aligned} Q^*(\mathbf{X}) \equiv Q^*(\mathbf{Z}) &= (\boldsymbol{\mu} + B\mathbf{Z})'A(\boldsymbol{\mu} + B\mathbf{Z}) + \mathbf{a}'(\boldsymbol{\mu} + B\mathbf{Z}) + d \\ &= \boldsymbol{\mu}'A\boldsymbol{\mu} + 2\boldsymbol{\mu}'A'B\mathbf{Z} + \mathbf{Z}'B'AB\mathbf{Z} + \mathbf{a}'B\mathbf{Z} + \mathbf{a}'\boldsymbol{\mu} + d\,. \end{aligned}$$

Let P be an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, with $\lambda_1, \ldots, \lambda_r$ being the eigenvalues of $B'AB$, $PP' = P'P = I$, $\mathbf{m}' = \mathbf{a}'BP$, $\mathbf{b}^{*'} = \boldsymbol{\mu}'ABP$ and $c_1 = \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d$ and $\mathbf{W} = P'Z \sim \mathcal{N}_r(\mathbf{0}, I)$. Then, assuming that $B'AB \neq O$, one has

$$\begin{aligned} Q^*(\mathbf{X}) \equiv Q^*(\mathbf{W}) &= \mathbf{W}'P'B'ABP\mathbf{W} + 2\boldsymbol{\mu}'ABP\mathbf{W} + \mathbf{a}'BP\mathbf{W} + \boldsymbol{\mu}'A\boldsymbol{\mu} \\ &\quad + \mathbf{a}'\boldsymbol{\mu} + d \\ &= \mathbf{W}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_r)\mathbf{W} + (2\mathbf{b}^{*'} + \mathbf{m}')\mathbf{W} + c_1\,. \end{aligned}$$

Thus we have the following representation.

Representation 4.1. Let $A = A'$, $\mathbf{X}$ be a p-dimensional normal vector with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\mathrm{Cov}(\mathbf{X}) = \Sigma \geq 0$, $\mathrm{rank}(\Sigma) = r \leq p$, $\Sigma = BB'$ where B is a $p \times r$ matrix, $\mathbf{a}$ is a p-dimensional vector, $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, $PP' = I$, $\lambda_1, \ldots, \lambda_{r_1}$ be the positive eigenvalues of $B'AB$, $\lambda_{r_1+1} = \cdots = \lambda_{r_1+\theta} = 0$, $\lambda_{r_1+\theta+1}, \ldots, \lambda_r$ be the negative eigenvalues of $B'AB$, $\mathbf{m}' = \mathbf{a}'BP$, $\mathbf{b}^{*'} = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}'ABP$, and d is a real constant. Assume that $B'AB \neq O$, then

$$\begin{aligned} Q^*(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d \\ &\equiv Q^*(\mathbf{W}) = \sum_{j=1}^{r} \lambda_j W_j^2 + 2\sum_{j=1}^{r}(\frac{1}{2}m_j + b_j^*)W_j + c_1 \\ &= \sum_{j=1}^{r_1} \lambda_j W_j^2 + 2\sum_{j=1}^{r_1} n_j W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| W_j^2 + 2\sum_{j=r_1+\theta+1}^{r} n_j W_j \\ &\quad + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + c_1 \\ &= \sum_{j=1}^{r_1} \lambda_j \Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j \\ &\quad + \Big(c_1 - \sum_{j=1}^{r_1} \frac{n_j^2}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{n_j^2}{\lambda_j}\Big) \end{aligned}$$

$$\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1$$
$$\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1, \qquad (6)$$

where $\mathbf{W}' = (W_1, \ldots, W_r) \sim \mathcal{N}_r(\mathbf{0}, I)$, $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are positive definite quadratic forms with

$$\mathbf{W}^+ = (W_1 + n_1/\lambda_1, \ldots, W_{r_1} + n_{r_1}/\lambda_{r_1})' \sim \mathcal{N}_{r_1}(\boldsymbol{\nu}_1, I),$$
$$\mathbf{W}^- = (W_{r_1+\theta+1} + n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, W_r + n_r/\lambda_r)' \sim \mathcal{N}_{r-r_1-\theta}(\boldsymbol{\nu}_2, I),$$

$\boldsymbol{\nu}_1 = (n_1/\lambda_1, \ldots, n_{r_1}/\lambda_{r_1})'$ and $\boldsymbol{\nu}_2 = (n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, n_r/\ \lambda_r)'$, θ being number of null eigenvalues, $n_j = \frac{1}{2}m_j + b_j^*$, $c_1 = \boldsymbol{\mu}' A \boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d$, $\kappa_1 = \Big(c_1 - \sum_{j=1}^{r_1} n_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\Big)$ and

$$T_1 = (2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1) \sim \mathcal{N}(\kappa_1\,,\, 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2)\,.$$

When $\boldsymbol{\mu} = \mathbf{0}$, one has

$$\begin{aligned}
Q^*(\mathbf{X}) \equiv Q^*(\mathbf{W}) &= \sum_{j=1}^{r} \lambda_j W_j^2 + \sum_{j=1}^{r} m_j W_j + d \\
&= \sum_{j=1}^{r_1} \lambda_j W_j^2 + \sum_{j=1}^{r_1} m_j W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| W_j^2 + \sum_{j=r_1+\theta+1}^{r} m_j W_j \\
&\quad + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + d \\
&= \sum_{j=1}^{r_1} \lambda_j \Big(W_j + \frac{m_j}{2\lambda_j}\Big)^2 - \sum_{j=r+\theta+1}^{r} |\lambda_j| \Big(W_j + \frac{m_j}{2\lambda_j}\Big)^2 \\
&\quad + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \Big(d - \sum_{j=1}^{r_1} \frac{m_j^2}{4\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{m_j^2}{4\lambda_j}\Big) \\
&\equiv Q_1(\mathbf{W}_1^+) - Q_2(\mathbf{W}_1^-) + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \kappa_1^* \\
&\equiv Q_1(\mathbf{W}_1^+) - Q_2(\mathbf{W}_1^-) + T_1^*, \qquad (7)
\end{aligned}$$

where $Q_1(\mathbf{W}_1^+)$ and $Q_2(\mathbf{W}_1^-)$ are positive definite quadratic forms with

$$\mathbf{W}_1^+ = (W_1 + m_1/2\lambda_1, \ldots, W_{r_1} + m_{r_1}/2\lambda_{r_1})' \sim \mathcal{N}_{r_1}(\boldsymbol{\mu}_1, I),$$
$$\boldsymbol{\mu}_1 = (m_1/2\lambda_1, \ldots, m_{r_1}/2\lambda_{r_1})',$$

$$\begin{aligned}
\mathbf{W}_1^- &= (W_{r_1+\theta+1} + m_{r_1+\theta+1}/2\lambda_{r_1+\theta+1}, \ldots, W_r + m_r/2\lambda_r)' \\
&\sim \mathcal{N}_{r-r_1-\theta}(\boldsymbol{\mu}_2, I), \\
\boldsymbol{\mu}_2 &= (m_{r_1+\theta+1}/2\lambda_{r_1+\theta+1}, \ldots, m_r/2\lambda_r)', \\
\kappa_1^* &= \left(d - \sum_{j=1}^{r_1} m_j^2/4\lambda_j - \sum_{j=r_1+\theta+1}^{r} m_j^2/4\lambda_j \right)
\end{aligned}$$

and

$$T_1^* = \left(\sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \kappa_1^* \right) \sim \mathcal{N}\left(\kappa_1^*, \sum_{j=r_1+1}^{r_1+\theta} m_j^2 \right).$$

4.2. Cumulants and moments of quadratic expressions in singular normal vectors. The cumulant generating function of $Q^* = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ and $Q = \mathbf{X}'A\mathbf{X}$ where $A = A'$, $\mathbf{X}$ has a singular p-variate normal density with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\mathrm{Cov}(\mathbf{X}) = \Sigma = BB'$, B is $p \times r$ of rank r, $\mathbf{a}$ is a p–dimensional constant vector and d is a scalar constant, are respectively

$$\begin{aligned}
\ln(M_{Q^*}(t)) = t(d + \mathbf{a}'\boldsymbol{\mu} + \boldsymbol{\mu}'A\boldsymbol{\mu}) &+ \frac{1}{2}\sum_{j=1}^{\infty} \frac{(2t)^j}{j} \mathrm{tr}(A\Sigma)^j \\
&+ \sum_{j=0}^{\infty} (2t)^{j+2} \Big\{ \frac{1}{8}\, \mathbf{a}'(\Sigma A)^j \Sigma\, \mathbf{a} + \frac{1}{2}\, \boldsymbol{\mu}'\, (A\Sigma)^{j+1} A\, \boldsymbol{\mu} \\
&+ \frac{1}{2}\, \mathbf{a}'(\Sigma A)^{j+1} \boldsymbol{\mu} \Big\}
\end{aligned}$$

and

$$\ln(M_Q(t)) = -\frac{1}{2} \sum_{j=1}^{r} \ln(1 - 2\,t\lambda_j) + c^* t + 2\,t^2 \sum_{j=1}^{r} \frac{b_j^{*2}}{(1 - 2\,t\lambda_j)}$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $B'AB$, $B'AB \neq O$, $c^* = \boldsymbol{\mu}'A\boldsymbol{\mu}$, $\mathbf{b}^* = P'B'A\boldsymbol{\mu}$, and $PP' = I$.

It is also shown in [18] that s-th cumulant of Q^* is

$$\begin{aligned}
k^*(s) = &\ 2^{s-1} s! \left\{ (1/s)\, \mathrm{tr}(B'AB)^s + (1/4)\, \mathbf{a}'B\, (B'AB)^{s-2} B'\mathbf{a} \right. \\
&\left. + \boldsymbol{\mu}'AB(B'AB)^{s-2}B'A\boldsymbol{\mu} + \mathbf{a}'B(B'AB)^{s-2}B'A\,\boldsymbol{\mu} \right\} \\
= &\ 2^{s-1} s! \left\{ (1/s)\mathrm{tr}(A\Sigma)^s + (1/4)\mathbf{a}'(\Sigma A)^{s-2}\Sigma\, \mathbf{a} \right. \\
&\left. + \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu} + \mathbf{a}'(\Sigma A)^{s-1}\boldsymbol{\mu} \right\}, \text{ for } s \geq 2 \\
= &\ \mathrm{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d\,, \text{ when } s = 1\,.
\end{aligned}$$

The moments of $Q^*(\mathbf{X})$ can then be readily determined via the recursive relationship given in Equation (4).

5. Approximating the distribution of quadratic expressions

Since the representations of indefinite quadratic expressions involve difference $Q_1 - Q_2$ where Q_1 and Q_2 are independently distributed positive definite quadratic forms, some approximations to the density function of $Q_1 - Q_2$ are provided in Sections 5.1 and 5.2. An algorithm describing proposed methodology is provided in Section 5.3.

Letting $Q(\mathbf{X}) = Q_1(\mathbf{X}_1) - Q_2(\mathbf{X}_2)$ and $h_Q(q)\,\mathcal{I}_{\Re}(q)$, $f_{Q_1}(q_1)\,\mathcal{I}_{(\tau_1,\infty)}(q_1)$ and $f_{Q_2}(q_2)\,\mathcal{I}_{(\tau_2,\infty)}(q_2)$ respectively denote the approximate densities of $Q(\mathbf{X}), Q_1(\mathbf{X}_1) > 0$ and $Q_2(\mathbf{X}_2) > 0$, where $\mathbf{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ and $\mathbf{X}_1'$ and $\mathbf{X}_2'$ are independently distributed, $\mathcal{I}_A(.)$ being the indicator function with respect to the set A, an approximation to density function of the indefinite quadratic form $Q(\mathbf{X})$ can be obtained as follows via the transformation variables technique:

$$h_Q(q) = \begin{cases} h_P(q) & \text{for } q \geq \tau_1 - \tau_2 \\ h_N(q) & \text{for } q < \tau_1 - \tau_2, \end{cases} \tag{8}$$

where

$$h_P(q) = \int_{q+\tau_2}^{\infty} f_{Q_1}(y) f_{Q_2}(y-q)\mathrm{d}y \tag{9}$$

and

$$h_N(q) = \int_{\tau_1}^{\infty} f_{Q_1}(y) f_{Q_2}(y-q)\mathrm{d}y\,. \tag{10}$$

Note that in the case of gamma-type density functions without location parameters, one should set τ_1 and τ_2 equal to zero in (8), (9) and (10).

Various types of quadratic expressions are represented as the difference of two positive definite quadratic forms plus linear combination of normal random variables in Equations (1), (5), (6) or (7). For example, in Equation (6), we know that $T_1 \sim \mathcal{N}(\kappa_1\,, 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2)$ and we can use Equation (8) to approximate the density function of $Q = Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-)$. Since Q and T_1 are independently distributed, $f_{Q,T_1}(q,t) = h_Q(q)\eta(t)$ where $\eta(t)$ is density function of T_1. To determine an approximation to distribution of $V = Q + T_1$, we apply transformation variables technique. Letting $U = T_1$, the joint density function of U and V is found to be $g_{V,U}(v\,, u) = f_{Q,T_1}(v - u\,, u)|J|$ where the Jacobian J is equal to one. Thus the density function of V is

$$g(v) = \int_{-\infty}^{\infty} g_{V,U}(v\,, u)\mathrm{d}u\,. \tag{11}$$

5.1. Approximations via generalized gamma distributions. Positive definite quadratic forms are approximated by gamma-type distributions in this section. First, let us consider the gamma distribution whose density function is given by

$$\psi(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}\,\mathcal{I}_{(0\,,\,\infty)}(x) \tag{12}$$

where $\alpha > 0$ and $\beta > 0$ can be specified as follows on the basis of $\mu(1)$ and $\mu(2)$, the first two raw moments of the distribution being approximated:

$$\alpha = \mu(1)^2/(\mu(2) - \mu(1)^2) \quad \text{and} \quad \beta = \mu(2)/\mu(1) - \mu(1)\,.$$

The generalized gamma density function that we are considering has the following parameterization:

$$\psi(x) = \frac{\gamma}{\beta^{\alpha\,\gamma}\Gamma(\alpha)}\,x^{\alpha\,\gamma-1}e^{-(x/\beta)^{\gamma}}\,\mathcal{I}_{(0\,,\,\infty)}(x)$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Denoting its raw moments by $m(j)$, $j = 0, 1, \ldots,$ one has

$$m(j) = \frac{\beta^{j}\,\Gamma(\alpha + j/\gamma)}{\Gamma(\alpha)}\,.$$

Its three parameters can readily be determined by solving the equations,

$$\mu_Q(i) = m(i)\,, \qquad \text{for } i = 1, 2, 3,$$

numerically, where $\mu_Q(i)$ denotes the i-th moment of a positive definite quadratic form Q.

A four-parameter gamma density, also called shifted generalized gamma density function, is given by

$$\psi(x) = \frac{\gamma}{\beta^{\alpha\,\gamma}\Gamma(\alpha)}\,(x-\tau)^{\alpha\,\gamma-1}e^{-(\frac{x-\tau}{\beta})^{\gamma}}\,\mathcal{I}_{(\tau,\infty)}(x)$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. One can determine the moments of the shifted generalized gamma distribution by applying the binomial expansion to the moments of the generalized gamma distribution.

Let $Q_1(\mathbf{Y}^+)$ and $Q_2(Y^-)$ be two independently distributed positive definite quadratic forms such as those defined in Equation (2). Then, an approximate density function for $Q_1(\mathbf{Y}^+)-Q_2(\mathbf{Y}^-)$ can be obtained from Equation (8). Let $\tau_1 = \tau_2 = 0$ and consider the gamma distribution whose density function is given in Equation (12). Let α_i and β_i be determined from the first two moments of $Q_i(\mathbf{X})$, $i = 1, 2$. It is assumed that the constants α_i are such that the arguments of the gamma functions appearing in the following expressions are neither negative integers nor zero. In this case, the negative part of the density function of $Q(\mathbf{X})$ is

$$
\begin{aligned}
h_N(q) &= \int_{-q}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y \\
&= \int_0^{\infty} \frac{(y)^{\alpha_1-1}\,(y-q)^{\alpha_2-1}\,e^{-y/\beta_1}\,e^{-(y-q)/\beta_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\,\mathrm{d}y \\
&= \frac{e^{q/\beta_2}\,\beta_1^{\alpha_2-1}\,\beta_2^{\alpha_1-1}\,(\beta_1+\beta_2)^{-\alpha_1-\alpha_2}}{q\,\Gamma(\alpha_1)\,\Gamma(1-\alpha_2)\,\Gamma(\alpha_2)}\Big(q\,(\beta_1+\beta_2)\,\Gamma(1-\alpha_2) \\
&\quad \times \Gamma(\alpha_1+\alpha_2-1)\,{}_1F_1\Big(1-\alpha_2;-\alpha_1-\alpha_2+2;-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big) \\
&\quad -\beta_1\,\beta_2\Big(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\,\Gamma(-\alpha_1-\alpha_2+1) \\
&\quad \times {}_1F_1\Big(\alpha_1;\alpha_1+\alpha_2;-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)\Big)\,,
\end{aligned}
$$

the positive part of the density being

$$
\begin{aligned}
h_P(q) &= \int_{q}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y \\
&= \int_q^{\infty} \frac{(y)^{\alpha_1-1}\,(y-q)^{\alpha_2-1}\,e^{-y/\beta_1}\,e^{-(y-q)/\beta_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\,\mathrm{d}y \\
&= \frac{e^{q/\beta_2}\,q^{\alpha_2-1}\,\beta_1^{-\alpha_1}\,\beta_2^{-\alpha_2}}{\Gamma(\alpha_1)}\,\frac{1}{\Gamma(1-\alpha_1)}\,\Gamma(-\alpha_1-\alpha_2+1) \\
&\quad \times {}_1F_1\left(\alpha_1;\alpha_1+\alpha_2;-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\right) q^{\alpha_1} \\
&\quad + \frac{1}{\Gamma(\alpha_2)}\left(\frac{\beta_1\,\beta_2}{\beta_1+\beta_2}\right)^{\alpha_1+\alpha_2-1}\Gamma(\alpha_1+\alpha_2-1) \\
&\quad \times {}_1F_1\left(1-\alpha_2;-\alpha_1-\alpha_2+2;-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\right) q^{1-\alpha_2}
\end{aligned}
$$

where ${}_1F_1(a,b,z)=\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\,\Gamma(b)\,z^k}{\Gamma(a)\,\Gamma(b+k)\,k!}$. Thus, the density function of $Q_1(\mathbf{Y}^+)$ $-Q_2(\mathbf{Y}^-)$ is

$$h_Q(q) = h_N(q)\,\mathcal{I}_{(-\infty,0)}(q) + h_P(q)\,\mathcal{I}_{(0,\infty)}(q)\,. \tag{13}$$

The approximate cumulative distribution function of $Q(\mathbf{X})$ for $y \le 0$ is then given by

$$
\begin{aligned}
F_N(y) &= \int_{-\infty}^{y} h_N(q)\mathrm{d}q \\
&= \int_{-\infty}^{y}\left(\frac{e^{q/\beta_2}\,\beta_1^{\alpha_2-1}\,\beta_2^{\alpha_1-1}\,(\beta_1+\beta_2)^{-\alpha_1-\alpha_2}}{q\,\Gamma(\alpha_1)\,\Gamma(1-\alpha_2)\,\Gamma(\alpha_2)}\Big(q\,(\beta_1+\beta_2)\,\Gamma(1-\alpha_2)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \Gamma(\alpha_1+\alpha_2-1)\,{}_1F_1\Big(1-\alpha_2;-\alpha_1-\alpha_2+2;-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)\\
&-\beta_1\,\beta_2\,\Big(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\,\Gamma(-\alpha_1-\alpha_2+1)\\
&\times{}_1\,F_1\Big(\alpha_1;\alpha_1+\alpha_2;-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)\Big)\Bigg)\mathrm{d}q\\
=&\int_{-\infty}^{y}\Bigg(\frac{e^{q/\beta_2}\,\beta_1^{\alpha_2-1}\,\beta_2^{\alpha_1-1}\,(\beta_1+\beta_2)^{-\alpha_1-\alpha_2}}{q\,\Gamma(\alpha_1)\,\Gamma(1-\alpha_2)\,\Gamma(\alpha_2)}\Bigg(q\,(\beta_1+\beta_2)\,\Gamma(1-\alpha_2)\Gamma(\alpha_1\\
&+\alpha_2-1)\sum_{k=0}^{\infty}\Bigg(\frac{\Gamma(1-\alpha_2+k)\,\Gamma(-\alpha_1-\alpha_2+2)\,\Big(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)^k}{\Gamma(1-\alpha_2)\,\Gamma(-\alpha_1-\alpha_2+2+k)\,k!}\Bigg)\\
&-\beta_1\,\beta_2\,\Big(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\,\Gamma(-\alpha_1-\alpha_2+1)\\
&\times\sum_{k=0}^{\infty}\Bigg(\frac{\Gamma(\alpha_1+k)\,\Gamma(\alpha_1+\alpha_2)\,\Big(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)^k}{\Gamma(\alpha_1)\,\Gamma(\alpha_1+\alpha_2+k)\,k!}\Bigg)\Bigg)\Bigg)\mathrm{d}q\\
=&\sum_{k=0}^{\infty}\frac{\beta_1^{\alpha_2-1}\,\beta_2^{\alpha_1-1}\,(\beta_1+\beta_2)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\,\Gamma(1-\alpha_2)\,\Gamma(\alpha_2)}\Bigg((\beta_1+\beta_2)\,\Gamma(1-\alpha_2)\,\Gamma(\alpha_1+\alpha_2-1)\\
&\times\Bigg(\frac{\Gamma(1-\alpha_2+k)\,\Gamma(-\alpha_1-\alpha_2+2)\,\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^k}{\Gamma(1-\alpha_2)\,\Gamma(-\alpha_1-\alpha_2+2+k)\,k!}\Bigg)\int_{-\infty}^{y}(-q)^k\,e^{q/\beta_2}\mathrm{d}q\\
&+\beta_1\,\beta_2\,\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\,\Gamma(-\alpha_1-\alpha_2+1)\\
&\times\Bigg(\frac{\Gamma(\alpha_1+k)\,\Gamma(\alpha_1+\alpha_2)\,\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^k}{\Gamma(\alpha_1)\,\Gamma(\alpha_1+\alpha_2+k)\,k!}\Bigg)\Bigg)\int_{-\infty}^{y}(-q)^{k+\alpha_1+\alpha_2-1}\,e^{q/\beta_2}\mathrm{d}q\\
=&\sum_{k=0}^{\infty}\frac{\beta_1^{\alpha_2-1}\,\beta_2^{k+\alpha_1}\,(\beta_1+\beta_2)^{-\alpha_1-\alpha_2}\,\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^k}{k!\,\Gamma(\alpha_1)^2\,\Gamma(1-\alpha_2)^2\,\Gamma(k-\alpha_1-\alpha_2+2)\,\Gamma(\alpha_2)\,\Gamma(k+\alpha_1+\alpha_2)}\\
&\times\Big((\beta_1+\beta_2)\,\Gamma(k+\alpha_1)\,\Gamma(1-\alpha_2)^2\,\Gamma(k-\alpha_1-\alpha_2+2)\,\Gamma(\alpha_1+\alpha_1-1)\\
&\times\Gamma(\alpha_1+\alpha_1)\,\Gamma(1+k,-y/\beta_2)+\beta_1\,\beta_2^{\alpha_1+\alpha_2}\,\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{\alpha_1+\alpha_1}\Gamma(\alpha_1)^2\\
&\times\Gamma(k-\alpha_2+1)\,\Gamma(-\alpha_1-\alpha_2+1)\,\Gamma(-\alpha_1-\alpha_2+2)\\
&\times\Gamma(k+\alpha_1+\alpha_2)\,\Gamma(k+\alpha_1+\alpha_2,-y/\beta_2)\Big)\,.
\end{aligned}
$$

It can be similarly determined that for $y \geq 0$, the approximate cumulative distribution function of $Q(\mathbf{X})$ is expressible as follws:

$$
\begin{aligned}
F_P(y) &= F_N(0) + \int_0^y h_P(q)\mathrm{d}q \\
&= F_N(0) + \int_0^y \Bigg(\frac{e^{q/\beta_2}\, q^{\alpha_2-1}\, \beta_1^{-\alpha_1}\, \beta_2^{-\alpha_2}}{\Gamma(\alpha_1)} \Big(\frac{1}{\Gamma(1-\alpha_1)}\, \Gamma(-\alpha_1-\alpha_2+1) \\
&\quad \times {}_1F_1\Big(\alpha_1; \alpha_1+\alpha_2; -\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)\, q^{\alpha_1} + \frac{1}{\Gamma(\alpha_2)} \Big(\frac{\beta_1\,\beta_2}{\beta_1+\beta_2}\Big)^{\alpha_1+\alpha_2-1} \\
&\quad \times \Gamma(\alpha_1+\alpha_2-1) \quad {}_1F_1\Big(1-\alpha_2; -\alpha_1-\alpha_2+2; -\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\Big)\, q^{1-\alpha_2} \Bigg)\mathrm{d}q \\
&= F_N(0) + \int_0^y \Bigg(\frac{e^{q/\beta_2}\, q^{\alpha_2-1}\, \beta_1^{-\alpha_1}\, \beta_2^{-\alpha_2}}{\Gamma(\alpha_1)} \Big(\frac{1}{\Gamma(1-\alpha_1)}\, \Gamma(-\alpha_1-\alpha_2+1) \\
&\quad \times \sum_{k=0}^{\infty} \left(\frac{\Gamma(\alpha_1+k)\,\Gamma(\alpha_1+\alpha_2)\left(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\right)^k}{\Gamma(\alpha_1)\,\Gamma(\alpha_1+\alpha_2+k)\,k!} \right) q^{\alpha_1} \\
&\quad + \frac{1}{\Gamma(\alpha_2)} \Big(\frac{\beta_1\,\beta_2}{\beta_1+\beta_2}\Big)^{\alpha_1+\alpha_2-1} \Gamma(\alpha_1+\alpha_2-1) \\
&\quad \times \sum_{k=0}^{\infty} \left(\frac{\Gamma(1-\alpha_2+k)\,\Gamma(-\alpha_1-\alpha_2+2)\left(-\frac{q(\beta_1+\beta_2)}{\beta_1\,\beta_2}\right)^k}{\Gamma(1-\alpha_2)\,\Gamma(-\alpha_1-\alpha_2+2+k)\,k!} \right) q^{1-\alpha_2} \Bigg)\mathrm{d}q \\
&= F_N(0) + \sum_{k=0}^{\infty} \frac{\beta_1^{-\alpha_1}\,\beta_2^{-\alpha_2}}{\Gamma(\alpha_1)} \Bigg(\frac{1}{\Gamma(1-\alpha_1)}\, \Gamma(-\alpha_1-\alpha_2+1) \\
&\quad \times \left(\frac{\Gamma(\alpha_1+k)\,\Gamma(\alpha_1+\alpha_2)\left(-\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\right)^k}{\Gamma(\alpha_1)\,\Gamma(\alpha_1+\alpha_2+k)\,k!} \right) \int_0^y q^{k+\alpha_1+\alpha_2-1}\, e^{q/\beta_2}\mathrm{d}q \\
&\quad + \frac{1}{\Gamma(\alpha_2)} \Big(\frac{\beta_1\,\beta_2}{\beta_1+\beta_2}\Big)^{\alpha_1+\alpha_2-1} \Gamma(\alpha_1+\alpha_2-1) \\
&\quad \times \left(\frac{\Gamma(1-\alpha_2+k)\,\Gamma(-\alpha_1-\alpha_2+2)\left(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\right)^k}{\Gamma(1-\alpha_2)\,\Gamma(-\alpha_1-\alpha_2+2+k)\,k!} \right) \Bigg) \int_0^y (-q)^k\, e^{q/\beta_2}\mathrm{d}q \\
&= F_N(0) \; + \sum_{k=0}^{\infty} \frac{\beta_1^{-\alpha_1}\,\beta_2^{-\alpha_2}}{k!\,\Gamma(\alpha_1)} \Big(-\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^k \\
&\quad \times \Bigg(\frac{\left(\frac{\beta_1\,\beta_2}{\beta_1+\beta_2}\right)^{\alpha_1+\alpha_2-1} \Gamma(1-\alpha_2+k)\,\Gamma(-\alpha_1-\alpha_2+2)}{\Gamma(1-\alpha_2)\,\Gamma(k-\alpha_1-\alpha_2+2)\,\Gamma(\alpha_2)}
\end{aligned}
$$

$$\times\,\Gamma(\alpha_1+\alpha_2-1)\Big(\Gamma(k+1,-y/\beta_2)-k\,\Gamma(k)\Big)\,\beta_2^{k+1}$$
$$+\,\frac{(-1/\beta_2)^{-k-\alpha_1-\alpha_2}\,\Gamma(k+\alpha_1)\,\Gamma(\alpha_1+\alpha_2)}{\Gamma(1-\alpha_1)\,\Gamma(k+\alpha_1+\alpha_2)\,\Gamma(\alpha_1)}\,\Gamma(-\alpha_1-\alpha_2+1)$$
$$\times\Big(\Gamma(k+\alpha_1+\alpha_2)-\Gamma(k+\alpha_1+\alpha_2,-y/\beta_2)\Big)\Bigg).$$

Consider the singular quadratic expression $Q^*(\mathbf{X})$ which is decomposed in Equation (6) into $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)+T_1$ where the approximate density function of $Q=Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)$ is as given in Equation (13) and $T_1=(2\sum_{j=r_1+1}^{r_1+\theta}n_jW_j+\kappa_1)\sim\mathcal{N}(\kappa_1\,,\,4\sum_{j=r_1+1}^{r_1+\theta}n_j^2)$ with $\kappa_1=\Big(c_1-\sum_{j=1}^{r_1}n_j^2/\lambda_j-\sum_{j=r_1+\theta+1}^{r}n_j^2/\lambda_j\Big)$, T_1 being distributed independently of $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$. In this case, the density function of T_1 is

$$\eta(t)=\Big(1\Big/\Big(\sqrt{2\pi}\sigma\Big)\Big)\ e^{-(t-\kappa_1)^2/\left(2\sigma^2\right)}$$

where $\sigma^2=4\sum_{j=r_1+1}^{r_1+\theta}n_j^2$. Then, it follows from Equation (11) that the approximate density function of $V=Q+T_1$ is

$$\begin{aligned}g(v)&=\int_{-\infty}^{\infty}g_{V,U}(v\,,\,u)\mathrm{d}u\\&=\int_{-\infty}^{\infty}h_Q(v-u)\eta(t)\mathrm{d}u\\&=\int_{-\infty}^{\infty}\big(h_N(v-u)\,\mathcal{I}_{(-\infty,0)}(v-u)\eta(t)\\&\quad+h_P(v-u)\,\mathcal{I}_{(0,\infty)}(v-u)\eta(t)\big)\,\mathrm{d}u\\&=\int_{-\infty}^{v}h_N(v-u)\,\eta(t)\,\mathrm{d}u+\int_{v}^{\infty}h_P(v-u)\,\eta(t)\,\mathrm{d}u\\&\equiv g_n(v)\ +g_p(v)\end{aligned}$$

where

$$\begin{aligned}g_n(v)&=\int_{-\infty}^{0}h_N(v-u)\,\eta(u)\,\mathrm{d}u\\&=\int_{-\infty}^{0}\sum_{k=0}^{\infty}\Bigg(e^{-\frac{(u-\kappa_1)^2}{2\sigma^2}-\frac{u}{\beta_2}+\frac{v}{\beta_2}}\beta_1^{\alpha_2-2}\beta_2^{\alpha_1-2}b^{-a+1}\left(\zeta(u-v)\right)^{k-1}\\&\quad\times\Big(\beta_1\beta_2\Gamma(\alpha_1)^2\Gamma(k-\alpha_2+1)\Gamma(-a+1)\Gamma(-a+2)\Gamma(k+a)\\&\quad\times(\zeta\,(u-v))^a+(u-v)\,b\,\Gamma(k+\alpha_1)\,\Gamma(1-\alpha_2)^2\,\Gamma(k-a+2)\\&\quad\times\Gamma(a-1)\Gamma(a)\Big)\Big/\Big((\sqrt{2\pi}\sigma k!\Gamma(\alpha_1)^2\Gamma(1-\alpha_2)^2\Gamma(k-a+2)\end{aligned}$$

$$
\begin{aligned}
&\times \Gamma(\alpha_2)\Gamma(k+a))\Big)\Bigg)\mathrm{d}u \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{\pi}k!\Gamma(\alpha_2)}2^{\frac{k}{2}-2}e^{-\frac{(v-\kappa_1)^2}{2\sigma^2}}\beta_1^{\alpha_2}\zeta^k\beta_2^{\alpha_1-2}b^{-a}\,\sigma^{k-2}\right. \\
&\times\left(\frac{1}{\Gamma(1-\alpha_2)^2\Gamma(k-a+2)}2^{\frac{a}{2}}\beta_2\Gamma(k-\alpha_2+1)\Gamma(-a+1)\right. \\
&\times\Gamma(-a+2)\Bigg(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{1}{2}(k+a)\right)\,{}_1F_1\left(\frac{1}{2}(k+a);\frac{1}{2};\gamma\right) \\
&-2\left(\sigma^2+v\beta_2-\beta_2\,\kappa_1\right)\Gamma\left(\frac{1}{2}(k+a+1)\right)\,{}_1F_1\Big(\frac{1}{2}(k+a+1); \\
&\times\frac{3}{2};\gamma\Big)\Bigg)(\zeta\sigma)^a+\frac{1}{\beta_1\Gamma(\alpha_1)^2\Gamma(k+a)}\sqrt{2}\beta_2\sigma\Gamma(k+\alpha_1)\Gamma(a-1) \\
&\times\Gamma(a)\left(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\,{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\gamma\right)\right. \\
&\left.\left.-2\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right)\,{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\gamma\right)\right)\right) \\
&+\frac{1}{\Gamma(\alpha_1)^2\Gamma(k+a)}2\sigma\Gamma(k+\alpha_1)\Gamma(a-1)\Gamma(a) \\
&\times\left(\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\,{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\gamma\right)\right. \\
&\left.\left.-\sqrt{2}\left(\sigma^2+v\beta_2-\beta_2\,\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right)\,{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\gamma\right)\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
g_p(v) &= \int_v^{\infty} h_P(v-u)\,\eta(t)\,\mathrm{d}u \\
&= \int_v^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2\pi}\sigma k!\Gamma(\alpha_1)}e^{\frac{v-u}{\beta_2}-\frac{(u-\kappa_1)^2}{2\sigma^2}}(v-u)^{\alpha_2-1}\beta_1^{-\alpha_1}\beta_2^{-\alpha_2}\right. \\
&\times(\zeta(u-v))^k\left(\frac{\Gamma(k+\alpha_1)\Gamma(-a+1)\Gamma(a)(v-u)^{\alpha_1}}{\Gamma(1-\alpha_1)\Gamma(\alpha_1)\Gamma(k+a)}\right. \\
&\left.\left.+\left(\frac{\left(\frac{1}{\zeta}\right)^{a-1}\Gamma(k-\alpha_2+1)\Gamma(-a+2)\Gamma(a-1)(v-u)^{1-\alpha_2}}{\Gamma(1-\alpha_2)\Gamma(k-a+2)\Gamma(\alpha_2)}\right)\right)\right)\mathrm{d}u \\
&= \sum_{k=0}^{\infty}\left\{\frac{\frac{1}{\sqrt{\pi}\sigma^2k!\Gamma(\alpha_1)^2}2^{\frac{k}{2}-2}e^{-\frac{(v-\kappa_1)^2}{2\sigma^2}}\beta_1^{-\alpha_1}\beta_2^{-\alpha_2-2}}{\Gamma(1-\alpha_2)\Gamma(k-a+2)\Gamma(\alpha_2)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\frac{1}{\Gamma(1-\alpha_1)\Gamma(k+a)} 2^{\frac{a}{2}} \beta_2 \sigma^{k+a} \Gamma(k+\alpha_1)\Gamma(-a+1)\Gamma(a) \right.\\
&\times \left(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{1}{2}(k+a)\right) {}_1F_1\left(\frac{1}{2}(k+a); \frac{1}{2}; \gamma\right) \right.\\
&+ 2\left(\sigma^2 + v\beta_2 - \beta_2\kappa_1\right)\Gamma\left(\frac{1}{2}(k+a+1)\right)\\
&\times {}_1F_1\left(\frac{1}{2}(k+a+1); \frac{3}{2}; \gamma\right) \left.\right)(-\zeta)^k + \left(2\beta_2\left(\frac{1}{\zeta}\right)^{a-1}\sigma\,(\zeta\sigma)^k\right.\\
&\times \Gamma(\alpha_1)\Gamma(k-\alpha_2+1)\Gamma(-a+2)\Gamma(a-1)\Bigg(\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\\
&\times {}_1F_1\left(\frac{k+1}{2}; \frac{1}{2}; \gamma\right) + \sqrt{2}\left(\sigma^2 + v\beta_2 - \beta_2\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right)\\
&\times {}_1F_1\left(\frac{k+2}{2}; \frac{3}{2}; \gamma\right)\Bigg)\Bigg)\Bigg]\Bigg\}
\end{aligned}
$$

where $a = \alpha_1 + \alpha_2$, $b = \beta_1 + \beta_2$, $\zeta = \frac{\beta_1+\beta_2}{\beta_1\beta_2}$ and $\gamma = \frac{(\sigma^2+v\beta_2-\beta_2\kappa_1)^2}{2\beta^2\sigma^2}$.

5.2. Polynomially adjusted density functions. It is explained in this section that the density approximations can be adjusted with polynomials whose coefficients are such that their first n moments coincide with the first n moments of the positive definite quadratic forms being approximated.

In order to approximate the density function of a noncentral quadratic form $Q(\mathbf{X})$, one should first approximate the density functions of the two positive definite quadratic forms, $Q_1(\mathbf{Y}^+)$ and $Q_2(\mathbf{Y}^-)$. On making use of the recursive relationship given in (4), the moments of the positive definite quadratic form $Q_1(\mathbf{Y}^+)$ denoted by $\mu_{Q_1}(\cdot)$ can be obtained from its cumulants. Then, on the basis of the first n moments of $Q_1(\mathbf{Y}^+)$, a density approximation of the following form is assumed for $Q_1(\mathbf{Y}^+)$:

$$f_n(x) = \varphi(x) \sum_{j=0}^{n} \xi_j x^j$$

where $\varphi(x)$ is an initial density approximant referred to as base density function, which could be a gamma, generalized gamma or shifted generalized gamma density function.

In order to determine the polynomial coefficients, ξ_j, we equate the h-th moment of $Q_1(\mathbf{X})$ to the h-th moment of the approximate distribution specified by $f_n(x)$ for $h = 0, 1, \ldots, n$. That is,

$$\begin{aligned}\mu_{Q_1}(h) &= \int_{\tau_1}^{\infty} x^h \varphi(x) \sum_{j=0}^{n} \xi_j x^j \mathrm{d}x = \sum_{j=0}^{n} \xi_j \int_{\tau_1}^{\infty} x^{h+j} \varphi(x) \mathrm{d}x \\ &= \sum_{j=0}^{n} \xi_j \; m(h+j), \qquad h = 0, 1, \ldots, n,\end{aligned}$$

where $m(h+j)$ is the $(h+j)$-th moment determined from $\varphi(x)$. This leads to a linear system of $(n+1)$ equations in $(n+1)$ unknowns whose solution is

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} m(0) & m(1) & \cdots & m(n-1) & m(n) \\ m(1) & m(2) & \cdots & m(n) & m(n+1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m(n) & m(n+1) & \cdots & m(2n-1) & m(2n) \end{bmatrix}^{-1} \begin{bmatrix} \mu_{Q_1}(0) \\ \mu_{Q_1}(1) \\ \vdots \\ \mu_{Q_1}(n) \end{bmatrix}.$$

The resulting representation of the density function of $Q_1(\mathbf{X})$ will be referred to as an n-th degree polynomially adjusted density approximant. As long as higher moments are available, more accurate approximations can always be obtained by making use of additional moments.

The density function for $Q_2(\mathbf{X})$ can be similarly approximated, so that approximations for both positive definite quadratic forms, $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$, are available. The density approximant to the noncentral indefinite quadratic expressions in nonsingular and singular normal vectors are then obtained by making use of Equations (11) in conjunction with Equation (13).

5.3. The algorithm. The following algorithm can be utilized to approximate the density function of the quadratic expression $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma \geq 0$, A is an indefinite symmetric real matrix, $\mathbf{a}$ is a p-dimensional constant vector and d is a scalar constant. When Σ is a singular matrix, the symmetric square root does not exist. In this case, we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements are positive, the remaining diagonal elements being equal to zero. Next, we let $B^*_{p\times p} = UW^{1/2}$ and remove the $p-r$ last columns of B^*, which are null vectors, to obtain the matrix $B_{p\times r}$. Then, it can be verified that $\Sigma = BB'$.

1. The eigenvalues of $B'AB$ are denoted by $\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_{r+\theta} = 0 > \lambda_{r+\theta+1} \geq \cdots \geq \lambda_p$, and the *corresponding normalized* eigenvectors, $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p$, are determined, and we let $P = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p)$.

2. In the singular case, one can decompose $Q^*(\mathbf{X})$ as $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)+T_1$ where $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are positive definite quadratic forms with $\mathbf{W}^+=(W_1+n_1/\lambda_1,\ldots,W_{r_1}+n_{r_1}/\lambda_{r_1})'\sim\mathcal{N}_{r_1}(\boldsymbol{\nu}_1,I)$, $\boldsymbol{\nu}_1=(n_1/\lambda_1,\ldots,n_{r_1}/\lambda_{r_1})'$, $\mathbf{W}^-=(W_{r_1+\theta+1}+n_{r_1+\theta+1}/(\lambda_{r_1+\theta+1}),\ldots,W_r+n_r/(\lambda_r))'\sim\mathcal{N}_{r-r_1-\theta}(\boldsymbol{\nu}_2,I)$, $\boldsymbol{\nu}_2=(n_{r_1+\theta+1}/(\lambda_{r_1+\theta+1}),\ldots,\,n_r/(\lambda_r))'$, θ being number of null eigenvalues, $\mathbf{b}^{*'}=(b_1^*,\ldots,b_r^*)=\boldsymbol{\mu}'ABP$, $n_j=\frac{1}{2}m_j+b_j^*$, $c_1=\boldsymbol{\mu}'A\boldsymbol{\mu}+\mathbf{a}'\boldsymbol{\mu}+d$ and $\mathbf{W}'=(W_1,\ldots,W_r)$. Letting $\kappa_1=\left(c_1-\sum_{j=1}^{r_1}n_j^2/\lambda_j-\sum_{j=r_1+\theta+1}^{r}n_j^2/\lambda_j\right)$, $T_1=(2\sum_{j=r_1+1}^{r_1+\theta}n_jW_j+\kappa_1)\sim\mathcal{N}(\kappa_1\,,\,4\sum_{j=r_1+1}^{r_1+\theta}n_j^2)$. Clearly, $\mathbf{b}^*=\mathbf{0}$ whenever $\boldsymbol{\mu}=\mathbf{0}$ and in that case, there is no need to determine the matrix P.

3. The cumulants and the moments of Q_1 and Q_2 are obtained from Equations (3) and (4), respectively.

4. Density approximants are determined for each of the positive definite quadratic forms Q_1 and Q_2 on the basis of their respective moments and denoted by $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$.

5. Given $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$, we first approximate density of $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)$ by using Equation (8) and then, determine the density function of $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)+T_1$ by making use of Equation (11).

6. A polynomial adjustment, which improves the accuracy of the approximations, can also be applied to the density approximations for $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ as explained in Section 5.2. Then, an approximate density function for $Q^*(\mathbf{X})$ is obtained as explained in Step 5.

6. Numerical examples

Example 6.1. Consider the following general linear combination of independently distributed central chi-square random variables:

$$Q(\mathbf{X})=\sum_{i=1}^{s}\lambda_i'\,T_i-\sum_{j=s+1}^{t}|\lambda_j'|\,T_j\,,$$

where $s=6$, $t=10$, $\lambda_k'=\lambda_{k/2}$, $k=1,\ldots,10$, and the random variables T_i and T_j are independently distributed chi-square random variables, each having *two* degrees of freedom and $\lambda_1=\lambda_2=23.1$, $\lambda_3=\lambda_4=4.5$, $\lambda_5=\lambda_6=6.8$, $\lambda_7=\lambda_8=8.13$, $\lambda_9=\lambda_{10}=10.3$, $\lambda_{11}=\lambda_{12}=20.1$, $\lambda_{13}=\lambda_{14}=-3.4$, $\lambda_{15}=\lambda_{16}=-12.4$, $\lambda_{17}=\lambda_{18}=-2$ and $\lambda_{19}=\lambda_{20}=-1.3$.

Since the eigenvalues occur in pairs, the exact density of $Q(\mathbf{X})$ can be determined as explained in the Appendix. In this example, we compare the exact density and distribution functions of $Q(\mathbf{X})$ with various approximations. Some exact and approximate percentiles are listed in Tables 1 and 2 and the best approximation for a given percentile is indicated by an asterisk.

TABLE 1. Three approximations to the distribution function of $Q(\mathbf{X})$ evaluated at certain exact percentiles (Exact %)

CDF	Exact %	Gamma	Gen. Gam.	Sh. Gen. Gam.
0.0001	−147.47	0.000040	0.000127	0.000103*
0.001	−90.366	0.000689	0.001041	0.000985*
0.01	−33.257	0.010198	0.009811	0.009886*
0.05	7.0176	0.055784	0.049952*	0.049864
0.10	25.734	0.108681	0.100281	0.100013*
0.25	57.398	0.255312	0.250484	0.250396*
0.50	98.008	0.494008	0.499698	0.500128*
0.90	203.27	0.898124	0.900115	0.899893*
0.95	241.73	0.950857	0.950186	0.950052*
0.99	325.86	0.991558	0.990045*	0.990057
0.999	440.25	0.999399	0.998977	0.998997*
0.9999	551.20	0.999961	0.999889	0.999895*

TABLE 2. Three polynomially-adjusted approximations to the distribution function of $Q(\mathbf{X})$ evaluated at certain exact percentiles (Exact %)

CDF	Exact %	Gam. Poly	G. Gam. Poly	Sh. G. Gam. Poly
0.0001	−147.47	0.000097	0.000101*	0.000098
0.001	−90.366	0.001006*	0.000979	0.000975
0.01	−33.257	0.009993	0.010003*	0.010031
0.05	7.0176	0.050010	0.050009*	0.049983
0.10	25.734	0.099996*	0.100029	0.100153
0.25	57.398	0.249949	0.249998*	0.249980
0.50	98.008	0.500075	0.499928*	0.499725
0.90	203.27	0.899967	0.900005*	0.899984
0.95	241.73	0.950023	0.949979*	0.949858
0.99	325.86	0.989989	0.990005*	0.990030
0.999	440.25	0.998996	0.999003*	0.999004
0.9999	551.20	0.999901	0.999899*	0.999894

The results presented in Table 1 indicate that the approximations obtained from the shifted generalized gamma distribution are more accurate when no polynomial adjustments are being made. The results included in Table 2 show that after the polynomial adjustment, the generalized gamma distribution is more accurate, even for extreme percentage points.

Example 6.2. Consider the singular quadratic expression $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where $\mathbf{X} \sim \mathcal{N}_5(\boldsymbol{\mu}, \Sigma)$,

$$A = \begin{pmatrix} 1 & -0.9 & -1 & 0 & -5 \\ -0.9 & 1 & 1 & 2 & 1 \\ -1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 3 & -1 & 0 \\ -5 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$\boldsymbol{\mu} = (100, 0, -50, 150, 5)'$, $\mathbf{a}' = (-1, 2, 3, 1, 1)\,, d = 6$ and

$$\Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case the matrices B and P were found to be

$$B = \begin{pmatrix} 1.66591 & 0.39015 & 0 & -0.26929 \\ 1.66591 & 0.39015 & 0 & -0.26929 \\ 2.03287 & -0.92672 & 0 & 0.09291 \\ 1.18171 & 0.49418 & 0 & 0.59945 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} -0.97731 & 0.00042 & -0.14936 & -0.15022 \\ 0.05695 & -0.58347 & -0.72923 & 0.35290 \\ 0.13922 & 0.69384 & -0.66277 & -0.24484 \\ -0.14916 & 0.42208 & 0.08157 & 0.89048 \end{pmatrix}$$

respectively. The eigenvalues of $B'AB$ are $\lambda_1 = 31.2355$, $\lambda_2 = 3.80066$, $\lambda_3 = -2.92434$, $\lambda_4 = -2.51178$ and $n_1 = -693.095$, $n_2 = -317.337$, $n_3 = 443.115$, and $n_4 = -268.983$. Moreover, referring to Representation 4.1, $\mathbf{m}_1 = (-22.1894, -83.4953)'$, $\mathbf{m}_2 = (-151.526, 107.089)'$ and $c_1 = -48064$.

The approximate density functions of $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are obtained by making use of the gamma, generalized gamma and the shifted generalized gamma approximations. The resulting distribution functions are evaluated at certain simulated percentiles obtained on the basis of 1,000,000 replications.

The results presented in Tables 3 to 4 indicate that the gamma, generalized gamma and the shifted generalized gamma distribution all provide accurate approximations.

TABLE 3. Three approximations to the distribution of $Q^*(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simulated %)

CDF	Simulated %	Gamma	G. Gam.	Sh. G. Gam.
0.0001	−54663.55	0.0001217	0.0001217	0.0001216*
0.001	−53591.02	0.0010932	0.0010931	0.0010929*
0.01	−52256.04	0.0102818	0.0102818	0.0102805*
0.05	−51039.46	0.0504963	0.0504963	0.0504927*
0.10	−50389.24	0.1000880	0.1000880	0.1000830*
0.25	−49289.67	0.2490610	0.2490608	0.2491550*
0.50	−48053.09	0.4984397*	0.4984396	0.4984341
0.75	−46801.40	0.7492549*	0.7492548	0.7492500
0.90	−45661.40	0.9002350	0.9002350	0.9002310*
0.95	−44971.41	0.9505530	0.9505530	0.9505491*
0.99	−43679.37	0.9901870	0.9901870	0.9901841*
0.999	−42211.50	0.9990400	0.9990400	0.9990360*
0.9999	−40911.81	0.9999200	0.9999200	0.9999181*

TABLE 4. Two approximations with and without polynomial adjustments ($d = 7$) to the distribution of $Q^*(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simulated %)

CDF	Simulated %	Gamma	Gam. Poly	G. Gam.	G. Gam. Poly
0.0001	−54663.55	0.0001217	0.0001044	0.0001217	0.0001040*
0.001	−53591.02	0.0010932	0.0010010	0.0010932	0.0009997*
0.01	−52256.04	0.0102818	0.0099062	0.0102818	0.0099072*
0.05	−51039.46	0.0504963	0.0498882	0.0504963	0.0499204*
0.10	−50389.24	0.1000876	0.0996168	0.1000876*	0.0996930
0.25	−49289.67	0.2490610	0.2493520	0.2490608	0.2495601*
0.50	−48053.08	0.4984397	0.4993070	0.4984397	0.4997190*
0.75	−46801.40	0.7492549	0.7492760	0.7492548	0.7498871*
0.90	−45661.40	0.9002350	0.8992180	0.9002350	0.8999590*
0.95	−44971.41	0.9505530	0.9492760	0.9505530	0.9500640*
0.99	−43679.37	0.9901870	0.9890400	0.9901870	0.9898710*
0.999	−42211.50	0.9990400	0.9981590	0.9990400	0.9990009*
0.9999	−40911.81	0.9999200	0.9991150	0.9999200*	0.9999580

Tables 3 and 4 include various approximate cumulative distribution function values which are determined with and without polynomial adjustments. Table 3 indicates that the shifted generalized gamma provides the most accurate approximations for most points. Two of the approximations that are presented in Table 4 were adjusted with polynomials of degree 7. Also included in Table 4 are their non polynomially-adjusted counterparts. The numerical results indicate that the polynomially-adjusted generalized gamma approximations are generally more accurate than the other approximations.

Appendix. The exact density function of a certain type of quadratic forms

Consider a central quadratic form $Q(\mathbf{X})$ that is expressible as the following general linear combination of independently distributed central chi-square random variables:

$$Q(\mathbf{X}) = \sum_{i=1}^{r} \lambda_i Y_i - \sum_{j=r+\theta+1}^{p} |\lambda_j| Y_j \,,$$

where θ is the number of null eigenvalues and the Y_j's, $j = 1, \dots, p$ are independently distributed central chi-square random variables having one degree of freedom. Suppose that all the eigenvalues occur in pairs. Then, $Q(\mathbf{X})$ can be expressed as

$$Q(\mathbf{X}) = \sum_{i=1}^{s} \lambda'_i T_i - \sum_{j=s+1}^{t} |\lambda'_j| \, T_j \,,$$

where $s = r/2$, $t = p/2$, $\lambda'_k = \lambda_{k/2}$, $k = 1, \dots, t$, and the T_i's and T_j's are independently distributed chi-square random variables, each having *two* degrees of freedom. The following representation of the exact density function of $Q(\mathbf{X})$ is derived in [13]:

$$g(q) = \begin{cases} \displaystyle\sum_{j=1}^{s} \frac{(\lambda'_j)^{t-2}\, e^{-q/(2\lambda'_j)}}{2\left(\prod_{k=1,k\neq j}^{s}(\lambda'_j - \lambda'_k)\right)\left(\prod_{k=s+1}^{t}(|\lambda'_j| + |\lambda'_k|)\right)} \,, & q \geq 0 \\[2ex] \displaystyle\sum_{j=s+1}^{t} \frac{|\lambda'_j|^{t-2} e^{q/(2|\lambda'_j|)}}{2\left(\prod_{k=s+1,k\neq j}^{t}(|\lambda'_j| - |\lambda'_k|)\right)\left(\prod_{k=1}^{s}(\lambda'_j + \lambda'_k)\right)} \,, & q < 0. \end{cases}$$

This representation of the exact density of $Q(\mathbf{X})$ is used in Example 6.1 to assess the accuracy of the proposed approximations.

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