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On Morita equivalence of partially ordered semigroups with local units

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Abstract. We show that for two partially ordered semigroups S and T with common local units, there exists a unitary Morita context with surjective maps if and only if the categories of closed right S- and T-posets are equivalent.

1. Introduction

The Morita theory for semigroups has seen some significant recent advancement by Lawson in [9], who provided a definition of Morita equivalence in terms of "closed" acts and showed that such a definition is equivalent to both Talwar's original definition (see [11]) and, more importantly, to a number of other algebraic conditions. In a follow-up article to Lawson's (see [15], which is extended in [13]), we proved that Lawson's conditions except for the definition of Morita equivalence are still equivalent for partially ordered semigroups with local units. We defined what we called "strong Morita equivalence" (cf. [11], [12], [8]) as the strongest of those conditions, the existence of a (unitary) Morita context with surjective maps. At that time, we could only remark that we do not know whether some condition similar to Lawson's definition is also equivalent to the others. In this article, we will show that if we restrict the class of partially ordered semigroups under consideration, Lawson's definition can indeed be modified to be equivalent to the other conditions.

We use the symmetric monoidal closed category Pos of partial orders and monotone maps (with cartesian product for monoidal tensor product) and different categories enriched over Pos. For more details on Pos-categories, Pos-functors and Pos-equivalences, the reader is referred to [4]. When $\mathcal{C}$ is a

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Pos-category, we use the notation $\mathcal{C}_0$ for its class of objects instead of $\mathcal{C}$ in cases when it is not immediately clear from the context whether we mean objects or morphisms. By $\mathcal{C}(A, B)$ we denote the poset of morphisms in the Pos-category $\mathcal{C}$ from object A to object B. Categorial compositions will be written from right to left.

A *partially ordered semigroup* S (a *posemigroup* for short) is a (nonempty) semigroup that is endowed with a partial order so that its operation is monotone. One can get a pomonoid from a posemigroup by adjoining an external identity 1 and considering the order $\leq \cup\{(1,1)\}$. For a fixed posemigroup S, (one-sided) *S-posets* are partially ordered S-acts where the S-action is monotone in both arguments. In the sequel, the set of idempotents of a (po)semigroup S will be denoted by $E(S)$. A right S-poset X is said to be *unitary* if $X = XS$. We say that a right S-poset X is Pos-*unitary* if for all $x, y \in X$ such that $x \leq y$ there exist $s, t \in E(S)$ such that $s \leq t$, $xs = x$ and $yt = y$. The notions for left S-posets are dual. A poset is called an (S, T)-*biposet* if it is a left S- and a right T-poset and its S- and T-actions commute with each other. Moreover, (S, T)-biposets are called *unitary* (Pos-*unitary*) if they are unitary (Pos-unitary) as both left S- and right T-posets. *Posemigroup homomorphisms* are monotone semigroup homomorphisms. *S-poset morphisms* are monotone S-act homomorphisms. All such morphisms naturally form posets with respect to the pointwise order.

A posemigroup S is said to have *(weak) local units* if for any $s \in S$ there exist $e, f \in E(S)$ ($e, f \in S$) such that $es = s = sf$. A posemigroup has *common (weak) local units* (cf. [7]) if for any $s, s' \in S$ there are $e \in E(S)$ and $f \in E(S)$ ($e \in S$ and $f \in S$) such that $es = s = sf$ and $es' = s' = s'f$. We say that a posemigroup S has *ordered (weak) local units* if for all $s, s' \in S$, $s \leq s'$, there exist $e, e', f, f' \in E(S)$ ($e, e', f, f' \in S$) such that

$$es = s = sf,\ e's' = s' = s'f',\ e \leq e',\ f \leq f'.$$

A natural example of a posemigroup with ordered local units is a partially ordered band (a *poband*), e.g. $(\mathbb{N}, \min, \leq)$. The last example is noteworthy because it illustrates two important facts: (1) we can choose from many such pairs of local units and (2) not all of the local units are in the required order relation.

A posemigroup S is said to have a property *locally* (e.g. to be locally inverse) if every local subpomonoid eSe, $e \in E(S)$, has that property (e.g. is inverse). A posemigroup is called *factorizable* if each of its elements can be written as a product of two elements. Having local units implies having weak local units, which in turn implies factorizability. Also, a posemigroup with common (weak) local units has ordered (weak) local units, and one with ordered (weak) local units has (weak) local units. The converse implications do not hold in general.

Example 1.1. All lower semilattices (with their natural order) that lack upwards direction have ordered local units but not common local units. The smallest example of such is the three-element lower semilattice that is not a chain.

Example 1.2. Take two posets $U = \{1\}$ and $V = \{1 < 2\}$, the linearly ordered group $(\mathbb{Z}, +, \leq)$, $M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and construct the Rees matrix semigroup $\mathcal{M} = \mathcal{M}(\mathbb{Z}, U, V, M)$ with cartesian order. The elements $(1, z, 1)$, $z \in \mathbb{Z}$, have only one local right unit, namely $(1, 0, 1)$. Elements $(1, z, 2)$, $z \in \mathbb{Z}$, also have only $(1, -1, 2)$ as a local right unit. But then $(1, 0, 1) \leq (1, 0, 2)$, yet $(1, 0, 1) \not\leq (1, -1, 2)$, so $\mathcal{M}$ has local units but not ordered local units.

The *directed kernel* of a posemigroup homomorphism $f : S \to T$ is the relation

$$\overrightarrow{\operatorname{Ker}} f = \{(a, b) \in S^2 |\, f(a) \leq f(b) \text{ in } T\}.$$

For a morphism $f : S \to T$, we can construct a quotient posemigroup $S/\overrightarrow{\operatorname{Ker}} f$ if we factorize S by the semigroup congruence $\operatorname{Ker} f = \overrightarrow{\operatorname{Ker}} f \cap (\overrightarrow{\operatorname{Ker}} f)^{-1}$ and set

$$[s] \leq [s'] \text{ in } S/\overrightarrow{\operatorname{Ker}} f \iff (s, s') \in \overrightarrow{\operatorname{Ker}} f.$$

Theorem 1.3 of [3] shows that $S/\overrightarrow{\operatorname{Ker}} f \cong \operatorname{Im} f$ as posemigroups.

An *admissible preorder* on a posemigroup S is a preorder on S that is compatible with S-multiplication and which contains the order on S. The *admissible preorder generated by a relation* $H \subseteq S^2$ is the smallest admissible preorder on S that contains H. By Theorem 1.2 of [3], admissible preorders are exactly the directed kernels of posemigroup homomorphisms. Therefore we can use any admissible preorder ρ on S to construct the quotient posemigroup S/ρ as above.

If S is a posemigroup and ρ is a reflexive relation on S, one can define a preorder $\leq_\rho$ on S by setting $s \leq_\rho t$ if there exist $n \in \mathbb{N}$, $s_i, t_i \in S$, $1 \leq i \leq n$ such that

$$s \leq s_1 \,\rho\, t_1 \leq s_2 \,\rho\, t_2 \leq \ldots \leq s_n \,\rho\, t_n \leq t.$$

A *posemigroup congruence* on S is a semigroup congruence θ on S such that the *closed chains condition* holds:

$$\text{if } s \leq_\theta t \leq_\theta s, \text{ then } s\theta t.$$

The *posemigroup congruence generated by a relation* $H \subseteq S \times S$ (denoted by $\theta(H)$) is the smallest posemigroup congruence on S that contains H. We remark that if H is a semigroup congruence, then $\theta(H) = \leq_H \cap \geq_H$. The notions of admissible preorder and congruence for S-posets and (S, T)-biposets are analogous.

The *tensor product* $A \otimes_S B$ of a right S-poset A and a left S-poset B is the quotient poset $(A \times B)/\!\sim$ by the least poset congruence $\sim$ for which $(as, b) \sim (a, sb)$ for all $a \in A$, $b \in B$, $s \in S$. If A is a (T, S)-biposet, then $A \otimes_S B$ is a left T-poset, with the left T-action $t(a \otimes b) = (ta) \otimes b$. And $A \otimes_S B$ is similarly a right T-poset whenever B is an (S, T)-biposet.

A right S-poset X is unitary if and only if the canonical S-poset morphism

$$\mu_X : X \otimes_S S \to X,\ x \otimes s \mapsto xs,$$

is surjective. If it is also an order embedding (implying it is an order isomorphism), then X is said to be *closed*. Closed left S-posets are defined dually.

We use the following notation:

Pos – the category of posets and monotone maps,

Pos_S – the Pos-category of right S-posets and right S-poset morphisms,

${}_S\mathsf{Pos}$ – the Pos-category of left S-posets and left S-poset morphisms,

${}_S\mathsf{Pos}_T$ – the Pos-category of (S, T)-biposets and (S, T)-biposet morphisms,

UPos_S – the full Pos-subcategory of Pos_S generated by unitary S-posets,

FPos_S – the full Pos-subcategory of Pos_S generated by closed S-posets,

$\mathcal{IP}_S$ – the full Pos-subcategory of Pos_S generated by objects eS, $e \in E(S)$.

Let C be an (S, T)-biposet. We get a Pos-functor $- \otimes_S C : \mathsf{Pos}_S \to \mathsf{Pos}_T$ by taking $(- \otimes_S C)(A) = A \otimes_S C$. If $f : A \to B$ is an S-poset morphism, then $f \otimes_S C : A \otimes_S C \to B \otimes_S C$ is defined by $(f \otimes_S C)(a \otimes c) = f(a) \otimes c$ for any $a \in A$, $c \in C$. There is a similar Pos-functor $\mathsf{Pos}_T(C, -) : \mathsf{Pos}_T \to \mathsf{Pos}_S$ and it is easy to prove that these two Pos-functors are a Pos-adjoint pair, similar to how it is done in Proposition 2.5.19 of [5].

A posemigroup R is called an *enlargement* of a subposemigroup S (with inherited order) if $S = SRS$ and $R = RSR$. For posemigroups S, T and R, R is said to be a *joint enlargement* (cf. [9]) of S and T if it is an enlargement of its subposemigroups $S' \cong S$ and $T' \cong T$. In general, if it does not cause any confusion, we will ignore the isomorphisms $S' \cong S$ and $T' \cong T$ and write S instead of S' and T instead of T'.

The *Cauchy completion* of a posemigroup S (cf. [9]) is the (small) Pos-category $C(S)$ that has $C(S)_0 = E(S)$, morphism posets

$$C(S)(f, e) = \{(e, s, f) \mid s \in S, esf = s\},$$

with the order $(e, s, f) \leq (e, s', f) \iff s \leq s'$ in S and the composition rule $(e, s, f) \circ (f, s', g) = (e, ss', g)$.

A category $\mathcal{C}$ is called *strongly connected* if for all $A, B \in \mathcal{C}_0$ there always exists a morphism $f : A \to B$.

A *consolidation* on a strongly connected category $\mathcal{C}$ is a map $p:\mathcal{C}_0\times\mathcal{C}_0\to\mathcal{C}$ (denoted by $p_{B,A} := p(B,A) : A \to B$) such that $p_{A,A} = 1_A$. A small Pos-category $\mathcal{C}$ with a consolidation p can be made into a posemigroup if we define a multiplication $\diamond$ by

$$g \diamond f = g \circ p_{\mathrm{dom} g,\mathrm{cod} f} \circ f$$

and take the order from the morphism-poset order of $\mathcal{C}$. This posemigroup will be denoted by $\mathcal{C}^p$. It is easy to see that if the composite $g \circ f$ exists for morphisms f and g in $\mathcal{C}$, then $g \diamond f = g \circ f$.

Let S and T be two posemigroups. We say that a sextuple

$$(S, T, P, Q, \langle -, - \rangle, [-, -])$$

is a *Morita context* if the following conditions hold:

(M1) P is an (S, T)-biposet and Q is a (T, S)-biposet;
(M2) $\langle -, - \rangle : P \otimes_T Q \to S$ is an (S, S)-biposet morphism and $[-, -] : Q \otimes_S P \to T$ is a (T, T)-biposet morphism;
(M3) the following two conditions hold for all $p, p' \in P$ and $q, q' \in Q$:
(i) $\langle p, q\rangle p' = p[q, p']$, (ii) $q\langle p, q'\rangle = [q, p]q'$.

A Morita context is called *unitary* if the biposets P and Q are unitary. We say that two posemigroups S and T are *strongly Morita equivalent* (a notion introduced for unordered semigroups by Talwar in [12]) if there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ such that the mappings $\langle -, - \rangle$ and $[-, -]$ are surjective.

2. General observations

First, let us recall from [15] the ordered analogue of Theorem 1.1 of [9], showing the equivalence of the algebraic characterizations.

Theorem 2.1 (Theorem 2.1 of [15])**.** *Let S and T be posemigroups with local units. Then the following are equivalent:*

(1) *S and T are strongly Morita equivalent,*
(2) *S and T have a joint enlargement,*
(3) *the Pos-categories $C(S)$ and $C(T)$ are Pos-equivalent.*

Note that due to the following proposition, the Cauchy completion $C(S)$ can be considered as a Pos-subcategory of Pos_S or, dually, ${}_S\mathsf{Pos}$.

Proposition 2.2. *Let S be a posemigroup with local units. Then the Pos-category $C(S)$ is Pos-equivalent to $\mathcal{IP}_S$.*

Proof. Take a right S-poset morphism $h : eS \to fS$ for some $e, f \in E(S)$. Observe that $h(es) = h(e)es$ for every $s \in S$, whence $h = \lambda_{h(e)}$. Furthermore, we have $fh(e) = h(e) = h(e^2) = h(e)e$, so $(f, h(e), e) \in C(S)(e, f)$. It is clear that the map $\lambda_s : eS \to fS$ is a right S-poset morphism whenever

$(f,s,e) \in C(S)(e,f)$. Define a functor $F : C(S) \to \mathcal{IP}_S$ by taking $F(e) = eS$ and $F(f,s,e) = \lambda_s$. It is easy to see that we obtain a Pos-functor, that it provides poset isomorphisms $\mathcal{IP}_S(eS, fS) \cong C(S)(e,f)$ and that F is (essentially) surjective on objects. □

If the posemigroups have ordered local units, then the context maps turn out to be biposet isomorphisms.

Lemma 2.3. *Let S and T be posemigroups with ordered weak local units. If $(S,T,P,Q,\langle -,-\rangle,[-,-])$ is a unitary Morita context with surjective maps, then $\langle -,-\rangle : P\otimes_T Q \to S$ and $[-,-] : Q\otimes_S P \to T$ are biposet isomorphisms.*

Proof. We will prove only the claim for $[-,-]$, since the proof for $\langle -,-\rangle$ is symmetric. It is sufficient to show that the mapping $[-,-]$ reflects order. Let $q\otimes p, q'\otimes p' \in Q\otimes P$ be such that $[q,p] \leq [q',p']$ in T. As Q is unitary, T has ordered weak local units and the mapping $[-,-]$ is surjective, we can find elements $t'' \in T$, $q_0, q_0', q'', q_1, q_2 \in Q$, $p_0, p_0', p_1, p_2 \in P$ such that $q = tq'' = [q_0,p_0]tq'' = [q_0,p_0]q$, $q' = [q_0',p_0']q'$, $[q,p] = [q,p][q_1,p_1]$, $[q',p'] = [q',p'][q_2,p_2]$ and $[q_1,p_1] \leq [q_2,p_2]$. Then in $Q\otimes_S P$

$$\begin{aligned}
q\otimes p &= [q_0,p_0]q\otimes p = q_0\langle p_0,q\rangle\otimes p = q_0\otimes\langle p_0,q\rangle p = q_0\otimes p_0[q,p][q_1,p_1]\\
&\leq q_0\otimes p_0[q,p][q_2,p_2] = q_0\otimes\langle p_0,q\rangle p[q_2,p_2] = q_0\langle p_0,q\rangle\otimes p[q_2,p_2]\\
&= [q_0,p_0]q\otimes\langle p,q_2\rangle p_2 = q\langle p,q_2\rangle\otimes p_2 = [q,p]q_2\otimes p_2 \leq [q',p']q_2\otimes p_2\\
&= q'\langle p',q_2\rangle\otimes p_2 = [q_0',p_0']q'\otimes\langle p',q_2\rangle p_2 = q_0'\langle p_0',q'\rangle\otimes p'[q_2,p_2]\\
&= q_0'\otimes\langle p_0',q'\rangle p'[q_2,p_2] = q_0'\otimes p_0'[q',p'][q_2,p_2] = q_0'\otimes p_0'[q',p']\\
&= q_0'\otimes\langle p_0',q'\rangle p' = q_0'\langle p_0',q'\rangle\otimes p' = [q_0',p_0']q'\otimes p' = q'\otimes p'.
\end{aligned}$$

□

Now, let us recall some constructions from [15].

Take a Pos-category $\mathcal{C}$. We say that $\mathcal{C}$ is *bipartite* with left part $\mathcal{A}$ and right part $\mathcal{B}$ (cf. [10]) and write $\mathcal{C} = [\mathcal{A},\mathcal{B}]$ if the following conditions hold:

- $\mathcal{A}$ and $\mathcal{B}$ are disjoint full Pos-subcategories of $\mathcal{C}$ and $\mathcal{A}_0\cup\mathcal{B}_0 = \mathcal{C}_0$,
- for every $A\in\mathcal{A}_0$ there is an isomorphism $f : A\to C$ with $C\in\mathcal{B}_0$, and for each $B\in\mathcal{B}_0$ there is an isomorphism $g : D\to B$ with $D\in\mathcal{A}_0$.

The Pos-category $\mathcal{C}$ is therefore made up of four kinds of morphisms: those of $\mathcal{A}$, those of $\mathcal{B}$, and those with either domain in $\mathcal{A}_0$ and codomain in $\mathcal{B}_0$ or domain in $\mathcal{B}_0$ and codomain in $\mathcal{A}_0$. The latter two are called *heteromorphisms*. We will denote the classes of such morphisms by $\mathcal{AA}$, $\mathcal{BB}$, $\mathcal{BA}$ and $\mathcal{AB}$, respectively. If $\mathcal{A}$ and $\mathcal{B}$ are strongly connected, then so is $\mathcal{C}$.

Proposition 3.1 of [15] establishes the following connection between bipartite Pos-categories and Pos-equivalences.

Proposition 2.4. *Two* Pos*-categories* $\mathcal{A}$ *and* $\mathcal{B}$ *are* Pos*-equivalent if and only if there exists a bipartite* Pos*-category* $\mathcal{C}$ *with left part isomorphic to* $\mathcal{A}$ *and right part isomorphic to* $\mathcal{B}$*.*

We will explicitly use the constructions from the necessity part of the above result, so a brief review is in order. For a Pos-equivalence $F : \mathcal{A} \to \mathcal{B}$, take a skeleton $\overline{\mathcal{A}}$ of $\mathcal{A}$, put $\bar{F} : \overline{\mathcal{A}} \to \overline{\mathcal{B}}$ as the Pos-isomorphism obtained by restricting F to $\overline{\mathcal{A}}$, so $\overline{\mathcal{B}}$ is a skeleton of $\mathcal{B}$, and finally fix an isomorphism $\xi_A : A \to A_0$ with $A_0 \in \overline{\mathcal{A}}_0$ for every $A \in \mathcal{A}_0$. Then put $\mathcal{C}_0 = \mathcal{A}_0 \coprod \mathcal{B}_0$. For any $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$, define heteromorphisms by taking

$$\mathcal{C}(A, B) = \{\langle A, \alpha\rangle \,|\, \alpha \in \mathcal{B}(\bar{F}(A_0), B)\}$$

and

$$\mathcal{C}(B, A) = \{\langle \alpha, A\rangle \,|\, \alpha \in \mathcal{B}(B, \bar{F}(A_0))\}.$$

Those sets inherit the order from the posets $\mathcal{B}(\bar{F}(A_0), B)$ and $\mathcal{B}(B, \bar{F}(A_0))$, respectively. We compose as follows:

1. for $f : A \to A'$ in $\mathcal{A}$, $\langle A', \alpha'\rangle \circ f = \langle A, \alpha' \bar{F}(\xi_{A'} f \xi_A^{-1})\rangle$, $f \circ \langle \alpha, A\rangle = \langle \bar{F}(\xi_{A'} f \xi_A^{-1})\alpha, A'\rangle$,
2. for $g : B \to B'$, $h : B' \to B$ in $\mathcal{B}$, $g \circ \langle A, \alpha\rangle = \langle A, g\alpha\rangle$, $\langle \alpha, A\rangle \circ h = \langle \alpha h, A\rangle$,
3. $\langle A, \alpha'\rangle \circ \langle \alpha, A\rangle = \alpha'\alpha$,
4. $\langle \alpha, A\rangle \circ \langle A', \alpha'\rangle = \xi_A^{-1} \bar{F}^{-1}(\alpha\alpha')\xi_{A'}$,
5. the compositions on $\mathcal{A}$ and $\mathcal{B}$ remain the same.

Note that for a heteromorphism $\langle A, \alpha\rangle \in \mathcal{C}(A, B)$ we do have

$$\langle A, \alpha\rangle = 1_B \circ \langle A, \alpha\rangle \quad \text{and} \quad \langle A, \alpha\rangle = \langle A, \alpha\rangle \circ 1_A. \tag{1}$$

Lemma 2.5. *Let* H *be a binary relation on a posemigroup* S *and let* ρ *be the admissible preorder generated by* H*. Then* $s \,\rho\, t$ *if and only if there exist* $n \in \mathbb{N}$ *and for* $1 \leq i \leq n$ *elements* $s_i, t_i, u_i, v_i \in S$, $x_i, y_i \in S^1$ *such that* $s = s_1$, $s_i \leq t_i$, $t_n = t$*; and if* $1 \leq i < n$*, then* $t_i = x_i u_i y_i$, $s_{i+1} = x_i v_i y_i$ *and* $(u_i, v_i) \in H$*.*

Take a small bipartite Pos-category $\mathcal{C} = [\mathcal{A}, \mathcal{B}]$ with strongly connected parts $\mathcal{A}$ and $\mathcal{B}$. Let p be a consolidation on $\mathcal{A}$ and q be a consolidation on $\mathcal{B}$. Fix an isomorphism $\xi : B_0 \leftarrow A_0 \in \mathcal{BA}$ and define a consolidation r on $\mathcal{C}$ as follows:

$$r_{Y,X} = \begin{cases} p_{Y,X} & \text{if } X, Y \in \mathcal{A}_0, \\ q_{Y,X} & \text{if } X, Y \in \mathcal{B}_0, \\ q_{Y,B_0}\xi p_{A_0,X} & \text{if } X \in \mathcal{A}_0, Y \in \mathcal{B}_0, \\ p_{Y,A_0}\xi^{-1} q_{B_0,X} & \text{if } X \in \mathcal{B}_0, Y \in \mathcal{A}_0. \end{cases}$$

Then r is said to be the *natural extension of* p *and* q *to* $\mathcal{C}$ *via* ξ.

We will now extend the following result.

Theorem 2.6 (Theorem 3.1 of [13]). *Let S and T be posemigroups with local units. If their Cauchy completions $C(S)$ and $C(T)$ are* Pos*-equivalent, then S and T have a joint enlargement R.*

Again, we first outline the constructions used in [13]. Let $C = [C(S), C(T)]$ be the small bipartite Pos-category that exists due to Proposition 2.4. We define consolidations p on $C(S)$ and q on $C(T)$ by taking

$$p_{e,f} = (e, ef, f) \quad \text{and} \quad q_{i,j} = (i, ij, j)$$

for all idempotents $e, f \in S$ and all idempotents $i, j \in T$.

Take an isomorphism $\xi : e_0 \to i_0 \in C(T)C(S)$ and define the natural extension r of p and q to C via ξ. There are congruences ρ_1 and ρ_2 for which

$$\begin{aligned} (e, s, f) \ \rho_1 \ (e', s', f') &\iff s \leq s' \text{ in } S, \\ (i, t, j) \ \rho_2 \ (i', t', j') &\iff t \leq t' \text{ in } T, \end{aligned}$$

for $e, e', f, f' \in E(S)$, $s, s' \in S$, $i, i', j, j' \in E(T)$, $t, t' \in T$. In this way, we get posemigroup isomorphisms $C(S)^p/\rho_1 \cong S$ and $C(T)^q/\rho_2 \cong T$. Take ρ as the admissible preorder generated by $\rho_1 \cup \rho_2$. Then $R = C^r/\rho$ is a joint enlargement of S and T via the isomorphic copies $C(S)^r/\rho \cong C(S)^p/\rho_1 \cong S$ and $C(T)^r/\rho \cong C(T)^q/\rho_2 \cong T$. Note that in order to show this, we use the fact that $\rho \cap (C(S) \times C(S)) = \rho_1$ and $\rho \cap (C(T) \times C(T)) = \rho_2$.

Proposition 2.7. *If both the posemigroups S and T in Theorem 2.6 have common local units, then R also has common local units.*

Proof. Let the posemigroups S and T have common local units and take comparable $[c]_{\rho\cap\rho^{-1}} \leq [c']_{\rho\cap\rho^{-1}}$ in $R = C^r/\rho$, i.e. $c \, \rho \, c'$. As ρ is generated by ρ_1 and ρ_2, we can use Lemma 2.5 to find $n \in \mathbb{N}$ and for $1 \leq i \leq n$ elements $c_i, c'_i, u_i, v_i \in C^r$, $x_i, y_i \in (C^r)^1$ such that $c = c_1$, $c_i \leq c'_i$, $c'_n = c'$, and if $1 \leq i < n$ then

$$c'_i = x_i \diamond u_i \diamond y_i, \quad c_{i+1} = x_i \diamond v_i \diamond y_i \quad \text{and} \quad (u_i, v_i) \in \rho_1 \cup \rho_2.$$

We have four possibilities:

(1) $c \in C(S)$,
(2) $c \in C(T)$,
(3) $c \in C(T)C(S)$,
(4) $c \in C(S)C(T)$.

It is sufficient to consider only cases (1) and (3). If $c \in C(S)$, then $c'_1 \in C(S)$ as well, because the order on C^r is derived from the morphism-poset order of C. Now there are four subcases:

(a) $x_1 \neq 1$, $y_1 \neq 1$,
(b) $x_1 = 1$, $y_1 \neq 1$,
(c) $x_1 \neq 1$, $y_1 = 1$,
(d) $x_1 = 1$, $y_1 = 1$.

For (a), we have $x_1 \in C(S)*$ and $y_1 \in *C(S)$, where $* \in \{C(S), C(T)\}$, so we get $c_2 = x_1 \diamond v_1 \diamond y_1 \in C(S)$. In subcase (b), we have $u_1 \in C(S)*$, $y_1 \in *C(S)$, whence $u_1 \rho_1 v_1$ in $C(S)$ and consequently $c_2 = v_1 \diamond y_1 \in C(S)$. Subcases (c) and (d) similarly yield $u_1 \rho_1 v_1$ in $C(S)$ (and $x_1 \in C(S)*$ in subcase (c)), so $c_2 \in C(S)$ again. These two steps can be repeated to get $c_1, c_1', c_2, \ldots, c_n, c_n' = c' \in C(S)$.

For case (3), we have the same subcases (a)-(d) and can again perform a similar analysis to get $c_1' \in C(T)C(S)$, $x_1 \in C(T) * \cup \{1\}$, $y_1 \in *C(S) \cup \{1\}$. Once more we have $u_1 \rho_j v_1$ for subcases (c) (with $j = 1$) and (b) (with $j = 2$). Subcase (d) never arises. Thus $c_2 \in C(T)C(S)$ as well and we can again repeat the two steps until we have $c_1, c_1', c_2, \ldots, c_n, c_n' = c' \in C(T)C(S)$.

Since the assumptions of Theorem 2.6 are still fulfilled, we can deduce that $\rho \cap (C(S) \times C(S)) = \rho_1$ and $\rho \cap (C(T) \times C(T)) = \rho_2$. So case (1) reduces to $c \, \rho_1 \, c'$, i.e. the relation $[c]_{\rho \cap \rho^{-1}} \leq [c']_{\rho \cap \rho^{-1}}$ in the subposemigroup $C^p/\rho_1 \cong S$. Since S has common local units, this concludes case (1).

For case (3), we have already shown that $c = c_1, c' = c_n' \in C(T)C(S)$, i.e. they are heteromorphisms. Then there exist elements $e, e' \in E(S)$ and $j, j' \in E(T)$ such that $c \in C(e, j)$ and $c' \in C(e', j')$. Therefore we can write $c = \langle e, (j, t, i) \rangle$ and $c' = \langle e', (j', t', i') \rangle$, where $(j, t, i) \in C(T)(i, j)$, $(j', t', i') \in C(T)(i', j')$ and $i = \bar{F}(e_0)$, $i' = \bar{F}(e'_0)$, using the notation of Proposition 2.4. As S and T have common local units, there exist $e'' \in E(S)$ and $j'' \in E(T)$ such that $j''t = t$, $j''t' = t'$, $ee'' = e$ and $e'e'' = e'$. Then we have $(j'', t, i) \in C(T)(i, j'')$ and $(j'', t', i') \in C(T)(i', j'')$, implying that $\langle e, (j'', t, i) \rangle \in C(e, j'')$ and $\langle e', (j'', t', i') \rangle \in C(e', j'')$. Moreover, clearly $(e, e, e'') \in C(S)(e'', e)$ and $(e', e', e'') \in C(S)(e'', e')$, so we have that $\xi_e(e, e, e'')\xi_{e''}^{-1} \in C(S)(e''_0, e_0)$ and $\xi_{e'}(e', e', e'')\xi_{e''}^{-1} \in C(S)(e''_0, e'_0)$ as well. If we denote $\bar{F}(e''_0) = i''$, then there exist elements $t'' \in iTi''$ and $t''' \in i'Ti''$ such that we can write $\bar{F}(\xi_e(e, e, e'')\xi_{e''}^{-1}) = (i, t'', i'')$ and $\bar{F}(\xi_{e'}(e', e', e'')\xi_{e''}^{-1}) = (i', t''', i'')$.

Observe that we have $(j, t, i) \, \rho \cap \rho^{-1} \, (j'', t, i)$, $(j', t', i') \, \rho \cap \rho^{-1} \, (j'', t', i')$, $(e, e, e) \, \rho \cap \rho^{-1} \, (e, e, e'')$ and $(e', e', e') \, \rho \cap \rho^{-1} \, (e', e', e'')$. So

$$\begin{aligned} c \quad &= \quad \langle e, (j, t, i) \rangle = (j, t, i) \diamond \langle e, (i, i, i) \rangle \diamond (e, e, e) \\ \rho \cap \rho^{-1} \quad & \quad (j'', t, i) \diamond \langle e, (i, i, i) \rangle \diamond (e, e, e'') = \langle e, (j'', t, i) \rangle \diamond (e, e, e'') \\ &= \quad \langle e'', (j'', t, i)\bar{F}(\xi_e(e, e, e'')\xi_{e''}^{-1}) \rangle = \langle e'', (j'', t, i)(i, t'', i'')) \rangle \\ &= \quad \langle e'', (j'', tt'', i'') \rangle \end{aligned}$$

and

$$\begin{aligned} c' \quad &= \quad \langle e', (j', t', i') \rangle = (j', t', i') \diamond \langle e', (i', i', i') \rangle \diamond (e', e', e') \\ \rho \cap \rho^{-1} \quad & \quad (j'', t', i') \diamond \langle e', (i', i', i') \rangle \diamond (e', e', e'') = \langle e', (j'', t', i') \rangle \diamond (e', e', e'') \\ &= \quad \langle e'', (j'', t', i')\bar{F}(\xi_{e'}(e', e', e'')\xi_{e''}^{-1}) \rangle = \langle e'', (j'', t', i')(i', t''', i'')) \rangle \\ &= \quad \langle e'', (j'', t't''', i'') \rangle. \end{aligned}$$

Therefore

$$c\,\rho\cap\rho^{-1}\,\langle e'',(j'',tt'',i'')\rangle \text{ and } c'\,\rho\cap\rho^{-1}\,\langle e'',(j'',t't''',i'')\rangle$$

in C^r. Equations (1) yield that

$$\begin{aligned}[c]_{\rho\cap\rho^{-1}}[(e'',e'',e'')]_{\rho\cap\rho^{-1}} &= [\langle e'',(j'',tt'',i'')\rangle \diamond (e'',e'',e'')]_{\rho\cap\rho^{-1}}\\ &= [\langle e'',(j'',tt'',i'')\rangle]_{\rho\cap\rho^{-1}} = [c]_{\rho\cap\rho^{-1}}\end{aligned}$$

and

$$\begin{aligned}[(j'',j'',j'')]_{\rho\cap\rho^{-1}}[c]_{\rho\cap\rho^{-1}} &= [(j'',j'',j'') \diamond \langle e'',(j'',tt'',i'')\rangle]_{\rho\cap\rho^{-1}}\\ &= [\langle e'',(j'',tt'',i'')\rangle]_{\rho\cap\rho^{-1}} = [c]_{\rho\cap\rho^{-1}}.\end{aligned}$$

Similarly

$$[c']_{\rho\cap\rho^{-1}}[(e'',e'',e'')]_{\rho\cap\rho^{-1}} = [c']_{\rho\cap\rho^{-1}},$$

and

$$[(j'',j'',j'')]_{\rho\cap\rho^{-1}}[c']_{\rho\cap\rho^{-1}} = [c']_{\rho\cap\rho^{-1}}.$$

This concludes case (3) and our proof is complete. □

Corollary 2.8. *Let S and T be posemigroups with common local units. If their Cauchy completions $C(S)$ and $C(T)$ are* Pos*-equivalent and R is the joint enlargement of S and T constructed in Theorem 2.6, then the biposets $P = SRT \in {}_S\mathsf{Pos}_T$ and $Q = TRS \in {}_T\mathsf{Pos}_S$ are* Pos*-unitary.*

Proof. We merely observe that $Q = TRS$ is a subposet of $C(T)C(S)/\rho$. The last part of the proof of Theorem 2.6 demonstrates that the elements of $C(T)C(S)/\rho$ have common local units, which are a stronger version of the ordered local units required in the definition of Pos-unitarity. The case for P is symmetrical via $C(S)C(T)/\rho$. □

For illustrative purposes, we present a simple example of the construction of a joint enlargement. This example also demonstrates that the requirement for both S and T to have common local units is not necessary to get Pos-unitary biposets SRT and TRS.

Example 2.9. Let $(S,\leq)$ be a left zero posemigroup and $(T,\leq)$ be another left zero posemigroup. Take the rectangular poband $C = (S\coprod T)\times(S\coprod T)$ with the discrete order $\preceq$. Consider the binary relation

$$(r_1,r_2)\rho(r_1',r_2') \Longleftrightarrow (r_1\leq r_1')\wedge(r_2,r_2'\in S\vee r_2,r_2'\in T).$$

It is easy to see that ρ is an admissible preorder. Consider $R = C/\rho$. Then $R\cong(S\coprod T)\times\{S,T\}$ is a joint enlargement of the two subposemigroups $S\cong S\times\{S\}\subseteq(S\coprod T)\times\{S,T\}$ and $T\cong T\times\{T\}\subseteq(S\coprod T)\times\{S,T\}$. It is easy to verify this directly because every rectangular band satisfies the identity $xyx = x$, yielding the required equalities $R = RSR$, $S = SRS$, $R = RTR$ and $T = TRT$. In the same way, the biposet SRT can be written as $SRT\cong S\times\{T\}\subseteq(S\coprod T)\times\{S,T\}$. If S has only ordered local units but not common local units, then SRT is still Pos-unitary. Similarly, if T has ordered local units but not common local units, then the biposet $TRS \cong T \times \{S\} \subseteq (S \coprod T) \times \{S, T\}$ is also Pos-unitary.

3. Closed S-posets

We extend to posemigroups Banaschewski's observation (Proposition 5 of [1]) that one cannot define two semigroups to be Morita equivalent if and only if their categories of right posets are equivalent. We also remark on an issue that may seem to arise from this fact.

Theorem 3.1. *Let S and T be arbitrary posemigroups. If the Pos-categories Pos_S and Pos_T are Pos-equivalent, then S and T are isomorphic.*

Proof. We can just use Theorem 7 of [6] and repeat Banaschewski's original argument. □

So we cannot use Pos_S and Pos_T to define Morita equivalence and need to restrict ourselves to some Pos-subcategory of Pos_S. A good candidate is the Pos-category of closed right S-posets FPos_S. But there is the following slightly alarming observation, the consequences of which are dealt with in Remark 3.4.

Lemma 3.2. *Let S be a posemigroup with common weak local units. Then all unitary right S-posets are closed.*

Proof. Let X be a unitary right S-poset. We have to verify that the morphism $\mu_X : X \otimes_S S \to X$ reflects order. Take $x, x' \in X$, $s, s' \in S$ and let $xs \leq x's'$ in X. Then there exists $e \in E(S)$ such that $s = se$ and $s' = s'e$. Thus
$$x \otimes s = x \otimes se = xs \otimes e \leq x's' \otimes e = x' \otimes s'e = x' \otimes s'.$$
□

Proposition 3.3. *Let S and T be posemigroups with common local units. Then the following are equivalent:*

(1) UPos_S *and* UPos_T *are* Pos-*equivalent,*
(2) FPos_S *and* FPos_T *are* Pos-*equivalent,*
(3) $C(S)$ *and* $C(T)$ *are* Pos-*equivalent.*

Proof. First, Lemma 3.2 provides (1)⇔(2). The equivalence (2)⇔(3) will be proved for posemigroups with ordered local units in Theorem 4.10 and Corollary 5.4. □

Remark 3.4. While Lemma 3.2 shows that all unitary right S-posets are closed, there are always non-unitary S-posets. For example, any poset with at least two elements and with S-actions that all map every element to a single fixed element are non-unitary. Such posets are, of course, not pomonoid S-posets if S happens to be a pomonoid. Therefore we do not have the situation where if S and T are posemigroups with common local units, then all right S-posets are closed and so $S \cong T$ by Theorem 3.1. This would, for example, trivialize a number of our results on Morita invariants (see [14]).

4. From closed S-posets to Cauchy completions

We will now modify Lawson's proof (see Theorem 1.1 of [9]) that Morita equivalence implies the equivalence of Cauchy completions and show the same for posemigroups.

Lemma 4.1 (Lemma 4.1 of [13])**.** *Let T, P and Q be subposemigroups of some posemigroup R. Furthermore, let P be a right T-poset and Q a left T-poset with respect to actions defined by multiplication in R. If $p \otimes q \leq p' \otimes q'$ in $P \otimes_T Q$, then $pq \leq p'q'$ in R.*

Lemma 4.2. *Let S be a posemigroup with ordered weak local units. Then the right S-poset S_S is closed.*

Proof. Proving that the map $s \otimes s' \mapsto ss'$ is a right S-poset morphism is easy if one keeps in mind Lemma 4.1. It is surjective because S is factorizable. Finally, take $ss' \leq tt'$ for some $s, s', t, t' \in S$. Then there exist $e, e', g, g' \in S$ such that $ess' = ss'$, $e'tt' = tt'$, $e \leq e'$, $s'g = s'$ and $t'g' = t'$. So

$$s \otimes s' = ess' \otimes g = e \otimes ss' \leq e' \otimes tt' = e'tt' \otimes g' = t \otimes t'.$$

□

Lemma 4.3. *Let S be a posemigroup such that S_S is closed and let X be a right S-poset. Then $X \otimes_S S$ is a closed right S-poset.*

Proof. Because the right S-poset S_S is closed, we have a right S-poset isomorphism $(X \otimes_S S) \otimes_S S \to X \otimes_S (S \otimes_S S) \to X \otimes_S S$, where $(x \otimes s) \otimes s' \mapsto x \otimes (s \otimes s') \mapsto x \otimes (ss')$. But this is exactly $\mu_{X \otimes_S S}$, since $\mu_{X \otimes_S S}((x \otimes s) \otimes s') = (x \otimes s)s' = x \otimes (ss')$. Therefore $X \otimes_S S$ is closed. □

We can now extend Proposition 2.3 of [9] as follows.

Lemma 4.4. *Let S be a posemigroup such that S_S is closed and let X be a unitary right S-poset. Then the following are equivalent:*

1) *X is closed;*
2) *$X \otimes_S S \cong X$ for some isomorphism in Pos_S.*

Proof. Implication 1) ⇒ 2) is trivial. For the converse, let $\varphi : X \otimes_S S \to X$ be a right S-poset isomorphism. Then $\varphi \otimes_S S : (X \otimes_S S) \otimes_S S \to X \otimes_S S$ must also be an isomorphism in Pos_S because Pos-functors preserve isomorphisms.

Observe that $\mu_X : X \otimes_S S \to X$ is natural in X, since for any $f : X \to X'$ and $x \in X$, $s \in S$ we have

$$(\mu_{X'} \circ (f \otimes_S S))(x \otimes s) = f(x)s = f(xs) = (f \circ \mu_X)(x \otimes s).$$

Therefore we have a commutative diagram in Pos_S:

$$\begin{array}{ccc} (X \otimes_S S) \otimes_S S & \xrightarrow{\mu_{X \otimes_S S}} & X \otimes_S S \\ \Big\downarrow{\scriptstyle \varphi \otimes_S S} & & \Big\downarrow{\scriptstyle \varphi} \\ X \otimes_S S & \xrightarrow[\mu_X]{} & X \end{array} .$$

Now, $X \otimes_S S$ is closed by Lemma 4.3 and therefore $\mu_{X \otimes_S S}$ is an isomorphism. Thus μ_X must also be an isomorphism. □

The previous lemma allows us to ignore checking that an isomorphism $X \otimes_S S \cong X$ is actually μ_X if we need to show that a unitary right S-poset X is closed.

Lemma 4.5. *Coproducts in Pos_S and FPos_S are constructed as disjoint unions with componentwise order and action.*

Proof. For Pos_S, this is essentially proved in Section 2 of [2]. Observe that the Pos-functor $- \otimes_S S : \mathsf{Pos}_S \to \mathsf{Pos}_S$ as a left adjoint preserves colimits. Thus if we take $X_i \in \mathsf{FPos}$, $i \in I$, we get the following isomorphisms in Pos_S:

$$(\coprod_i X_i) \otimes_S S \cong \coprod_i (X_i \otimes_S S) \cong \coprod_i X_i.$$

So a Pos_S-coproduct of closed right S-posets is closed by Lemma 4.4. Because FPos_S is a full Pos-subcategory, the FPos_S-coproduct of closed right S-posets is the same as in Pos_S. □

The amalgamated coproduct of a right S-poset B with itself over an S-subposet $B' \subseteq B$ is the pushout of the embedding $B' \hookrightarrow B$ along itself. Let us recall from Section 2 of [2] that an amalgamated coproduct can be realized as the set $(\{1, 2\} \times (B \setminus B')) \cup B'$ with the action

$$(i, b)s = \begin{cases} (i, bs) & \text{if } bs \in B \setminus B' \\ bs & \text{if } bs \in B' \end{cases}, i = 1, 2, b \in B \setminus B', s \in S.$$

The order relation for $b_1, b_2 \in B \setminus B'$, $i, j \in \{1, 2\}$ is

$$(i, b_1) \leq (j, b_2) \iff (i = j \wedge b_1 \leq b_2) \vee (i \neq j \wedge b_1 \leq b'' \leq b_2 \text{ for some } b'' \in B').$$

For $b' \in B'$, $b \in B \setminus B'$, $i \in \{1, 2\}$, it is

$$b' \leq (i, b) \iff b' \leq b \quad \text{and} \quad (i, b) \leq b' \iff b \leq b'.$$

The amalgamated coproduct of B with itself over B' is denoted by $B\coprod_{B'}B$. The pushout morphisms $i_1 : B \to B\coprod_{B'}B$ and $i_2 : B \to B\coprod_{B'}B$ are defined by

$$i_j(b) = \begin{cases} (j,b) & \text{if } b \in B \setminus B' \\ b & \text{if } b \in B'. \end{cases}$$

We can now prove the ordered counterpart of Proposition 2.4 of [9].

Lemma 4.6. *Let S be a posemigroup such that S_S is closed. Then all epimorphisms in FPos_S are surjective.*

Proof. Let $f : A \to B$ be an epimorphism in FPos_S. Lemma 4.3 shows that $A\otimes_S S$ and $B\otimes_S S$ are closed. Since the same lemma also demonstrates that μ_X is natural in X, we have the following commutative diagram in FPos_S:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \mu_A}\uparrow & & \uparrow{\scriptstyle \mu_B} \\ A\otimes S & \xrightarrow[f\otimes S]{} & B\otimes_S S. \end{array}$$

Since μ_A is an epimorphism, $f\otimes_S S \in \mathsf{FPos}_S$ is a product of epimorphisms and therefore must also be an epimorphism. Assume that f is not surjective, put $B' = \operatorname{Im} f$, take the embedding $\iota : B' \hookrightarrow B$ and the amalgamated coproduct $C := B\coprod_{B'}B$ in Pos_S as constructed above and let $i_1, i_2 : B \to C$ be the pushout maps. Then obviously $i_1 f = i_2 f$. Yet for all $b \in B \setminus B'$ we get that $i_1(b) = (1,b) \neq (2,b) = i_2(b)$. Observe that due to Lemma 4.3 we have the following commutative diagram in FPos_S:

$$A\otimes_S S \xrightarrow{f\otimes_S S} B\otimes_S S \underset{i_2\otimes_S S}{\overset{i_1\otimes_S S}{\rightrightarrows}} C\otimes_S S.$$

Take $b \in B\setminus B' \neq \emptyset$ and $s \in S$ such that $bs = b$ (this can be done as B is unitary). If $(i_1\otimes_S S)(b\otimes s) = (i_2\otimes_S S)(b\otimes s)$, then $i_1(b)\otimes s = i_2(b)\otimes s$. Therefore we get that $\mu_C(i_1(b)\otimes s) = \mu_C(i_2(b)\otimes s)$, which implies that $i_1(b) = i_1(bs) = i_1(b)s = i_2(b)s = i_2(bs) = i_2(b)$, a contradiction. So f must be surjective. ☐

We say that an object A from some category $\mathcal{A}$ of either S-acts or S-posets is *indecomposable* if there do not exist non-initial objects $A_1, A_2 \in \mathcal{A}_0$ such that $A \cong A_1 \coprod A_2$. In particular, we use this in the cases $\mathcal{A} = \mathsf{Act}_S$, $\mathcal{A} = \mathsf{Pos}_S$ and $\mathcal{A} = \mathsf{FPos}_S$.

Next, we extend, respectively, Lemma 3.1 and Lemma 3.2 of [9] as follows.

Lemma 4.7. *Let S be a posemigroup such that S_S is closed. Then the right S-posets eS, $e \in E(S)$, are indecomposable and projective in FPos_S.*

Proof. First, the right S-posets eS are clearly unitary because $es = (ee)s$ for any $s \in S$. If we take $s, s', t, t' \in S$ and let $est \leq es't'$, then

$$es \otimes t = ees \otimes t = e \otimes est \leq e \otimes es't' = ees' \otimes t' = es' \otimes t'$$

in $eS \otimes_S S$. So eS are also closed.

Showing that eS are indecomposable and projective is standard due to Lemma 4.6 (cf. Lemma 1.5.9 and Proposition 3.17.2 of [5]). □

Lemma 4.8. *Let S be a posemigroup with ordered local units. For every $A \in \mathsf{FPos}_S$ there exists a projective $P \in \mathsf{FPos}_S$ and an epimorphism $\pi : P \to A$ in FPos_S.*

Proof. Take $A \in \mathsf{FPos}_S$. Since A is unitary and S has local units, for every $a \in A$ there exists $e_a \in E(S)$ such that $ae_a = a$. Form the coproduct $\coprod_{a \in A} e_a S$. Because $e_a S$ are closed and the left adjoint $- \otimes_S S : \mathsf{Pos}_S \to \mathsf{Pos}_S$ preserves coproducts, $\coprod_{a \in A} e_a S$ is closed by Lemma 4.4. Note that the use of Lemma 4.4 is legitimate due to Lemma 4.2. Since $e_a S$ are also projective and coproducts of projectives are projective, $\coprod_{a \in A} e_a S$ is a projective in FPos_S. As the S-poset morphism $\pi : \coprod_{a \in A} e_a S \to A$, defined by $\pi(e_a s) = as$, is an epimorphism in Pos_S, it is also an epimorphism in FPos_S. □

Once we have those two lemmas, we can easily get the ordered version of Proposition 3.3 of [9].

Proposition 4.9. *Let S be a posemigroup with ordered local units. Then a closed right S-poset is indecomposable and projective in FPos_S if and only if it is isomorphic to eS for some $e \in E(S)$.*

Proof. The proof is again standard (for details, see Theorem 3.17.8 of [5]), using Lemmas 4.2, 4.7 and 4.8. □

Finally, we complete the section by proving the ordered counterpart of Theorem 3.4 of [9].

Theorem 4.10. *Let S and T be posemigroups with ordered local units. If FPos_S and FPos_T are Pos-equivalent, then $C(S)$ and $C(T)$ are also Pos-equivalent.*

Proof. Let the Pos-functors $F : \mathsf{FPos}_S \to \mathsf{FPos}_T$ and $G : \mathsf{FPos}_T \to \mathsf{FPos}_S$ form a Pos-equivalence. Since F and G map indecomposable projectives to indecomposable projectives, the full Pos-subcategories FIP_S and FIP_T of indecomposable projectives are Pos-equivalent. Due to Proposition 4.9, each indecomposable projective in FPos_S is isomorphic to eS for some $e \in E(S)$, so those poset isomorphisms provide a Pos-equivalence between FIP_S and $\mathcal{IP}_S$. Similarly, FIP_T is Pos-equivalent to $\mathcal{IP}_T$. Proposition 2.2 shows that the Pos-categories $\mathcal{IP}_S$ and $C(S)$ (and $\mathcal{IP}_T$ and $C(T)$) are Pos-equivalent and therefore $C(S)$ and $C(T)$ are Pos-equivalent. □

5. From Cauchy completions to closed S-posets

We now prove the converse of what we showed in the previous section, namely that the Pos-equivalence of the Cauchy completions $C(S)$ and $C(T)$ implies the Pos-equivalence of the Pos-categories FPos_S and FPos_T. The prototype is still Theorem 1.1 of [9], although the latter does not explicitly cover this implication.

We start by proving the ordered version of Proposition 3.14 of [9].

Lemma 5.1. *Let S and T be two posemigroups with local units that have a joint enlargement R such that the biposets $P = SRT \in {}_S\mathsf{Pos}_T$ and $Q = TRS \in {}_T\mathsf{Pos}_S$ are Pos-unitary. Then the biposets P and Q constructed for the Morita context $(S, T, P, Q, \langle -, -\rangle, [-, -])$ in implication (2)⇒(1) of Theorem 2.1 are closed as right T- and S-posets .*

Proof. Since $P = SRT$ and $Q = TRS$ are Pos-unitary, they are also unitary. For them to be closed as well it is sufficient to prove that the mappings μ_P and μ_Q are order-reflecting. We will show this only for $\mu_P : P \otimes_T T \to P$, since the case for Q is essentially the same. Let $pt \leq p't'$ in P. As $P = SRT$ and T has weak local units, we can write

$$p = s_1 r_1 t_1, p' = s_1' r_1' t_1', t = tu_1, t' = t'u_2$$

for $s_1, s_1' \in S$, $t_1, t_1', u_1, u_2 \in T$, $r_1, r_1' \in R$. Because P is Pos-unitary, $S \subseteq R = RTR$ and T has weak local units, we can find $s_2, s_2' \in E(S)$, $r_3, r_4 \in R$, $t_2, u \in T$ so that

$$s_2 pt = pt, s_2' p't' = p't', s_2 \leq s_2', s_2' = r_3 t_2 r_4 \text{ and } t_2 = ut_2.$$

Then $t_2 r_4 p = t_2 r_4 s_1 r_1 t_1 \in TRSRT = TRT = T$, $t_2 r_4 p' = t_2 r_4 s_1' r_1' t_1' \in T$ and $s_2' r_3 u \in SRT = P$. We can now calculate in $P \otimes_T T$ that

$$\begin{aligned}
p \otimes t &= p \otimes tu_1 = pt \otimes u_1 = s_2 pt \otimes u_1 \leq s_2' pt \otimes u_1 = (s_2')^2 pt \otimes u_1 \\
&= s_2'(r_3 t_2 r_4) pt \otimes u_1 = s_2' r_3 (ut_2) r_4 pt \otimes u_1 = s_2' r_3 u t_2 r_4 p \otimes tu_1 \\
&= s_2' r_3 u t_2 r_4 p \otimes t = (s_2' r_3 u)(t_2 r_4 p) \otimes t = s_2' r_3 u \otimes (t_2 r_4 p) t \\
&= s_2' r_3 u \otimes (t_2 r_4)(pt) \leq s_2' r_3 u \otimes (t_2 r_4)(p't') = s_2' r_3 u \otimes (t_2 r_4 p') t' \\
&= s_2' r_3 u \otimes (t_2 r_4 p')(t' u_2) = s_2' r_3 u \otimes ((t_2 r_4 p') t') u_2 \\
&= (s_2' r_3 u)(t_2 r_4 p') t' \otimes u_2 = s_2' r_3 (ut_2) r_4 p' t' \otimes u_2 \\
&= s_2'(r_3 t_2 r_4) p' t' \otimes u_2 = (s_2')^2 (p't') \otimes u_2 = s_2'(p't') \otimes u_2 \\
&= p't' \otimes u_2 = p' \otimes t' u_2 = p' \otimes t'.
\end{aligned}$$

□

Corollary 5.2. *Let S and T be two posemigroups with ordered local units that have a joint enlargement R and* Pos*-unitary* ${}_SSRT_T \in {}_S\mathsf{Pos}_T$ *and* ${}_TTRS_S \in {}_T\mathsf{Pos}_S$*. Then there exists a unitary Morita context* $(S, T, P, Q, \langle -, - \rangle, [-, -])$ *with surjective maps where P and Q are closed right S- and T-posets.*

Proof. Due to implication (2)⇒(1) of Theorem 2.1, there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps. Since we have Pos-unitary $P = SRT$ and $Q = TRS$, we can apply Lemma 5.1 and get that P and Q are closed right S- and T-posets. □

It is now easy to get the following counterpart of Proposition 3.16 of [9].

Theorem 5.3. *Let S and T be posemigroups with ordered local units. If* $(S, T, P, Q, \langle -, - \rangle, [-, -])$ *is a unitary Morita context with surjective maps and P and Q are closed, then the* Pos*-categories* FPos_S *and* FPos_T *are* Pos*-equivalent via* Pos*-functors*

$$- \otimes_Q P : \mathsf{FPos}_S \to \mathsf{FPos}_T \quad \text{and} \quad - \otimes_T Q : \mathsf{FPos}_T \to \mathsf{FPos}_S.$$

Proof. The proof closely mirrors Proposition 3.16 of [9]. □

Corollary 5.4. *Let S and T be posemigroups with common local units. If the Cauchy completions $C(S)$ and $C(T)$ are* Pos*-equivalent, then the* Pos*-categories* FPos_S *and* FPos_T *are* Pos*-equivalent.*

Proof. Theorem 2.6 and Corollary 2.8 show that S and T have a joint enlargement R such that $P = SRT \in {}_S\mathsf{Pos}_T$ and $Q = TRS \in {}_T\mathsf{Pos}_S$ are Pos-unitary. By Corollary 5.2, there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps where P and Q are closed right S- and T-posets. Therefore we can apply Theorem 5.3 to get that the Pos-categories FPos_S and FPos_T are Pos-equivalent. □

Example 2.9 shows that our assumption of the existence of common local units is an artefact of the proof used in Theorem 2.6 and it is not necessary for satisfying the Pos-unitarity requirement of Corollary 5.2. Since most of our results hold for posemigroups with ordered local units and unordered semigroups have ordered local units if and only if they have local units, we have the following natural open question.

Problem 5.5. Let S and T be posemigroups with ordered local units. If the Cauchy completions $C(S)$ and $C(T)$ are Pos-equivalent, are the Pos-categories FPos_S and FPos_T also Pos-equivalent?

We have now the ordered version of Lemma 3.17 of [9].

Lemma 5.6. *Let S and T be two posemigroups with ordered local units. If* $(S, T, P, Q, \langle -, - \rangle, [-, -])$ *is a unitary Morita context with surjective maps* *and P and Q are closed as right posets, then they are also closed as left posets.*

Proof. By Lemma 2.3, we have the following (S,T)-biact isomorphisms:

$$S \otimes P \cong (P \otimes Q) \otimes P \cong P \otimes (Q \otimes P) \cong P \otimes T \cong P \quad \text{and}$$
$$T \otimes Q \cong (Q \otimes P) \otimes Q \cong Q \otimes (P \otimes Q) \cong Q \otimes S \cong Q. \qquad \square$$

This allows us to get the ordered equivalent of Theorem 3.18 of [9].

Proposition 5.7. *Let S and T be two posemigroups with common local units. If the* Pos*-categories* FPos_S *and* FPos_T *are* Pos*-equivalent, then the* Pos*-categories* ${}_S\mathsf{FPos}$ *and* ${}_T\mathsf{FPos}$ *are also* Pos*-equivalent.*

Proof. By Theorem 2.1, Theorem 4.10 and Corollary 5.4, the Pos-categories FPos_S and FPos_T are Pos-equivalent if and only if there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps and P and Q closed as right posets. Dualizing this result to the Pos-categories ${}_S\mathsf{FPos}$ and ${}_T\mathsf{FPos}$ yields a similar condition with P and Q closed as left posets. Now we only need to apply Lemma 5.6. $\square$

Theorem 4.10 and Corollary 5.4 allow us to refine Theorem 2.1 to

Theorem 5.8. *Let S and T be posemigroups with common local units. Then the following are equivalent:*

(1) *S and T are strongly Morita equivalent,*
(2) *S and T have a joint enlargement,*
(3) *the* Pos*-categories $C(S)$ and $C(T)$ are* Pos*-equivalent,*
(4) *the* Pos*-categories* FPos_S *and* FPos_T *are* Pos*-equivalent.*

Due to this, it is reasonable to define that two posemigroups S and T are *Morita equivalent* if the Pos-categories FPos_S and FPos_T are Pos-equivalent.

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